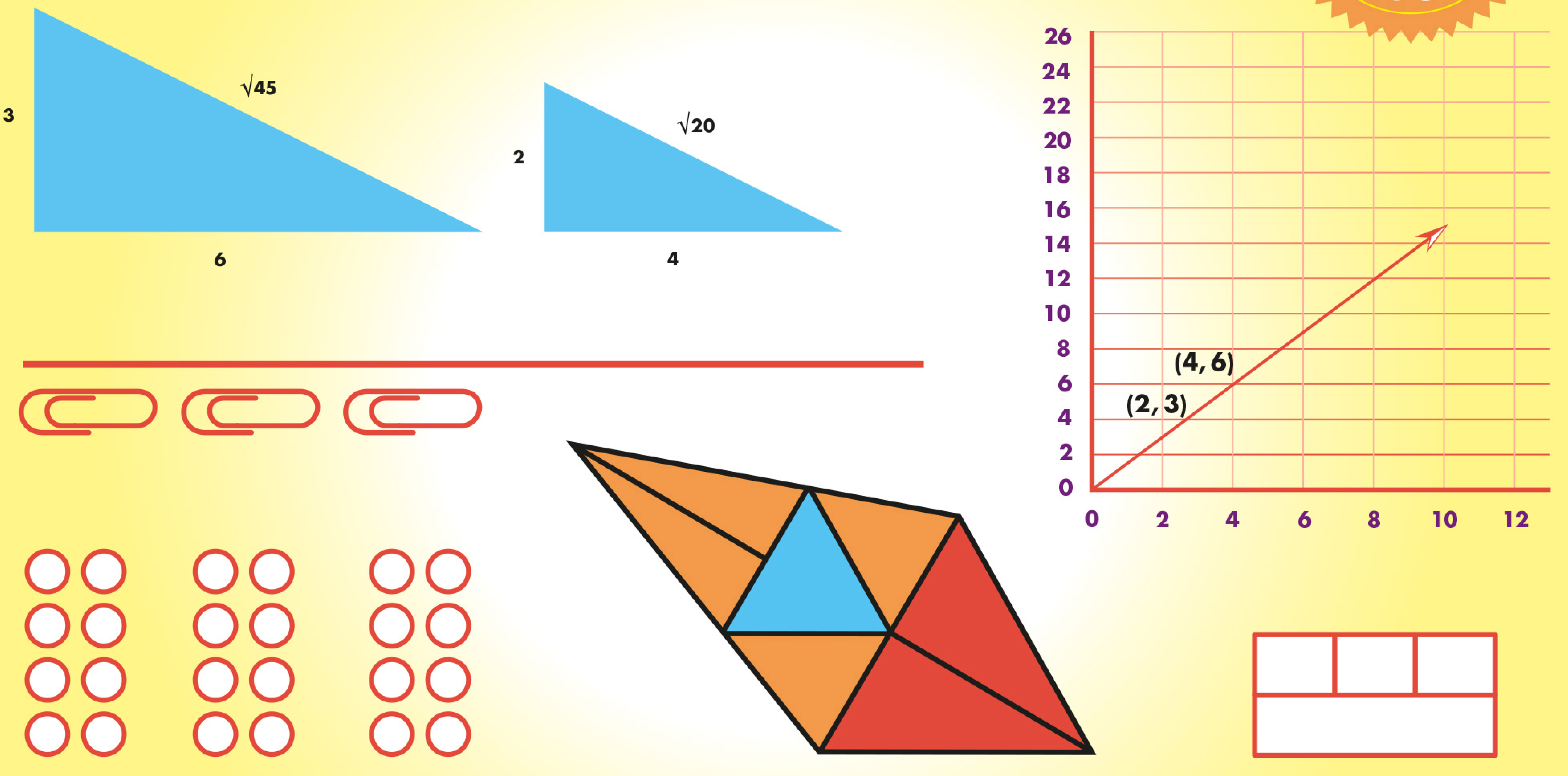
BUILDING PROPORTIONAL REASONING

Across Grades and Math Strands, K–8

A CCSS RESOURCE



Marian Small

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CONTENTS

PART II FOCUSING ON PROPORTIONAL THINKING

Preface

ORGANIZATION OF THE BOOK

Although proportional thinking is often thought of as a middle school topic, its roots start in the early elementary years. Bringing attention to important fundamental ideas in proportional thinking in the earlier grades will benefit students in developing essential proportional thinking skills.

This resource is organized by grade level around the Common Core State Standards for Mathematics related to proportional thinking and those that can be related to proportional thinking through the use of appropriate questioning. The grades covered in this resource begin with Kindergarten, where exposure to proportional thinking is more implicit, and end with Grade 8, where the focus on proportional thinking is much more explicit.

In Grades K–2, prior to the formal introduction of multiplication, important ideas related to proportional thinking can be brought up in a number of strands. These ideas are addressed in this resource by highlighting approaches to proportional thinking that can be taken when dealing with particular standards for those grades, as well as by suggesting some good questions to ask to bring the ideas out. For Grades 3–8, where specific standards directly target multiplicative thinking and proportional thinking, portions of those standards as well as others where proportional thinking can be addressed are presented, again followed by a delineation of important underlying ideas and good questions to ask to bring those ideas out.

Underlying ideas might provide the following:

- Background on the mathematics of the standard related to proportionality
- Suggestions for appropriate representations, including manipulatives, for those specific mathematical ideas
- Suggestions for explaining ideas to students, or
- Cautions about misconceptions or situations to avoid

Following the discussion of the sets of underlying ideas are groups of questions that can be used for either classroom instruction, practice, or assessment relating to the ideas presented. In addition, specific reference is often made to the Common Core State Standards for Mathematical Practice.

For Whom Is This Book Useful and Why?

This resource is designed to support math teachers, Kindergarten–Grade 8, as they strive to help students become more proficient and more comfortable in working with situations involving multiplicative thinking and proportionality. It is also intended as a resource for math coaches as they assist teachers in their transition to teaching mathematics within the more demanding framework of the Common Core State Standards. These new standards challenge all of us to help students become mathematical *thinkers*, not just mathematical "doers." The goal is to develop students who can reason and represent mathematical situations in multiple ways and can explain their reasoning to others. I also hope that this book will be helpful to preservice teachers and their instructors as the preservice teachers prepare themselves to understand and teach math with a deep level of understanding.

Considering the Bigger Picture

While I would hope that all users would read the entire book, I particularly suggest this approach for math coaches and preservice teachers. For grade-level or grade-band teachers, I suggest reading the Introduction and the grade-level sections that apply most directly for their particular groups of students, but also becoming acquainted with the mathematics related to proportional thinking taught in grades directly below and above. As we all know, individual students in any classroom are at different levels of understanding, and it is helpful to be aware of missing prerequisite knowledge as well as directions for moving forward in order to differentiate instruction for them. Knowledge of the progressions preceding specific concepts can also be helpful in diagnosing and resolving problems students may be encountering.

Lastly, I hope that reading this book adds to a broader conception of what proportional thinking is all about and its utility.

ACKNOWLEDGMENTS

During the year prior to writing this resource, I had the opportunity to interact with teachers in many school districts as they worked to enhance their own and their students' understanding of and comfort with proportional reasoning. Many of the ideas in this resource were tried out in their classrooms.

I appreciate the feedback I received from these teachers and their willingness to grant me access to their and their students' thinking.

BUILDING
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Across Grades and Math Strands, K–8

Introduction

WHAT IS PROPORTIONAL THINKING?

Proportional thinking is based on recognizing and forming multiplicative comparisons between quantities. It involves thinking of numbers in relative terms rather than absolute terms. For example, when comparing 4 to 10, thinking of 10 as $2\frac{1}{2}$ fours rather than as 6 more than 4 is proportional thinking. Similarly, deciding that a price increase from \$2 to \$4 (a \$2 increase) is a more dramatic change than an increase from \$90 to \$100 (a \$10 increase), because the first price was doubled and the second was not nearly doubled, is thinking proportionally.

Another way to make sense of proportional thinking is to think of it as unitizing. Proportional thinking involves viewing one measurement (or amount) as so many units of another. It might be thinking of a set of 10 fingers as 2 units of 5 fingers, or it might be thinking of 1 m as 100 units of 1 cm.

Proportional thinking often requires transferring the use of a multiplicative relationship from one pair of numbers to another pair of numbers. For example, if you know that 3 identical items cost \$12 and want to know how many 6 will cost, you transfer the multiplicative relationship between 3 and 6 to a multiplicative relationship between 12 and some other number. Or you transfer the multiplicative relationship between 3 and 12 to the multiplicative relationship between 6 and another number. You are, in essence, determining equivalent ratios. In fact, in the National Council of Teachers of Mathematics (NCTM) resource *Developing Essential Understanding of Ratios, Proportions & Proportional Reasoning, Grades 6–8*, the big idea in proportional reasoning is described as a recognition that "when two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor" (Lobato, Ellis, Charles, & Zbiek, 2010, p. 11).

When Do We Use It?

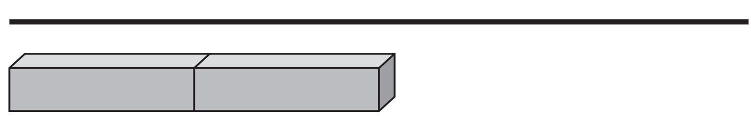
Proportional thinking, also often referred to as proportional reasoning, is used in everyday life in many ways. Just a few examples are listed on the next page:

- *Exchanging coins:* Every time you exchange quarters for nickels, or nickels for quarters, you change the unit of measure and, therefore, the number of units required. Because the exchange is always 5 nickels for 1 quarter, determining the number of nickels for so many quarters or vice versa is a use of proportional reasoning.
- *Changing measurement units:* Every time you change feet to inches or yards to feet or inches to yards, you change the unit of measure and, therefore, the number of units required. Changing a measurement from one unit to another is a use of proportional reasoning.
- *Calculating a best buy:* Every time you try to decide if so many gallons at one price is a better buy than a different number of gallons at a different price, you use proportional reasoning.
- *Planning a trip:* Every time you decide how many hours it will take for a trip based on an average speed, you use proportional reasoning.
- *Maps:* Every time you use a map with a given scale ratio to determine an actual distance, you use proportional reasoning.
- *Cooking:* Every time you adjust a recipe based on the number of people you want to feed or you figure out how many $\frac{1}{3}$ cup measures it would take to measure $\frac{1}{2}$ cup, you use proportional reasoning.

Where Is It in the Math Curriculum?

Although proportional reasoning is not formally mentioned as a topic in the Common Core math curriculum until 6th grade, its roots appear much earlier. Because proportional reasoning involves thinking of one number as a multiple of another (e.g., thinking of 6 as 2 threes or as 3 twos), it is applied as students begin to work in 3rd grade with simple multiplication and division. But proportional thinking actually begins even earlier than that.

For example, when students in earlier grades think about why they get to 50 quickly when they skip count by 10, but slowly when they skip count by 2, or when they think about why the whole line shown below is probably 4 rods long based on how far 2 rods extend along that line, they are building proportional reasoning.



Some work in place value can also be thought of in terms of proportional thinking. For example, realizing that 300 ones must be 3 hundreds since 100 ones is 1 hundred is an example of proportional thinking.

As will be illustrated throughout this resource, work in a wide range of areas involves proportional reasoning: work in probability, work in creating and interpreting graphs with scales, all work with fractions, work with multiplication and division, some work with patterns, some work with solving equations, and some work involving linear relationships.

ESSENTIAL UNDERSTANDINGS RELATED TO PROPORTIONAL THINKING

The list below outlines some important understandings underlying proportional reasoning and pre-proportional reasoning that are useful and that should be addressed throughout the grades:

- It is often useful to think of one amount as so many units of another amount, for example, 1 dollar as 4 quarters, 7 days as 1 week, 20 eggs as $1\frac{2}{3}$ dozen eggs, etc.
- If you use a bigger unit, you need fewer of them to express a quantity. For example, it takes only 10 tens to make 100, but it takes 20 fives. Or, it only takes 1 yard to measure 3 feet. Or, 1 is 5 fifths, but only 2 halves.

  Another way to rephrase this idea is the following: Any amount can be a small amount of a big unit or a big amount of a small unit. For example, 5 is half of a 10, but only a quarter of a 20.
- If units are related, you can use that relationship to predict how many of one unit you will have if you know how many there are of the other. For example, since 4 quarters make 1 dollar, you can predict that 12 quarters makes 3 dollars.
- Any two numbers can be compared multiplicatively, even if one is less than the other. For example, just as 6 is two 3s, 3 is half of a 6.
- How far apart two numbers are additively is unrelated to how far apart they are multiplicatively. For example, the pair of numbers 3 and 6 and the pair of numbers 100 and 200 have the same multiplicative relationship, even though the first numbers are only 3 apart and the second are 100 apart.
- Using a fraction, a decimal, or a percent is a form of multiplicative comparison. For example, the reason $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$ is because in each case the numerator is $\frac{2}{3}$ of the denominator, or the denominator is $\frac{3}{2}$ of the numerator. For example, the decimal 0.231 is a way to compare 231 to 1000. For example, the percent 42% is a way to compare 42 to 100.

These ideas emerge and re-emerge through this resource, through the grades, in the sections on underlying ideas, as well as in the sections listing Good Questions to Ask.

FOCUSING ON THE CCSSM STANDARDS FOR MATHEMATICAL PRACTICE

The CCSSM Standards for Mathematical Practice derive from the processes of the National Council of Teachers of Mathematics (NCTM, 2000) and the strands of mathematical proficiency from *Adding It Up* (National Research Council, 2001). The standards for mathematical practice describe the mathematical environment in which it is intended that the Common Core State Standards for Mathematics are learned. These standards for mathematical practice are meant to influence the instructional stance that teachers take when presenting tasks to help students grasp the content standards. The standards for mathematical practice are addressed in this resource both in the underlying ideas presented for each topic and in the types of Good Questions suggested.

Listed below are just a few examples of attention to each standard for mathematical practice in this resource.

1. ***Make sense of problems and persevere in solving them.*** Students in Grade 3 are encouraged to begin to think of multiplication in terms of a change of unit to help make sense of certain problems (page 37). Students in Grade 7 need to make sense of a problem that asks them to figure out which dimension change has the most effect on the surface area of a prism (page 90).

2. ***Reason abstractly and quantitatively.*** Proportional thinking is all about reasoning, so there are many reasoning opportunities presented in this resource. For example, in Grade 2, students reason about the relationship between 60 – 40 and 6 – 4 when they are stimulated to recognize the fact that they can describe the first situation as essentially the second one with a simple change of unit (from tens to ones) (page 28). Grade 5 students reason abstractly and quantitatively as they compare the growth rates of two patterns (page 54) and as they consider the value of digits in a place value situation (page 55). Using double number lines helps Grade 6 students make sense of how percents of numbers other than 100 relate to those numbers (pages 72–73).

3. ***Construct viable arguments and critique the reasoning of others.*** Based on work with counters, students in Kindergarten discover that it is not possible to create two equal groups when working with certain numbers of counters (pages 11–12)

and students in Grade 1 see that when halves of objects are bigger, so are the whole objects, or vice versa (page 22). Students in Grade 6 are asked to create an argument to predict why certain percent situations are not possible (page 74).

4. ***Model with mathematics.*** In Grade 1, students are modeling with mathematics when they measure half a distance and predict the whole distance (page 21). In Grade 7, students use mathematics to model probability situations (page 91). In Grade 8, students use dilations to create similar shapes (pages 98–99) and they use lines of good fit to model real-life situations (pages 102–103).

5. ***Use appropriate tools strategically.*** In Kindergarten, students use counters to explain principles (page 11). The 100-chart is a useful tool in Grade 1 to help students become familiar with multiples of 10 (page 18). Base-ten blocks are useful at many levels for many purposes, but one purpose is to help Grade 2 students see that counting larger subgroups is an efficient way to count (page 26). In Grade 4, using appropriate tools helps students understand various ways to compare two ratios or fractions (pages 46–47), and in Grade 7, 100-grids or double number lines help students calculate percents (page 81).

6. ***Attend to precision.*** Students in Grade 3 must attend to precision when they try to model a number as trains of another number, using Cuisenaire rods (pages 35–36). Students in Grade 7 consider precision when using strategies to get a sense of the size of π (page 87).

7. ***Look for and make use of structure.*** Students in Grade 1 start to recognize that when counting equal groups of objects, there are always two ways to count—either the number of groups or the number of items; the structure of every situation involving equal groups—multiplication—leads to this conclusion (pages 15–16). Students in Grade 3 might begin to notice that halving one number and doubling another leads to the same result because of what multiplication means (page 39). Students in Grade 7 begin to realize that if one variable is proportional to another, it is because the equation is of the form $y = mx$ and the graph is a line that goes through the origin (page 85).

8. ***Look for and express regularity in repeated reasoning.*** Students in Grade 2 might notice that there are two reasonable definitions for the notion of even number: either a number made up of a lot of twos or a number made up of two of the same whole number (page 23). Students in Grade 7 are expected to use tables of values involving proportional variables to see that 0 always matches 0, that 1 always matches r, where r is a unit rate of some sort, and that the y-values increase by r as the x-values increase by 1 (pages 83–85).

FOCUSING ON THE NCTM PRINCIPLES TO ACTIONS

Recently, the National Council of Teachers of Mathematics released *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014), articulating a vision for the conditions, structures, and policies that are critical to move mathematics education forward. Included are eight teaching practices, some of which focus on the tasks that teachers set and others on the pedagogical approach of the teacher. This resource directly supports many of those suggested practices.

For example, beyond the standards themselves, the underlying ideas articulated in this resource establish mathematical goals to focus learning and frequently use and connect mathematical representations. The tasks suggested as Good Questions are purposeful and promote reasoning and problem solving and meaningful discourse. The underlying ideas often require complex and non-algorithmic thinking.

FOCUSING ON THE CCSSM STANDARDS FOR MATHEMATICAL CONTENT

By its very organizational structure, this resource focuses on the mathematical content standards related to proportional thinking in the Common Core State Standards for Mathematics.

PART I

Opportunities to Build a Foundation for Proportional Thinking

KINDERGARTEN

Counting and Comparing Equal Groups

Counting and Cardinality	CCSSM K.CC
Count to tell the number of objects.	
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.	
Compare numbers.	
6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.	

IMPORTANT UNDERLYING IDEAS

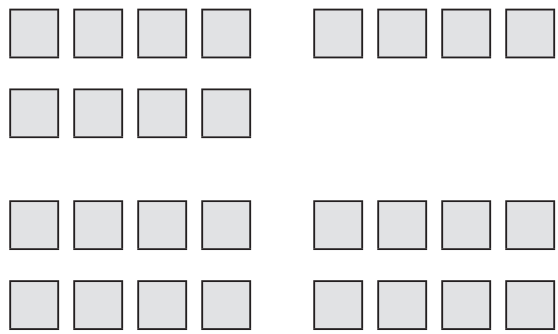
- ***Representing amounts as groups of other amounts.*** Much of the work we do with number in Kindergarten focuses students on counting amounts that are provided and representing requested amounts. Sometimes, instead of asking to see 4 counters or 10 counters, etc., we could ask to see two 2s or two 5s, encouraging students to think in units. Once the student has represented, for example, two 5s, we could then let him or her count the items to realize that 10 is what two 5s is.

- ***Comparing different numbers of groups of the same size or the same number of different-sized groups.*** As young students compare quantities, we might ask whether two 5s is more or less than three 5s, whether four 4s is more or less than three 4s, or whether four 2s is more or less than four 3s, without having them determine the actual totals each time, but rather encouraging them to continue to think in groups.

For example, three 4s is less than four 4s because there are extra items—the extra group of 4 after the three 4s have been matched with the first three 4s among the four 4s.

Three 4s

Four 4s

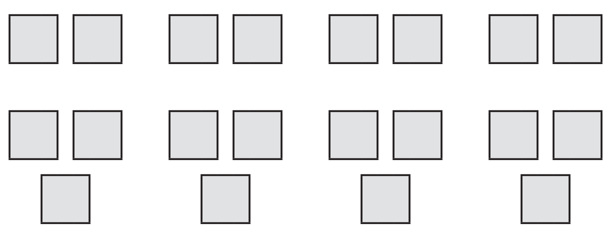


It is stronger reasoning for students to realize that four groups of 4 is more than three groups of 4 just because there are more groups of the same amount, rather than needing to depend on figuring out that 16 is more than 12. Later, to decide if 16 is more than 12, students might use the grouping idea to work backward. This is an example of the mathematical practice standard of reasoning abstractly and quantitatively.

Similarly, we want students to realize that four 2s is less than four 3s because there is an extra bit in each group. The actual numbers should not be the focus, but, rather, the reasoning about the extra in each group. Essentially we are matching two items among the three in each of the four groups of three to the two items in each of the four groups of two.

Four 2s

Four 3s

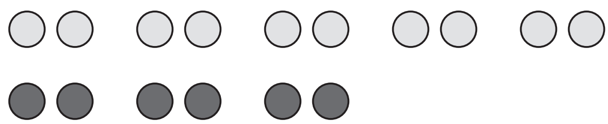


Later, when students are comparing 8 to 12, one of the rationales for deciding that 12 is more than 8 could be that 12 is four 3s whereas 8 is only four 2s.

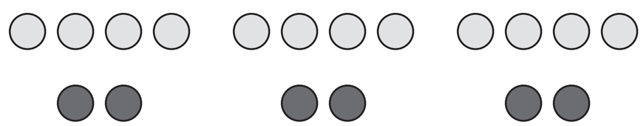
Comparing groups in this way ties into the notion that the same number of a bigger unit (vs. a smaller unit) is a bigger measure. That is why 2 nickels is less than 2 dimes or 2 short sticks make a length shorter than 2 long sticks.

Good Questions to Ask

- ***Ask:*** Show 5 counters. Next, show two 5s, then three 5s. [Encourage students to show two 5s by grouping 5 counters twice rather than counting out 10 individual items.]
- Tell students that Susie is holding 2 books in each of her hands and Liam is holding 3 books in each hand. ***Ask:*** Who is holding more books? Do you have to count to be sure? [We want students to realize that they could count, but they do not have to—each group is bigger, so the total is more.]
- ***Ask:*** What different things does this picture show about one amount being more than another amount? [It could be that five groups of 2 is more than three groups of 2 or that 10 is more than 6.]



- ***Ask:*** What different things does this picture show about one amount being more than another amount?



Decomposing Numbers into Equal Groups

Operations and Algebraic Thinking	CCSSM K.OA
Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.	
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).	

IMPORTANT UNDERLYING IDEAS

➢ ***Creating equal groups.*** As students decompose numbers 10 and under, make sure to draw attention to situations where the decomposition leads to equal groups. For example, 10 = 5 + 5, 8 = 4 + 4, etc. Ensure that students realize that only certain numbers can be shown this way when using counters, that is, 2, 4, 6, 8, 10, . . . , but not 1, 3, 5, 7, ...

Students might also be led to notice that when each of the equal-sized groups is increased by 1, the total is increased by 2, not 1.

Good Questions to Ask

- ***Ask:*** I showed a number as two equal groups. Tell me some numbers I might have been showing. Tell me a number I was not showing.
- ***Ask:*** Draw a picture or use your counters to show that 8 is two 4s.
- ***Ask:*** Show two equal groups with your counters. Write an equation to show what you did. Now add 1 to each of your groups. What equation do you write now? How did the numbers change from the first equation?

Comparing Measurements Relatively

Measurement and Data	**CCSSM K.MD**
Describe and compare measurable attributes.	
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating lengths to each other in relative terms.*** Although students at this level do not yet use units to describe or compare measurements, they could be encouraged, when comparing two lengths that are "foldable," such as strings, to see if the longer length is more than two of or four of the shorter length.

For example, the longer line at the left below is less than two of the shorter one because, when it is folded on itself, it is shorter than the short line.

But the longer line at the left below is more than two of the shorter one because, when it is folded on itself, it extends beyond the short line.

This type of comparison helps students look at relative differences.

We can help students think more relatively than absolutely by showing that the absolute amount longer may not be relevant. For example, Line 2 below is not much longer than Line 1, but it is still more than two of Line 1. Line 4 is a lot longer than Line 3, but it is less than two of Line 3.

Good Questions to Ask

- Provide two pieces of yarn of different colors, where one piece is more than double the length of the other. ***Ask:*** Which color string do you think is the longer one? Do you think it is more than two of the other color?
- Provide two pieces of yarn of different colors, where one piece is a little more than four times as long as the other. ***Ask:*** Fold the long string in half (so the ends meet) and then do that again. Is the folded string longer or shorter than the short string you have? How many of the short string do you think would fit into the long one? Why?
- Provide a long string, some yarn, and scissors. ***Ask:*** Cut a short string that is close to half the length of the long string.

Summary

Although work in proportional thinking is still very informal at this level, there is an opportunity for teachers to help children think in terms of one amount being units or groups of another, a concept that underlies proportional thinking. Approaches can include modeling and recognizing equal groups, comparing amounts by comparing the equal numbers of groups or the equal-sized groups that make them up, and thinking of one length relative to another length.

GRADE 1

Adding and Subtracting Using Equal Groups

Operations and Algebraic Thinking	CCSSM 1.OA
Represent and solve problems involving addition and subtraction.	
1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.	

IMPORTANT UNDERLYING IDEAS

➢ ***Combining equal groups.*** Ensure that some of the problems that students meet involve the combining of equal groups. For example, a problem such as this one might be posed: Sarah and Tom each held up all the fingers on one of their hands. Then Bruce joined in with the fingers on one of his hands. How many groups of fingers were in the air? How many fingers?

Students might be led to see that there are, in a way, two different additions describing the situation. There is 2 + 1 = 3, indicating the number of groups of fingers, but there is also 10 + 5 = 15, when the focus is on how many fingers.

Once students learn multiplication, in a higher grade, they will be able to solve the problem of the total number of fingers by multiplying the 3 from 2 + 1 by 5, rather than by doing the two separate operations to get 10 and 15.

The concept presented here could lead nicely later into skip counting by twos, since counting 8 + 10 by thinking 4 twos + 5 twos is, in essence, what you do when you skip count by twos by saying 4 numbers for the first group and 5 numbers for the second group.

➢ ***Removing some of a set of equal groups.*** Ensure that some of the problems that students solve involve the removal of some groups from a set of equal groups. For example, a problem like this might be presented: There are 4 tables, each seating

4 children. The children from two of the tables leave. How many tables of children are left? How many children?

Students might be led to see that there are two different subtractions describing the situation. There is 4 – 2 = 2, indicating the number of tables of children left, but there is also 16 – 8 = 8, when the focus is on how many children are left.

Once students learn multiplication, in a higher grade, they will be able to solve the problem asking how many children are left by multiplying the 2 from 4 – 2 by 4 rather than by doing the two separate operations to get 16 and 8. This is an example of the practice standard of looking for and making use of structure.

➢ ***Comparing different numbers of groups of the same size or the same number of different-sized groups.*** Some of the problems students solve that ask how many more one amount is than another could involve equal groups. For example, a problem such as this one might be given: There are 5 packs of 3 yellow tennis balls and 2 packs of 3 white tennis balls. How many more packs of yellow balls than white balls are there? How many more yellow balls than white balls are there?

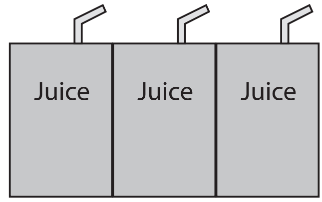
Again, students can be led to see that there are two different subtractions describing the situation. There is 5 – 2 = 3, which tells how many more packs of yellow balls there are than packs of white balls. But there is also 15 – 6 = 9, which tells how many more yellow balls than white balls there are.

Once students learn multiplication, they will be able to solve the problem of how many more yellow balls there are by calculating 5 – 2 and then multiplying the result by 3.

Another useful kind of problem involves comparing amounts that can both be represented as the same number of groups, but with group sizes that are different. For example, a problem like this might be addressed: Jojo got 3 pairs of socks and Leah got 3 packages with 4 socks in each package. How many more socks did Leah get?

Good Questions to Ask

- ***Ask:*** Kendra's mom bought some packs of juice boxes last week and some more packs this week. You decide how many packs she got each week. Nobody has drunk any of the juice yet. How many packs of juice are there altogether? How many boxes of juice?



Juice pack

- ***Ask:*** There were 2 packages of hot dogs in the fridge. Lee's mom bought a few more packages. How many packages of hot dogs might be in the fridge now? How many hot dogs?

- ***Ask:*** What might the problem have been if you had thought: 4 groups – 2 groups = 2 groups?
- ***Ask:*** Draw a picture or make a model that would *make it easy to see* how much more 5 groups of 4 is than 2 groups of 4. [This is an example of modeling with mathematics.]
- ***Ask:*** Draw a picture or make a model that would *make it easy to see* how much more 3 groups of 6 is than 3 groups of 4.

Multiples of Ten

Number and Operations in Base Ten	CCSSM 1.NBT
Understand place value.	
2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases: a. 10 can be thought of as a bundle of ten ones—called a "ten." c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones). 3. Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, = , and <.	

IMPORTANT UNDERLYING IDEAS

➢ ***Counting tens.*** When counting tens, it is important to encourage students to say 1 ten, 2 tens, 3 tens, . . . before too quickly moving to 10, 20, 30, ... Thinking of 30 as 3 tens will be critical when students move into computational situations.

One way of helping students see that 20 is 2 tens, 30 is 3 tens, etc., is to use the 100-chart. It is clear that the chart is organized in rows of ten, or this might be made clear by circling each row of ten and noticing the 10 numbers in the row. Students can see that the number that is 5 tens, 50, appears at the end of the 5th line of ten in the chart, just as the number that is 7 tens, 70, appears at the end of the 7th line.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Another strategy to help students focus on groups of ten is to fill 10-frames. For example, 4 tens, or 40, looks like this:

x	x	x	x	x
x	x	x	x	x

x	x	x	x	x
x	x	x	x	x

x	x	x	x	x
x	x	x	x	x

x	x	x	x	x
x	x	x	x	x

➢ ***Comparing tens.*** As students learn to compare two-digit numbers, it is helpful if they understand why one can look at the tens digits first to gather critical information to make the comparison.

To compare 60 and 40, we could say that 60 is greater because it comes later in the counting sequence, or because it is lower in the 100-chart, but fundamentally we want students to realize that 60 > 40 because 6 groups of ten is more than 4 groups of ten. Repeatedly returning to this approach of reasoning abstractly and quantitatively is useful.

When asking students to compare non-multiples of ten, for example, 23 and 35, we want them to realize that 23 is 2 groups of ten and less than another group of ten, but 35 is more than 3 groups of ten. More groups of the same amount results in a greater total.

Good Questions to Ask

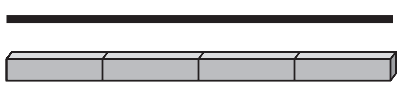
- ***Ask:*** A number is a lot of tens. What might it be? How could you draw it or model it so that it is easy to see that the number is a lot of tens?
- ***Ask:*** What numbers are you sure are more than 4□? Why are you sure?
- Tell students you filled in some 10-frames and that they are completely full with no counters left over. ***Ask:*** How many counters might you have had? Would you be able to completely fill in 5-frames with those same counters? Explain your thinking.

Measuring with Units

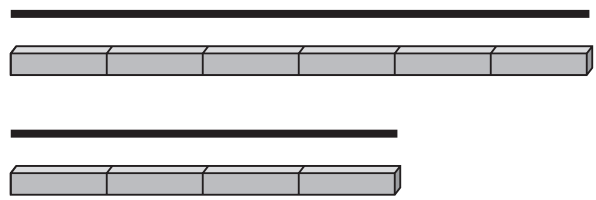
Measurement and Data	CCSSM 1.MD
Measure lengths indirectly and by iterating length units.	
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.	

IMPORTANT UNDERLYING IDEAS

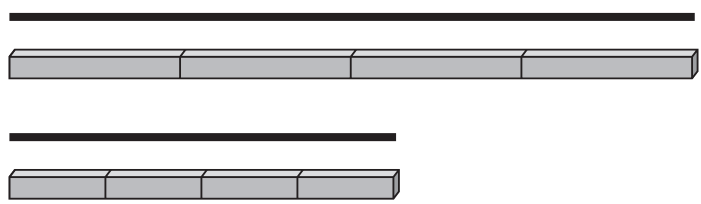
➢ ***Describing measurements using units.*** Measuring a length with copies of a unit is, in effect, considering that length as a number of equal groups, each unit being a "group." For example, the thick line below is 4 units long, where a unit is one of the gray rods shown, so the line is 4 groups of that unit.



The same kind of thinking that is used to compare more or fewer equal groups of counters or to compare the same number of bigger or smaller units of numbers also applies to measurements. For example, a measure that is 6 units long is longer than one that is 4 of those same units long, no matter what the unit is.



Or a measure that is 4 long units long is longer than a measure that is 4 short units long.

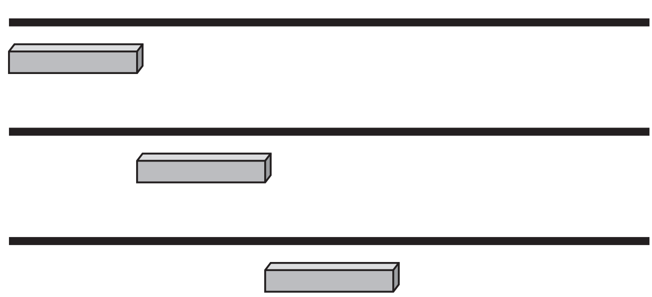


- ***Keeping only one of the unit or the object constant.*** It is important for students to have experiences where they measure the same length in different units to see that it takes fewer long units and more shorter units to measure it.

  Similarly, it is important for them to have experiences where they use the same number of different-sized units to see that the same number of longer units is a longer length.

- ***Estimating measurements based on partial measurements.*** Often when we ask students to measure lengths, we provide enough units to allow the children to measure the entire length. But there is value in sometimes providing an insufficient number of units to measure the full length, for two reasons:

  - First of all, it reinforces for students that you can move copies of units to measure with a fewer number of units.



- Secondly, it builds proportional thinking when students begin to use partial measurements to imagine full measurements. For example, when students see, as in the diagram below, that it takes 3 paper clips to extend as far as the clips go, they can start estimating that to cover the entire length would take about 6 paper clips.

Good Questions to Ask

- ***Ask:*** Will it take more pencils or more erasers to get across the table? Why? [This is an example of asking students to construct viable arguments.]
- ***Ask:*** Watch me take 4 steps. How many steps will it take me to get over there [as you point to a place about twice as far away]?
- ***Ask:*** It took a lot of units to measure this line. Show me what you think the unit looked like.

- ***Ask:*** A really long line is 5 units long. But a really short line is also 5 units long. Is that possible? How?

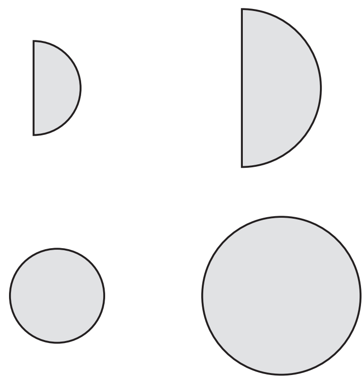
Introducing Halves and Quarters

Geometry	**CCSSM 1.G**
Reason with shapes and their attributes.	
3. Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.	

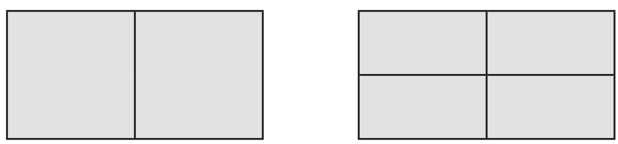
IMPORTANT UNDERLYING IDEAS

➢ ***Relating fractions to proportional reasoning.*** Partitioning shapes into two or four equal shares is a way of looking at an area as two or four units. Halving is about creating two units; quartering is about creating four units.

The same ideas about proportional thinking that arise in measuring lengths with units emerge here again. For example, students can see that if the half unit is bigger, the whole is bigger. That is, the same number of a larger unit is more than that number of a smaller unit.



They can also see that if the same whole is cut into two equal units (halved) and four equal units (quartered), the quarters are smaller than the halves. It takes more smaller units and fewer larger units to perform a measurement.



Good Questions to Ask

- Provide a circle. ***Ask:*** Show the circle as two equal amounts. What would each amount look like?
- Provide a non-square rectangle. ***Ask:*** Show the rectangle as four equal amounts. What would each amount look like?
- ***Ask:*** Why can every rectangle be thought of as a few smaller rectangles?

Summary

Although work in proportional reasoning remains informal at this level, there is an opportunity for teachers to help students think in terms of one amount being units or groups of another, a concept that underlies proportional reasoning. Approaches can include having students combine, separate, and compare amounts by focusing on their representation as equal groups, by introducing the multiples of ten as special numbers, by having students measure lengths with different-sized units and measure different lengths with the same unit, and by encouraging the concept of unit when students partition shapes into halves and quarters.

GRADE 2

Introducing Multiplication

Operations and Algebraic Thinking	CCSSM 2.OA
Work with equal groups of objects to gain foundations for multiplication.	

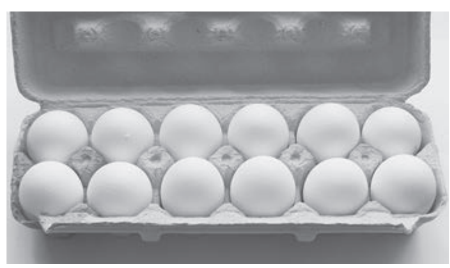
3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
4. Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

IMPORTANT UNDERLYING IDEAS

➢ ***Relating even and odd to grouping/unitizing.*** In many instances, children are taught that numbers are even or odd based on their rightmost digits. This is problematic because students often then misapply the "rule" and call a decimal like 2.4 even and a decimal like 2.5 odd. Of course, such an approach does not hold up, since 2.5 = 2.50, which those same students would then call even. It only makes sense to talk about oddness or evenness in terms of whole numbers (0, 1, 2, 3, 4, . . .) or integers (–1, –2, –3, . . . , as well as the whole numbers).

Students should understand that there are two fundamental ways to decide that a counting number is even: Either it can be decomposed into groups of 2 or it can be decomposed into two equal groups.

For example, consider the number 12. Twelve is even because it is made up of six 2s (if you look at pairs of eggs in the two rows) or because it is made up of two 6s (if you look at the two rows.)



So, fundamentally, evenness is about creating equal groups.

The reason that 11 is not even is because when you create groups of 2, there is one left over, or because when you try to create two equal groups of whole numbers, you cannot.

➢ ***Combining equal-sized groups.*** As students create arrays of objects and use addition to determine the total number, they are treating the total amount as if it were so many groups of some specific amount.

For example, the array below shows that 12 is made up of three equal groups of 4 if each row is a group, or of four equal groups of 3 if each column is a group.

x	x	x	x
x	x	x	x
x	x	x	x

Students should have opportunities to observe that if two arrays have the same number of rows but different numbers of columns, the one with more columns represents the greater number. This is another way of saying that a given number of bigger units is more than that same number of smaller units.

x	x	x	x
x	x	x	x
x	x	x	x

x	x	x	x	x
x	x	x	x	x
x	x	x	x	x

12 < 15 since three groups of 4 is less than three groups of 5.

Students should also have opportunities to see that if the same number of objects is arranged into an array with more rows, there are fewer columns. This is true because it takes more smaller units and fewer bigger units to describe an amount.

x	x	x	x
x	x	x	x
x	x	x	x

x	x
x	x
x	x
x	x
x	x
x	x

It takes only three 4s to make 12, but it takes six 2s.

These observations support the mathematical practice standard of reasoning abstractly and quantitatively.

Good Questions to Ask

- ***Ask:*** Choose an even number between 10 and 30 and model it with objects or an equation in two different ways that help show that it is even.
- ***Ask:*** Pick an odd number between 10 and 30 and model it with objects or an equation in two different ways that help show that it is odd.
- Tell students that a number has been written as 2 + 2 + 2 + 2 + 2 + 4. ***Ask:*** Is it an even number or not? Did you have to actually figure out what the number was to decide?

Creating Groups of Tens and Hundreds

Number and Operations in Base Ten	CCSSM 2.NBT
Understand place value.	
1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens—called a "hundred." b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones). 2. Count within 1000; skip-count by 5s, 10s, and 100s. 3. Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using >, =, and < symbols to record the results of comparisons.	

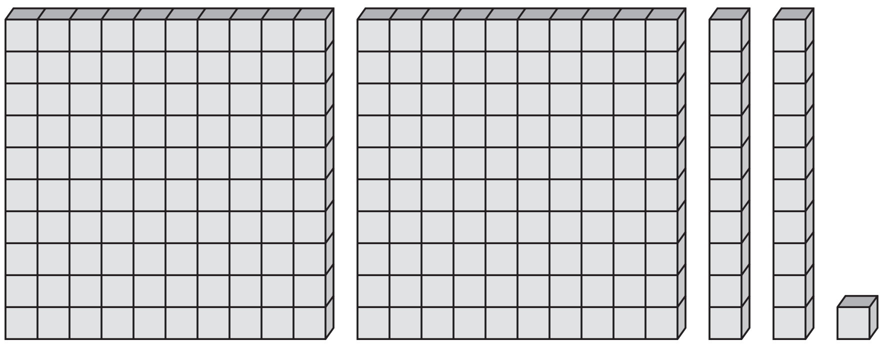
IMPORTANT UNDERLYING IDEAS

➢ ***Skip counting.*** One of the reasons we introduce students to skip counting is to make counting faster for them. It is quicker to count, for example, nine groups of 5, saying *5, 10, 15, 20, . . .* , than it is to count 45 individual items, *1, 2, 3, 4, 5, ...* Encouraging skip counting helps build proportional reasoning because it teaches students to think in terms of units (what we are skip counting by).

➢ ***Creating subgroups to make counting easier.*** Although the reasons we use a place value system are much more complex than the reason we use skip counting, one of the by-products of using the place value system for recording numbers is that it is quicker to count greater amounts by subgrouping the total into groups of tens and hundreds than it is to count the items individually.

Students learn that the numbers 200, 300, 400, 500, . . . are named as groups of 100. Working with hundreds reinforces the idea of grouping even more than working only with multiples of 10, because even the names of the multiples of 100 are □ hundreds (e.g., two hundred or four hundred).

When we use base-ten blocks, students do not count by 1s, but say, for the picture below, *100, 200, 210, 220, 221*; they can count this quickly because they are counting in groups.



➢ ***Comparing larger numbers by comparing equal groups that make them up.*** As students learn to compare three-digit numbers, it is important that they understand why they can look at the hundreds digit first to gather the most critical information they need to make the comparison.

To compare 500 and 300, we want students to think "5 groups of 100 is more than 3 groups of that same 100"; that is why 500 is more. When comparing, for example, 421 and 153, students should realize that they are comparing a number that is 4 groups of 100, but not quite 5 groups of 100, to a number that is 1 group of 100, but not quite 2 groups of 100. This is much more meaningful than just saying 4 is more than 1, particularly because the 4 is 421 is not really 4; it is 4 groups of 100.

Good Questions to Ask

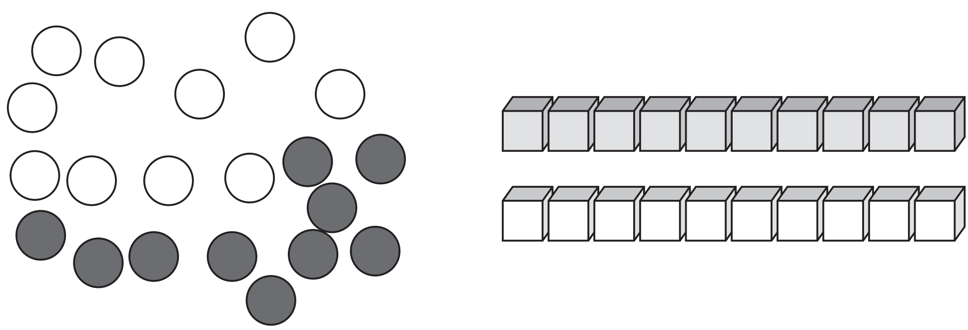
- ***Ask:*** Someone is skip counting and gets to the number 200 really fast. What do you think that person was skip counting by? What makes you think that?
- ***Ask:*** A number is a lot of hundreds. How could you draw it or model it so that this is easy to see?
- ***Ask:*** Cory said that 320 > 240 because 32 > 24. What about 320 is 32? What about 240 is 24? Does what he said make sense?

Decomposing into Units to Add and Subtract

Number and Operations in Base Ten	CCSSM 2.NBT
Use place value understanding and properties of operations to add and subtract.	
7. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.	

IMPORTANT UNDERLYING IDEAS

➢ ***Using grouping to make addition and subtraction easier.*** Consider the difference between adding the two amounts on the left in the picture below (the number of white circles to the number of dark ones) to adding the two amounts on the right. It is clearly much quicker to add the amounts on the right, especially knowing that each unit on the right is 10.



Students can realize that if they were adding, for example, 134 + 256, it would take a long time to figure out the total by adding 134 ones to 256 ones. But adding 1 hundred to 2 hundreds and 3 tens to 5 tens and 4 ones to 6 ones is much quicker. Essentially, the place value system lets us use our understanding of unitizing to make addition and subtraction easier.

Breaking up a number like 256 into 2 hundreds and 5 tens and 6 ones is not quite thinking of the number as copies of a single unit, as is the case with proportional situations, but it builds on recognizing that 200 is 2 hundreds and that 50 is 5 tens.

Good Questions to Ask

- ***Ask:*** How is figuring out these three calculations alike?

  6 – 4   60 – 40   600 – 400
- ***Ask:*** You are adding a few hundreds and a few tens to 348. What might the number you add be? What would the sum be?
- ***Ask:*** You are subtracting 4 tens from 358. Why is that not too complicated?

Measuring with Units

Measurement and Data	CCSSM 2.MD
Measure and estimate lengths in standard units.	
2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen. 3. Estimate lengths using units of inches, feet, centimeters, and meters.	

IMPORTANT UNDERLYING IDEAS

➢ ***Measuring the same object with different units.*** It is by measuring the same lengths with different units, particularly units of noticeably different lengths, that students come to terms with the idea that the same measurement can have very different descriptions. The same length might be 4 units long or even 20 units long, depending on the unit used. Students need to realize that a large number of small units or a smaller number of larger units might describe the same measure.

Eventually this understanding will help students make sense of why a length can be 36 units if the units are inches, but only 3 units if the units are feet. All unit conversion is based on the principle that the same measurement can be described by using different units.

➢ ***Developing benchmarks to estimate lengths.*** In order to estimate lengths, it is useful for students to have benchmarks for 1 inch, 1 foot, 1 meter, and 1 centimeter. This is because each benchmark can then be used to estimate any other length in relative terms.

But other benchmarks are also useful. For example, if a student has a sense that 1 meter describes the height of a door knob from the floor, it becomes easier to estimate the height of a room. This, again, is a case of thinking of a longer length as units of the benchmark. Estimation of length almost always involves thinking of one length as how many of another length.

➢ ***Estimating measurements based on partial measurements.*** Often when we ask students to measure lengths, we provide enough units to allow the child to measure the entire length. But there is value in sometimes providing an insufficient number of units to measure the full length. We help to build proportional reasoning when students use partial measurements to imagine full measurements. For example, we might ask students to estimate the full length of a wall but provide them with only a single 1-foot ruler. Then they would need to think about how many rulers (or units) would cover the full wall in order to estimate the measurement.

Good Questions to Ask

- ***Ask:*** It takes about 20 of this unit to get across a table.

  About how many of this unit will it take?

- ***Ask:*** Walk 10 steps. Predict where you will end up if you take 20 steps. Check your prediction.
- ***Ask:*** It did not take many units to measure this line. Show me what you think the unit looked like.

- ***Ask:*** Why are 5 cm and 5" not the same length? Which is longer? Why?

Relating Money Units

Measurement and Data	CCSSM 2.MD
Work with time and money.	
8. Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using \$ and ¢ symbols appropriately. *Example:* If you have 2 dimes and 3 pennies, how many cents do you have?	

IMPORTANT UNDERLYING IDEAS

➢ ***Counting out the same amount with different units.*** Just like we can measure a length with units of various sizes, we can count out many amounts of money with units of different sizes. For example, 50¢ can be modeled with 50 penny units, 10 nickel units, 5 dime units, or 2 quarter units. Teachers should draw students' attention to the parallel between this situation and the situation with length rather than treat them as two unrelated topics.

Certain amounts of money can be shown with only pennies if a single unit is required (e.g., 21¢), but any amount that is a multiple of 5¢ can be represented using at least two different units.

Just as with length units, students will see that fewer of a unit with a greater value (equivalent to a longer length unit) are required than of a unit with a lesser value (equivalent to a shorter length unit).

➢ ***Comparing two amounts using units.*** Just as we can compare two lengths by comparing how many of various units it takes to measure them, we can compare two amounts of money by focusing on units.

For example, 45¢ is worth 9 nickels, but 25¢ is worth only 5 nickels, so 45¢ is worth 4 nickels more. We can, of course, also say it is 20 pennies more, but that is neither more meaningful, nor more correct, than saying it is 4 nickels more.

➢ ***Estimating the value of certain amounts using an insufficient number of units.*** Suppose we asked students to figure out how many dimes make $1.50. If we provided 15 dimes, the students could simply count by tens and realize it takes 15 dimes. But if we give them, say, only 5 dimes, they are more likely to use proportional thinking. They would get up to 50, and then have to figure out how many units of 50 make 150, using proportional thinking, to finish the task.

It may be surprising, but sometimes providing *not quite enough* manipulatives can be very useful in building proportional reasoning.

Good Questions to Ask

- Provide students with 5 nickels. ***Ask:*** How many nickels would be needed to have 75¢. [This will encourage students to think about how many units of 25 are in 75.]
- ***Ask:*** Jeff has twice as many coins as Luke, but the total values are the same. What might each boy have?
- ***Ask:*** Choose a coin. How much are 6 of those coins worth? Could 6 different coins be worth less? More? Explain.
- ***Ask:*** Abby has 5 times as much money as Zayden. If Zayden has 8 nickels, what might Abby have? Do you have to figure out how much Zayden's money is worth before you can figure out what Abby might have?

Creating Fractions of Wholes

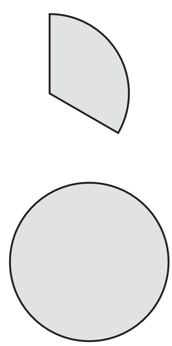
Geometry	CCSSM 2.G
Reason with shapes and their attributes.	

2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not be the same shape.

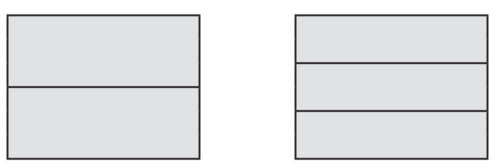
IMPORTANT UNDERLYING IDEAS

➢ ***Relating fractions to proportional reasoning.*** Partitioning shapes into two, three, or four equal shares is a way of looking at an area as two, three, or four units. Halving is about creating two units; taking one-third is about creating three units; quartering is about creating four units.

The same ideas about proportional reasoning that arise when measuring lengths with units emerge here again. For example, students can see that if the one-third unit is bigger, the whole is bigger. That is, the same number of a larger unit (in this case, 3) is more than that number of a smaller unit.



Students can see that you need more smaller units than larger units to partition the same whole. It takes 3 thirds to make a rectangle but only 2 halves to make the same rectangle since halves are bigger.



Good Questions to Ask

- Provide a circle. ***Ask:*** Show the circle as three equal amounts. How does each piece compare to one-quarter of the circle? To one-half of the circle? Why do they compare that way?
- ***Ask:*** Draw a rectangle. Now show the rectangle of which your first rectangle is one-third. Show the rectangle of which the first one is one-fourth. Which of the last two rectangles is bigger? Why?
- ***Ask:*** Can every rectangle and circle can be shown as three of something?

Summary

In Grade 2, work in proportional thinking remains informal, but there is an opportunity to help children think in terms of one amount being units or groups of another, a concept that underlies proportional thinking. This comes about as students consider the concept of even versus odd, as they think of larger numbers as groups of smaller ones both to compare them and to calculate with them, as they measure length and money amounts using units, and as they partition shapes into fractional units.

PART II

Focusing on Proportional Thinking

Grade 3

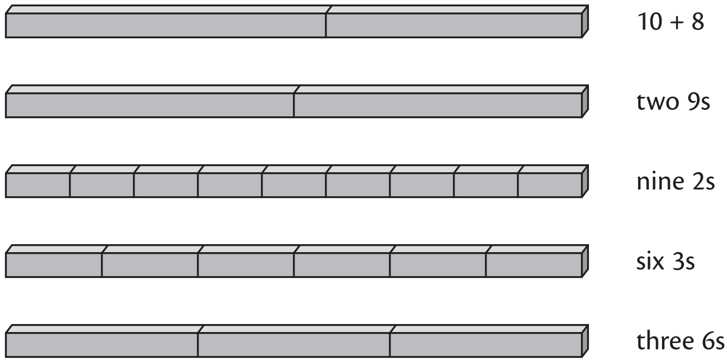
Interpreting a Number as Units of Another Number

Operations and Algebraic Thinking	CCSSM 3.OA
Represent and solve problems involving multiplication and division.	
1. Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7. 2. Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8. 3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.	

IMPORTANT UNDERLYING IDEAS

➢ ***Describing numbers as multiples.*** Starting at this grade level, the push for students to look at a number and realize it is so many units of another number is much greater than it was in earlier grades. For example, before, students looked at 18 and thought 10 + 8; even though that is still clearly the case, starting at this level, we want students to look at 18 and think *nine 2s* OR *two 9s* OR *three 6s* OR *six 3s*. The ability to think of numbers multiplicatively is the foundation for future success in math.

A useful tool for thinking multiplicatively is a set of Cuisenaire rods. Students might display 18 additively by using a 10-rod plus an 8-rod, but their new mission is to create trains of rods of a single color that will match that length. The diagram on the next page shows that 18 could be made up of 9s or 2s or 3s or 6s.



Students will need to be encouraged to think in this manner. For example, when the number 24 comes up, ask: *24 is a bunch of somethings. What might that something be?*

There is an opportunity for students to notice that when a bigger unit is used, fewer of them are required. For example, 24 takes six 4s, but only three 8s, since 4 is less than 8. They can also notice that 4 units of a smaller number is less than 4 units of a greater number. For example, they can know $4 \times 3 < 4 \times 6$ without actually calculating the values of 4×3 or 4×6. This is an illustration of the practice standard of reasoning abstractly and quantitatively.

➢ ***Thinking of division as asking how many units.*** When students meet a division question, such as $24 \div 6$, it helps to read the question as *How many 6s make 24?* rather than as *What is 24 divided by 6?* In this way, students start thinking about division as asking how many units of one number make up another number.

Students' attention should be drawn to the notion that dividing by a smaller amount gives a greater result (with positive numbers): for example, the result for $24 \div 3$ is greater than for $24 \div 6$ since there are more 3s in 24 than 6s in 24. Similarly, determining the number of units in a greater number results in a greater result. For example, $36 \div 6 > 24 \div 6$ since there are more units of 6 in 36 than in 24 because 36 is bigger than 24. Again, the actual values are not needed to determine this.

Good Questions to Ask

- ***Ask:*** Does it take more 8s or more 6s to make 48? Why does that make sense?
- ***Ask:*** Why is writing $2 \times 8 = 16$ a way to say that it takes two 8s to make 16?
- ***Ask:*** Why is writing $24 \div 3 = 8$ a way to say that it takes eight 3s to make 24?
- ***Ask:*** Suppose you know that $4 \times 8 = 32$. What does that tell you about ways to make up 32 using one size of unit?
- ***Ask:*** How could you show that 36 is six batches of 6?

Interpreting Multiplication as a Unit Change

Operations and Algebraic Thinking	CCSSM 3.OA
Represent and solve problems involving multiplication and division.	
1. Interpret products of whole numbers, e.g., interpret 5 × 7 as the number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7. 3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.	

IMPORTANT UNDERLYING IDEAS

➢ ***Reinterpreting a multiplication situation.*** We teach students that if there are 4 tables of children with 6 children at each table, we write 4 × 6 to describe the total number of children. Another way to look at this situation is as a unit change. There used to be 4 units; each unit was a table. But now, we want the unit to be children, so we have to multiply to convert to the new unit of children. The total number of objects we are describing is either 4 (tables with children) or 24 (children).

Although this may seem like an odd approach to bring up when first discussing multiplication, it makes a lot of sense to bring it up later, when unit conversions for measurement are occurring. We can help students see that writing 3 feet = 3 × 12 inches as a unit conversion is really the same as the converting from 5 baskets to 20 apples, if we know there are 4 apples in each basket.

More generally, students can think of a problem described by $4 \times 3 = \square$ as a unit conversion problem where the unit of groups (of which there are 4) was changed to a different unit (of which there are 12).

Good Questions to Ask

- ***Ask:*** Write a story problem that you could solve where the situation could be described as either 3 groups or as a lot of balloons.
- ***Ask:*** Cayley had 2 packages of cookies. Do you think it is better to describe the amount by saying 2 packages or by saying 24 cookies?
- ***Ask:*** There were 4 boxes of books. You know that 1 box of books = 12 books. What equation would you write to figure out how many books there are?

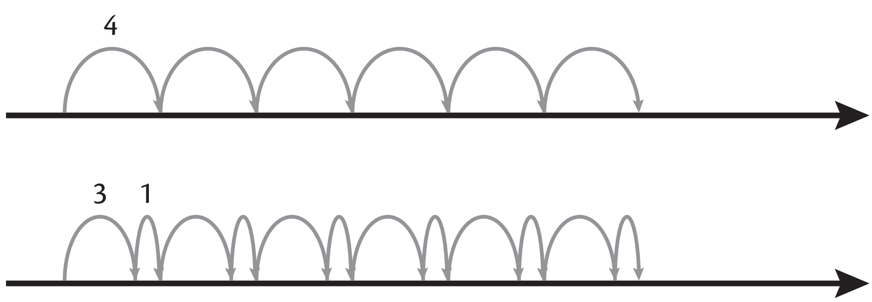
Recognizing Proportional Thinking in Multiplication/Division Properties

Operations and Algebraic Thinking	CCSSM 3.OA
Understand properties of multiplication and the relationship between multiplication and division.	
5. Apply properties of operations as strategies to multiply and divide. *Examples:* If 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (*Commutative property of multiplication.*) 3 × 5 × 2 can be found by 3 × 5 = 15, then 15 × 2 = 30, or by 5 × 2 = 10, then 3 × 10 = 30. (*Associative property of multiplication.*) Knowing that 8 × 5 = 40 and 8 × 2 = 16, one can find 8 × 7 as 8 × (5 + 2) = (8 × 5) + (8 × 2) = 40 + 16 = 56. (*Distributive property.*) 6. Understand division as an unknown-factor problem. For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.	

IMPORTANT UNDERLYING IDEAS

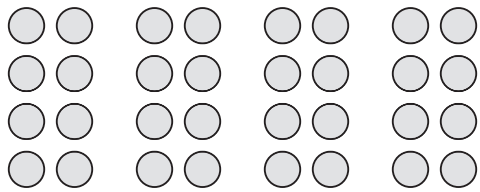
- ***Using the associative property.*** The associative property of multiplication is a clear example of the use of proportional thinking. For example, you can determine 3 × 5 × 4 either as (3 × 5) × 4 or as 3 × (5 × 4). This is an example of the idea that if one unit is $\frac{1}{5}$ of the size of another (e.g., a unit of 4 compared to a unit of 20), it takes 5 times as many of that unit to measure a given amount. In this case it takes either 15 units of 4 (i.e., (3 × 5) × 4) or 3 units of 20 (i.e., 3 × (5 × 4)) to measure 60.

- ***Using the distributive property.*** The distributive property of multiplication over addition can also be described by thinking in units. For example, the reason that 6 × 4 = (6 × 3) + (6 × 1) is because 6 units of 4 is the same as 6 units of 3 and 6 units of 1.



➢ ***Relating division to multiplication.*** Because $a \div b$ is asking how many units of b are in a, this is the same as asking what to multiply by b to get a, since we want a group of units of b to make the a.

The picture of dots below shows both $32 \div 8 = 4$ and $4 \times 8 = 32$ since both equations say there are 4 units of 8 in 32.



Good Questions to Ask

- ***Ask:*** Draw a picture that shows why $(3 \times 2) \times 6$ has to be equal to $3 \times (2 \times 6)$. But don't actually calculate what the value is.
- ***Ask:*** What multiplication or division do you see in this model? Or do you see both?

40							
5	5	5	5	5	5	5	5

- ***Ask:*** Draw a picture or build a model to show why $5 \times 7 = (3 \times 7) + (2 \times 7)$. Make sure that the picture does not rely on actually calculating that both sides are 35.

Measuring with Units

Measurement and Data	**CCSSM 3.MD**
Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.	
1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram. 2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating amounts of time.*** Measuring an amount of time is automatically a proportional situation. For example, if we say an event took 20 minutes, we are actually saying that it took 20 times as long as an event that took 1 minute. Students should be aware of this.

The particular standard here (3.MD1) that discusses time focuses on additive relationships between amounts of time, but there is still an opportunity to build in proportional thinking. For example, consider the problem, *I practiced piano 15 minutes today, but I practiced twice as much the day before. How long did I practice the day before?* This situation would be a chance to practice proportional reasoning.

➢ ***Relating volumes or masses.*** As with time, we want students to realize that when we suggest that a volume is 3 liters, we mean that it is 3 copies of 1 liter, or if a mass is 20 g, it is 20 copies of 1 g. More proportional reasoning can be cultivated by using multiplication and division problems involving volume and mass. For example, a problem might indicate that on a pan balance it took two 1-kg masses to balance a sack of potatoes, and students would need to figure out how many masses it would take to balance 3 sacks of potatoes. Or a problem might ask how many 50-g masses would be needed to balance 350 g on a pan balance.

Good Questions to Ask

- ***Ask:*** Set the hands of a clock to a time of your choice. Turn the hands 2 minutes per turn until 6 minutes have passed. How many turns did you have to make?
- ***Ask:*** Choose a time. Write it down. Suppose you walked for 12 minutes. What time would it be then? What if you walked twice as long?
- ***Ask:*** Sindy started playing ball at 5:30. Decide how long she played and when she finished. Liam played ball starting at 4:30 but played for only for half as long as Sindy. When did Liam finish?
- ***Ask:*** One ball's mass is twice as great as another's. What might the two masses be?
- ***Ask:*** Jennifer's mom's bowl held 3 times as much as Amy's mom's bowl. How many liters might each bowl have held?
- ***Ask:*** How many identical candy bars might have a total mass of about 170 g? Explain your thinking.

Graphing Using Scale

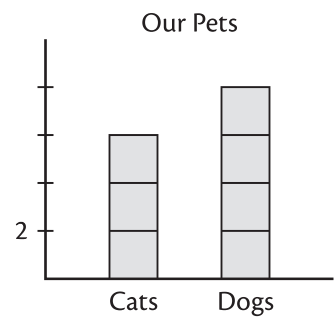
Measurement and Data	CCSSM 3.MD
Represent and interpret data.	
3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. For example, draw a bar graph in which each square in the bar graph might represent 5 pets.	

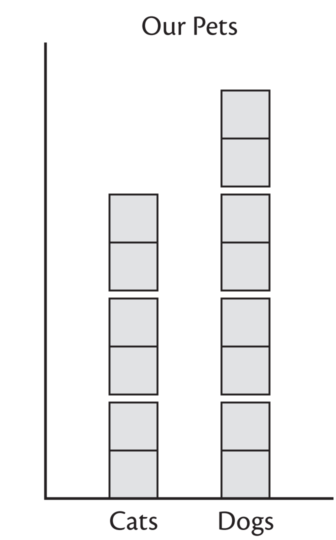
IMPORTANT UNDERLYING IDEAS

- ***Using a scale.*** In 3rd grade, we make a transition to bar graphs where one square represents more than one object. In a sense, this is the essence of proportional thinking—we look at 7 squares, but think 7 twos (or 7 fives or 7 of some other unit, depending on the scale).

  Even when counting how many cats or dogs are represented in the upper graph at the right, students are likely to think in units and say *2, 4, 6* or *2, 4, 6, 8*.

  To transition students to this sort of scaled graph, it might help them to see how the 6 and 8 units became 3 and 4 squares by beginning with 6 and 8 actual square tiles, then stacking them in 2s. The starting graph would have looked like the lower one at the right. When the tiles are stacked in pairs, students will see how the scaled graph (the upper graph) came to be. It also helps them to see clearly that a square in the scaled graph represents 2 of a unit.



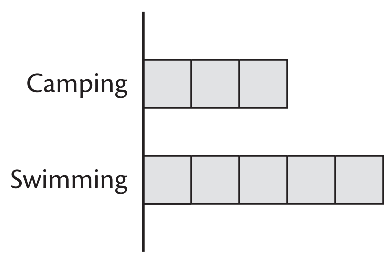


➢ ***Creating a scale.*** Sometimes students are provided data and must choose an appropriate scale for a graph. This involves looking at a set of numbers and thinking of each of them as 2 of something or 3 of something or 5 of something, etc. This too, promotes proportional thinking.

For example, to draw a scaled graph to show the number of students in various clubs, which might be 12, 16, 24, and 20, a student might think about the advantages of thinking of the numbers as 3, 4, 6, and 5 fours versus 6, 8, 12, and 10 twos. Students should deal with the fundamental proportional reasoning concept that if a larger unit is used (or a larger scale), then fewer units (or squares on a graph) are needed.

Good Questions to Ask

- ***Ask:*** You draw a graph with a scale of 2. Your friend uses the same information, but he draws a graph with a scale of 5. How will your graphs be different? How will they be alike?
- ***Ask:*** Suppose you were graphing these numbers of people who chose different favorite colors: 25 chose red, 40 chose blue, and 10 chose pink. What scale would you use? Why?
- ***Ask:*** Suppose you knew that 10 more people preferred swimming than camping. How does that help you know the scale of this graph?



Summary

The introduction of multiplication and division more formally in Grade 3 opens the door for much more substantial work in proportional thinking. At this level, students can be introduced to the concept that any number can be represented as many of another number (relating to multiplication and division), and they can think of multiplication as a unit change (moving from a group unit to an individual unit). They can also use proportional reasoning while applying multiplication/division strategies, while solving measurement problems, and while working with scaled graphs.

GRADE 4

Making Multiplicative Comparisons

Operations and Algebraic Thinking	CCSSM 4.OA
Use the four operations with whole numbers to solve problems.	

1. Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
2. Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison.

IMPORTANT UNDERLYING IDEAS

➢ ***Contrasting additive and multiplicative comparisons.*** When two numbers are compared, there is always more than one way to compare them. For example, looking at the numbers 25 and 5, the choice is to focus on the fact that 25 is 20 more than 5 or that 25 is 5 fives.

An interesting way to contrast these two situations might be this: Two children have bank accounts. One child deposited $10, and his account increased from $50 to $60. Another deposited $6, and his account increased from $2 to $8. Which account grew the most?

Certainly, from one perspective, the first account grew more; it increased by $10 rather than $6. This viewpoint focuses on additive comparisons. From another perspective, the second account grew more; the second original amount was multiplied by 4, whereas the first original amount was not even doubled. This approach focuses on multiplicative comparisons.

Multiplicative comparisons become more important as students get older, and so it is important to regularly ask for comparisons that are multiplicative.

It is usually easier for students to make these comparisons if one number is a whole number multiple of another (e.g., as 10 is of 5), but multiplicative comparisons apply in other situations as well. For example, 4 is $\frac{4}{10}$ of 10; this is a multiplicative comparison. Or 10 is $2\frac{1}{2}$ fours; this, too, is a multiplicative comparison.

➢ ***Comparing multiplicatively.*** There are many situations where students could be encouraged to think multiplicatively. They might solve simple problems such as *Jeff had 3 times as many coins as Amy. If Amy had 4 coins, how many did Jeff have?* OR *Jeff had 3 times as many coins as Amy. If Jeff had 12 coins, how many did Amy have?* OR *Jeff had 12 coins and Amy had 4. How many times as many coins as Amy did Jeff have?*

Students might compare bars on bar graphs to decide which bar is 3 times as high as which other. They might compare measurements of a shape and decide if the length of a rectangle is closer to double, triple, or 4 times the width. They might compare two angles to decide whether one angle is closer to double, triple, or 4 times another.

➢ ***Multiplying both sides of a multiplication equation by a factor.*** Realizing that an equation like $5 \times 4 = 20$ means that 20 is 5 fours should make it clear why doubling both sides of an equation does not change its truth. Certainly if 5 fours is 20, then 10 fours (which is double 5 fours) must be 20 + 20, which is double 20. Similarly, 15 fours would be triple 5 fours; it is 5 fours + 5 fours + 5 fours, or triple 20. Therefore $3 \times (5 \times 4) = 3 \times 20$. This leads to the very important algebraic generalization that one way to solve an equation is to multiply or divide both sides by the same amount. As an example, for the equation $2x = 40$, dividing both sides by 2 solves the equation.

Good Questions to Ask

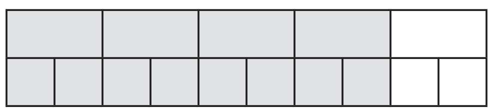
- ***Ask:*** The number *A* is 9 times the number *B*. What could *A* and *B* be?
- ***Ask:*** The number *A* is 24 times the number *B*. How does *A* compare with $2 \times B$? With $4 \times B$? With $6 \times B$? [This is an example of the mathematical practice standard of reasoning abstractly and quantitatively.]
- ***Ask:*** Which change do you think is more dramatic and why? A school's population increased from 100 students to 225 students? Or, a school's population increased from 150 students to 300 students?
- ***Ask:*** The sum of two numbers is 3 times their difference. What could the numbers be?
- ***Ask:*** 4 is more than 4 times as big as some number. What could the number be, and how many times as big as the number is 4?

Interpreting Fraction Equivalence Using Multiplicative Comparisons

Number and Operations—Fractions	CCSSM 4.NF
Extend understanding of fraction equivalence and ordering.	
1. Explain why a fraction $\frac{a}{b}$ is equivalent to a fraction $\frac{n \times a}{n \times b}$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.	

IMPORTANT UNDERLYING IDEAS

➢ ***Thinking of equivalence as maintaining a multiplicative comparison.*** Students learn that the reason that $\frac{4}{5} = \frac{8}{10}$ is because if $\frac{4}{5}$ were part of a whole, and we subdivided each existing section into two sections, we would actually be looking at $\frac{8}{10}$.



But there is another way to look at equivalence that relates to the multiplicative comparison between the numerator and the denominator of a fraction. If one defines the fraction $\frac{1}{2}$ as the ratio of two numbers where the second number is double the first, then clearly the fractions $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}$, etc. maintain that ratio. It is for this reason that they are equivalent to $\frac{1}{2}$.

Another way to say this is that a fraction can only be equivalent to, for example, $\frac{2}{3}$, if the numerator is $\frac{2}{3}$ of the denominator or the denominator is $1\frac{1}{2}$ ($\frac{3}{2}$) of the numerator. That means that fractions like $\frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{20}{30}$, etc. are all equivalent to $\frac{2}{3}$. A student might realize that the reason that $\frac{4}{5}$ and $\frac{5}{6}$ are not equivalent is because 5 is $1\frac{1}{4}$ copies of 4, but 6 is not $1\frac{1}{4}$ copies of 5.

Although this approach may appear to be more abstract for many students than the drawings of wholes, it is fundamentally a critical concept. Fractions are equivalent because they maintain the same multiplicative comparison between numerator and denominator or denominator and numerator.

Good Questions to Ask

- ***Ask:*** List 10 fractions where the denominator is $2\frac{1}{2}$ times the numerator. What fraction of the denominator is each numerator? What do you notice about the fractions?
- ***Ask:*** Is it reasonable to interpret the fraction $\frac{3}{5}$ as 3 out of 5? If you do, why is $\frac{30}{50}$ equivalent to $\frac{3}{5}$?
- ***Ask:*** Connor says that since $5 \times 7 = 35$ and $8 \times 7 = 56$, then $\frac{5}{8} = \frac{35}{56}$. Why does that make sense?

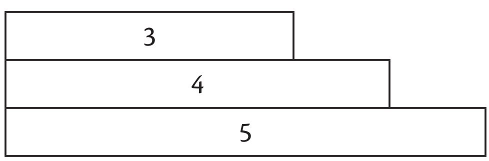
Ordering Fractions

Number and Operations—Fractions	CCSSM 4.NF
Extend understanding of fraction equivalence and ordering.	
2. Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $\frac{1}{2}$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, <, and justify the conclusions, e.g., by using a visual fraction model.	

IMPORTANT UNDERLYING IDEAS

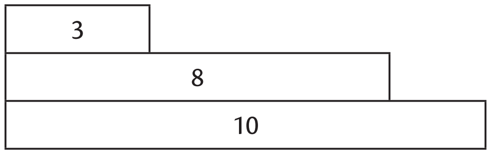
➢ ***Interpreting fraction size by considering multiplicative comparisons.*** We know that $\frac{4}{5} > \frac{3}{5}$ based on many interpretations of what fractions mean. Clearly if we take 4 out of 5 pieces of area, we are taking more than if we only took 3 out of those 5 pieces. As well, 4 jumps of $\frac{1}{5}$ from 0 takes us farther up the number line than 3 jumps of $\frac{1}{5}$ from 0.

But another way to see why $\frac{4}{5}$ is more than $\frac{3}{5}$ is because 4 is a bigger portion of the number 5 than 3 is. Whereas 4 is very close to all of the 5, 3 is not.



Similarly, we could compare $\frac{3}{8}$ and $\frac{3}{10}$ in many different ways. If a whole were divided into 8 pieces, each piece would be larger than if that whole were divided into 10 pieces. So 3 bigger pieces is more than 3 smaller pieces, and $\frac{3}{8} > \frac{3}{10}$. On a number line, a jump of $\frac{1}{8}$ is larger than a jump of $\frac{1}{10}$, since dividing the segment from 0 to 1 into 8 pieces results in bigger pieces than dividing the segment into 10 pieces. Three bigger jumps takes us farther up the number line than 3 smaller jumps.

But another way to see that $\frac{3}{8}$ is more than $\frac{3}{10}$ is to realize that 3 is a bigger portion of an 8 than it is of a 10.



Using these ideas, even when numbers are less comfortable, a student could argue that $\frac{29}{31} > \frac{29}{40}$, because 29 uses most of the 31 and 21 does not use even $\frac{3}{4}$ of the 40.

Sometimes, we compare fractions where neither the numerators nor the denominators are the same. One approach is to use benchmarks. Using benchmarks often involves making multiplicative comparisons.

For example, we could decide that $\frac{3}{8} < \frac{5}{9}$ without even looking at a model because we can reason that $\frac{3}{8} < \frac{1}{2}$ and $\frac{5}{9} > \frac{1}{2}$. But we know this only because we are implicitly using the ideas that

$$\frac{4}{8} = \frac{1}{2},$$

since 4 is half of 8 (just as 1 is half of 2), and that

$$\frac{4\frac{1}{2}}{9} = \frac{1}{2},$$

since $4\frac{1}{2}$ is half of 9 (just as 1 is half of 2). Therefore, because

$$\frac{3}{8} < \frac{4}{8} \text{ and } \frac{5}{9} > \frac{4\frac{1}{2}}{9}, \text{ then } \frac{3}{8} < \frac{5}{9}.$$

Even if we do not say all these words to ourselves, that is, in fact, exactly what we are thinking.

We might, similarly, decide that $\frac{4}{15} > \frac{1}{4}$, since 1 out of 4 is the same as 4 out of 16, so $\frac{1}{4} = \frac{4}{16}$. We know that $\frac{4}{15}$ is more than $\frac{4}{16}$, since 4 is a bigger part of a smaller number; that is, 4 is a bigger part of 15 than it is of 16.

Using multiplicative comparison is often an effective way to compare unfamiliar fractions.

Good Questions to Ask

- ***Ask:*** How could comparing 11 to 15 and 11 to 20 help you figure out why $\frac{11}{15}$ is more than $\frac{11}{20}$?
- ***Ask:*** How could you know that $\frac{5}{19}$ is close to $\frac{1}{4}$ without drawing or describing a picture?
- ***Ask:*** Without referring to pictures, how could you explain why $\frac{7}{12}$ must be less than $\frac{4}{5}$?

Interpreting Decimals as Ratios

Number and Operations—Fractions	CCSSM 4.NF
Understand decimal notation for fractions, and compare decimal fractions.	
6. Use decimal notation for fractions with denominators 10 or 100. For example, rewrite 0.62 as $\frac{62}{100}$; describe a length as 0.62 meters; locate 0.62 on a number line diagram. 7. Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.	

IMPORTANT UNDERLYING IDEAS

➢ ***Viewing decimal tenths or hundredths as multiplicative comparisons to 10 or 100.*** When students learn to read, for example, 0.4 as *four tenths*, they are, in essence, thinking about the comparison between 4 and 10. The decimal 0.4 is a bit less than half because 4 is a bit less than half of 10. Similarly, reading 0.23 as *twenty-three hundredths* is a way to describe the comparison between 23 and 100. Since 23 is a bit less than 25, 0.23 is slightly less than $\frac{25}{100}$, which is $\frac{1}{4}$.

Reasoning like this might help students who mistakenly think that $0.4 < 0.19$, and, unfortunately, many students do. Teachers often tell these students to change 0.4 to 0.40 to make it more obvious that 0.4 is more, but this is just a mechanical fix. We want students to more deeply understand that 0.4 is comparing 4 to 10 and 4 is almost half of 10, but 0.19 is comparing 19 to 100 and 19 is not even close to half of 100.

Good Questions to Ask

- ***Ask:*** How does comparing 73 to 100 help you explain why 0.73 is close to $\frac{3}{4}$?
- ***Ask:*** Why is $0.8 > 0.11$ even though $8 < 11$?
- ***Ask:*** A decimal tenth is just a little more than a decimal hundredth. What could each be? Explain why the decimal tenth is more.

Converting Measurements by Changing Units

Measurement and Data	CCSSM 4.MD
Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.	
1. Know relative sizes of measurement units within one system of units, including km, m, cm; kg, g; lb, oz; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...	

IMPORTANT UNDERLYING IDEAS

➢ ***Viewing measurement conversions as proportional reasoning.*** Changing from one measurement unit to another is an example of proportional thinking; in fact, it is one of the most fundamental examples. If, for instance, we decide that we would rather measure in inches than feet, we use the relationship that 1 ft = 12 in to recognize that the new unit is $\frac{1}{12}$ the size of the old one, so it will take 12 times as many of them to measure the same distance. This is quite similar to realizing that there are only 3 twelves in 36, but there are 36 ones.

Similarly, if we change 4 m to 400 cm, we are using the notion that since 1 cm = $\frac{1}{100}$ the size of 1 m, we need 100 times as many centimeters to describe the same distance that had previously been described in meters. This is quite similar to realizing that there are only 4 thousands in 4000, but there are 400 tens.

Good Questions to Ask

- ***Ask:*** How many feet are there in 1 mile? Without just saying that there are 12 inches in a foot, why are there 12 times as many inches in one mile?
- ***Ask:*** Suppose a new unit of weight measurement was invented called the duoune. It has the same weight as 12 ounces. How would you figure out how many ounces 8 duounes is?
- ***Ask:*** You want to know how many 3s are in 480. What measurement unit conversion problem does this remind you of? [It might be how many yards are in 480 feet.] What if you wanted to know how many 1000s are in 23,000?

Relating Angle Measurements

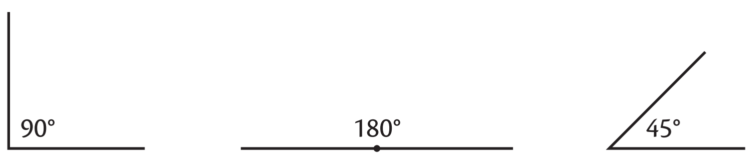
Measurement and Data	CCSSM 4.MD

Geometric measurement:
Understand concepts of angle and measure angles.

6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

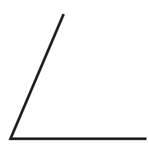
IMPORTANT UNDERLYING IDEAS

➢ ***Estimating angle size using benchmarks.*** Often, before we measure an angle with a protractor, we estimate its size by comparing it to angles we are very familiar with, for example, 90° angles, 180° angles, or 45° angles.

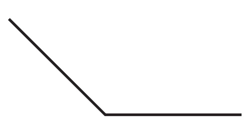


This is a good idea because it tells us whether our measurement is reasonable or not.

But using benchmark angles is actually an example of proportional reasoning. For example, I might estimate the measure of the angle below as 67° because it looks like it is about $1\frac{1}{2}$ 45° angles. I am using the 45° angle as a unit.



I can also use fractional units. For example, the angle below seems to be about $\frac{3}{4}$ of a 180° angle, so I estimate the measure to be about 135°.



➤ ***Exploring double and triple angles.*** Students might be asked to create angles that are double or triple the size of existing angles. This, too, is an example of using proportional reasoning.

Good Questions to Ask

- ***Ask:*** Draw a 45° angle. Now draw an angle that looks like it is about $\frac{1}{3}$ as big without using your protractor. Test with your protractor.
- ***Ask:*** An angle is about $\frac{5}{3}$ as big as a right angle. Estimate to draw it. Then test with your protractor.
- ***Ask:*** One angle is about 4 times as big as another. Draw those two angles without using your protractor. Then measure to see how close you were.

Summary

Proportional reasoning underlies much of the work in multiplication and division at this grade level. In particular, thinking of multiplication as a way to compare two numbers moves students from comparing numbers additively to comparing them multiplicatively. As well, work with fractions and decimals and work with measurement conversions and angle measurements naturally involve opportunities to use proportional thinking.

GRADE 5

Comparing Pattern Growth

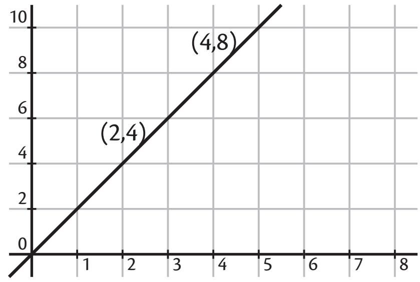
Operations and Algebraic Thinking	CCSSM 5.OA
Analyze patterns and relationships.	
3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. For example, express the calculation "add 8 and 7, then multiply by 2" as 2 × (8 + 7). Recognize that 3 × (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.	

IMPORTANT UNDERLYING IDEAS

➢ ***Comparing multiplication table patterns.*** It is probably easiest for students to begin comparing two patterns where the connections are fairly obvious, for example:

2, 4, 6, 8, 10, . . .
vs. 4, 8, 12, 16, 20, . . .

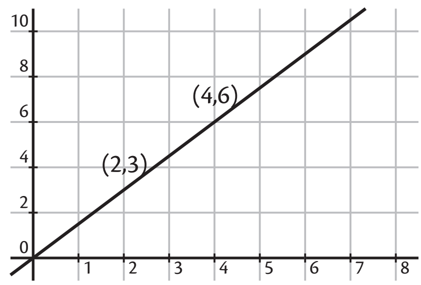
Clearly any term in the second pattern is double the corresponding term in the first pattern. This is the same as saying that any term in the first pattern is half the corresponding term in the second pattern. If corresponding terms are graphed as ordered pairs, a line is formed that goes through the origin of the Cartesian plane.



Next, students could move on to compare a pair of patterns with a multiplicative relationship that does not involve a whole number multiple, such as this one:

2, 4, 6, 8, 10, . . .
vs. 3, 6, 9, 12, 15, . . .

where any term in the second pattern is $1\frac{1}{2}$ times the corresponding term in the first pattern and any term in the first pattern is $\frac{2}{3}$ of the corresponding term in the second pattern. Again, a line is formed that goes through the origin of the Cartesian plane.



➢ ***Comparing pattern growth.*** An important issue for students to consider in regard to patterns is how fast they grow. Students need to learn that even though one pattern might start at a much higher number, if it grows more slowly, ultimately the other pattern will catch up and surpass it.

For example, the pattern 101, 102, 103, . . . takes 100 terms to reach the value of 200, whereas the pattern 5, 10, 15, 20, . . . only takes 40 terms.

Good Questions to Ask

- ***Ask:*** How can you use a term in Pattern 1 to get the value of the corresponding term in Pattern 2? What about going from Pattern 2 to Pattern 1?

  Pattern 1: 6, 12, 18, 24, 30, . . .
  Pattern 2: 15, 30, 45, 60, 75, . . .

- ***Ask:*** You choose two patterns and graph the corresponding terms. The term in the first pattern is the first coordinate and the term in the second pattern is the second coordinate. A line is formed that goes through the origin. The line is very steep. What might the two patterns be?
- ***Ask:*** How long does it take the pattern 4, 8, 12, 16, . . . to get past 250? How about the pattern 2, 4, 6, 8, 10, . . . ? Why do the two different values make sense?
- ***Ask:*** Which pattern gets to 1000 faster: 10, 15, 20, 25, . . . or 502, 504, 506, 508, . . . ?
- ***Ask:*** A pattern that starts low gets to 200 a lot faster than 105, 110, 115, 120, ... What might that pattern be?

Comparing Powers of Ten

Number and Operations in Base Ten	CCSSM 5.NBT
Understand the place value system.	
1. Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and $\frac{1}{10}$ of what it represents in the place to its left. 2. Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating powers of ten.*** Most students realize that when we calculate with numbers, if the calculation produces more than 10 units in any column of the result, 10 of those units are traded for 1 unit in the next column (e.g., adding 352 to 265 gives 11 units in the tens column; 10 of those are traded to give 1 additional unit in the hundreds column, for 3 + 2 + 1 hundreds). But that does not guarantee that students actually think of powers of ten in relation to other powers of ten. For example, a student should realize that 10,000 is 100 hundreds, since the 10,000 column is two columns to the left of the 100 column. Similarly, they should realize that 7 ten thousands (i.e., 7 in the 10,000 column) is worth 100 times as much as 7 hundreds (or 7 in the 100 column).

Ultimately, students could figure out that if there are two 4s three columns apart in a number, the leftmost 4 is worth 1000 times as the rightmost 4, no matter which column each appears in. For example, the 4 thousands in 34,214 are worth 1000 times as much as the 4 ones.

Good Questions to Ask

- ***Ask:*** How many tens does it take to make 100,000? How do you know?
- ***Ask:*** How many copies of 70 would you need to have 700,000?
- ***Ask:*** Create numbers to meet each condition:
  - One digit in the number is worth 100 times another
  - One digit in the number is worth $\frac{1}{1000}$ of another
  - One digit in the number is worth 10,000 of another

Comparing Decimals Using Proportional Reasoning

Number and Operations in Base Ten	CCSSM 5.NBT
Understand the place value system.	
3. Read, write, and compare decimals to thousandths. b. Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.	

IMPORTANT UNDERLYING IDEAS

➢ ***Interpreting a decimal as a ratio.*** A decimal such as 0.235 is actually a way to describe the ratio comparing 235 to 1000. Similarly, the decimal 0.14 is a way to describe the ratio comparing 14 to 100.

This sort of thinking should help students when comparing decimals. For example, the reason that 0.58 > 0.124 even though 58 < 124 is because 0.58 compares 58 to 100; 58 is a bit more than half of 100. But 0.124 compares 124 to 1000 (not 100), and 124 is not even close to half of 1000.

So, instead of trying to follow a "rule" that they may or may not really understand for how to compare two decimals, students may find this ratio approach much more meaningful.

Good Questions to Ask

- ***Ask:*** A decimal thousandth is slightly less than $\frac{1}{3}$. What might it be?
- ***Ask:*** What digits might you put in the blanks to make this expression true? 0.□8 > 0.79□.
- ***Ask:*** Explain why 0.□ could be more than 0.□□□.

Interpreting a Fraction as a Division

Number and Operations—Fractions	CCSSM 5.NF

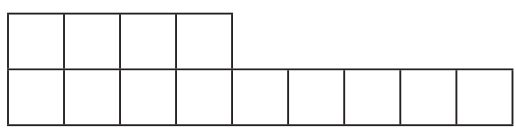
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator ($\frac{a}{b} = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For example, interpret $\frac{3}{4}$ as the result of dividing 3 by 4, noting that $\frac{3}{4}$ multiplied by 4 equals 3, and that when 3 wholes are shared equally among 4 people each person has a share of size $\frac{3}{4}$. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?

IMPORTANT UNDERLYING IDEAS

➢ ***Interpreting $\frac{a}{b}$ as a ÷ b by thinking of multiplicative comparisons.*** One of the ways students learn to think of $\frac{a}{b}$ as $a \div b$ is by thinking of sharing. But another way to help students understand this relationship is to help them think proportionally. For example, one of the ways to think of $\frac{3}{6}$ as $3 \div 6$ is to realize that $\frac{3}{6}$ can be interpreted as *How many 6s or how much of a 6 fits in a 3?* This is a standard meaning for division; for example, $12 \div 3$ asks *How many 3s fit in a 12?*

In a similar way, $\frac{4}{9}$ asks *How many 9s or how much of a 9 fits in a 4?* Clearly the answer is slightly less than half a 9, which is exactly what $\frac{4}{9}$ is, slightly less than half. The picture below shows that $\frac{4}{9}$ is exactly how much of a 9 fits in a 4.



Good Questions to Ask

- ***Ask:*** Draw a picture that helps show why $\frac{3}{8}$ is how much of an 8 fits in a 3.
- ***Ask:*** How could you use Cuisenaire rods to show why $\frac{4}{5} = 4 \div 5$?
- ***Ask:*** Suppose a fraction's denominator is $2\frac{1}{2}$ times its numerator. What fraction of its denominator is its numerator?

Interpreting Fraction Multiplication in Terms of Unitizing

Number and Operations—Fractions	CCSSM 5.NF
Apply and extend previous understandings of multiplication and division to multiply and divide fractions.	

5. Interpret multiplication as scaling (resizing), by:
   a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.
   b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $\frac{a}{b} = \frac{n \times a}{n \times b}$ to the effect of multiplying $\frac{a}{b}$ by 1.
6. Solve real-world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

IMPORTANT UNDERLYING IDEAS

➢ ***Interpreting a multiplication factor as a scale factor.*** Students are often surprised when they start multiplying by a fraction and find that the answer is less than the other factor, since they associate multiplication with making something bigger. But if we think of, for example, $\frac{2}{3} \times \frac{4}{5}$ as meaning $\frac{2}{3}$ of a $\frac{4}{5}$ unit (just as $2 \times \frac{4}{5}$ is two of a $\frac{4}{5}$ unit), it is clear why the result is not as much as $\frac{4}{5}$.

Unfortunately, some students then generalize that multiplying by any fraction decreases a value, but this, too, is not always true. For example, $\frac{5}{4} \times \frac{2}{3}$ is $\frac{5}{4}$ of a $\frac{2}{3}$ unit, which is more than one unit of $\frac{2}{3}$, so the result is greater than $\frac{2}{3}$. Ultimately, students need to realize that multiplying by a factor less than 1 scales the unit down and multiplying by a factor greater than 1 scales the unit up.

➢ ***Using scaling problems.*** Among the problems involving multiplication of fractions that students might solve are problems like these: *The length of a rectangle is scaled to $\frac{3}{4}$ its size and the width is scaled to $1\frac{1}{3}$ times its size. What happens to the area of the rectangle?* OR *Jeff has done $\frac{2}{3}$ as much work as Bruce. Andrew has done $\frac{3}{5}$ as much work as Jeff. What faction of the work that Bruce has done has Andrew done?*

Good Questions to Ask

- ***Ask:*** If 4 × 2 describes the length of a ribbon 4 times as long as a 2-unit ribbon, what does $\frac{4}{5} \times \frac{2}{3}$ describe?
- ***Ask:*** Without getting an answer, tell why it makes sense that $\frac{2}{3} \times \frac{4}{5}$ is the same amount as $\frac{4}{3} \times \frac{2}{5}$.
- ***Ask:*** Without using the rule to multiply the two fractions, how would you explain why $\frac{5}{4} \times \frac{2}{3}$ is more than $\frac{2}{3}$ but less than $\frac{5}{4}$?

Interpreting Fraction Division in Terms of Unitizing

Number and Operations—Fractions	CCSSM 5.NF

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.
   a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for $\frac{1}{3} \div 4$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $\frac{1}{3} \div 4 = \frac{1}{12}$ because $\frac{1}{12} \times 4 = \frac{1}{3}$.
   b. Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for $4 \div \frac{1}{5}$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $4 \div \frac{1}{5} = 20$ because $20 \times \frac{1}{5} = 4$.
   c. Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

IMPORTANT UNDERLYING IDEAS

➢ ***Dividing a whole number by a unit fraction.*** One way to interpret, for example, $3 \div \frac{1}{5}$ is to ask how many times as much as $\frac{1}{5}$ the number 3 is. Clearly, the number 1 is 5 times as much as $\frac{1}{5}$, so 3 must be $3 \times 5 = 15$ times as much as $\frac{1}{5}$. This approach involves interpreting division as asking for the missing multiplicative factor.

➢ ***Dividing a unit fraction by a whole number.*** One way to interpret, for example, $\frac{1}{5} \div 4$ is to ask how much of a 4 the number $\frac{1}{5}$ is. Since $\frac{1}{5}$ is $\frac{1}{5}$ of a 1, it can only be $\frac{1}{20}$ of a unit that is 4 times as big. That is why $\frac{1}{5} \div 4 = \frac{1}{20}$.

➢ ***Solving problems involving division with a unit fraction and a whole number.*** Among the types of problems involving division with a unit fraction that students might solve are problems like these: *Jeff has walked $\frac{1}{3}$ of a mile. What fraction of a 5 mile walk is this?* OR *How many laps of $\frac{1}{4}$ mile would someone have to run to run a total of 6 miles?*

Good Questions to Ask

- ***Ask:*** How are the interpretations of the question $\frac{1}{4} \div 5$ and $5 \div \frac{1}{4}$ alike? How are they different?
- ***Ask:*** What measurement problem might this division describe: $6 \div \frac{1}{4}$?
- ***Ask:*** Create a division problem involving measurement where the answer is $\frac{1}{48}$.

Measurement Conversion

Measurement and Data	CCSSM 5.MD
Convert like measurement units within a given measurement system.	
1. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real-world problems.	

IMPORTANT UNDERLYING IDEAS

➢ ***Viewing measurement conversions as proportional reasoning.*** Changing from one measurement unit to another is an opportunity to exercise proportional thinking. If, for instance, we decide that we would rather measure in inches than in feet, we use the relationship that 1 ft = 12 in to realize that the new unit is $\frac{1}{12}$ the size of the old one. So, if something measured 4 feet with the old unit, it would measure only $12 \times 4 = 48$ inches with the new unit.

Conversely, if we change 5000 m to 5 km, we are using the notion that since 1 m = $\frac{1}{1000}$ the size of 1 km, we use only $\frac{1}{1000}$ as many kilometers as meters to describe the same distance.

Good Questions to Ask

- ***Ask:*** How many centimeters are there in 1 kilometer? How does that help you write 4000 cm as ____ km?
- ***Ask:*** Suppose 600 ____ = 0.6 ____. What units might go in the blanks?
- ***Ask:*** You change a length measured in yards to one measured in miles. How will the number change? Why will it change that way?

Using Proportional Reasoning to Explore Volume

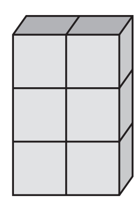
Measurement and Data	CCSSM 5.MD
Geometric measurement: Understand concepts of volume and relate volume to multiplication and to addition.	

5. Relate volume to the operations of multiplication and addition and solve real-world and mathematical problems involving volume.
   a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.
   b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real-world and mathematical problems.

IMPORTANT UNDERLYING IDEAS

➢ ***Volume as a multiplicative comparison.*** One way to interpret the volume of a rectangular prism is as multiple layers of the base. For example, if a prism has a height of 9 units, its volume is 9 times as much as the volume of its base.

➢ ***Related volumes.*** Students might also compare the volumes of two prisms. For example, they might compare volumes if one prism has a base of twice the area and a height of three times as much as another. Again they are thinking multiplicatively. In this case, if the volume of the original prism is 1 unit, the volume of the larger one would be 6 units.



Good Questions to Ask

- ***Ask:*** The volume of a rectangular prism is 8 × 6. Describe what the prism could look like. How do you know that?
- ***Ask:*** The volume of one rectangular prism is 10 times as much as the volume of another. What could the dimensions of each be?
- ***Ask:*** If a rectangular prism's height is tripled, what other change would you have to make to keep the original volume?

Summary

At the 5th-grade level, students have many opportunities to build proportional thinking. These include situations involving comparing number patterns, considering place value, interpreting decimals, interpreting fractions as divisions, interpreting fraction multiplication and division, converting measurements, and determining the volume of rectangular prisms.

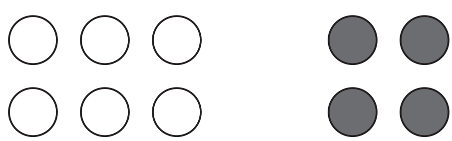
GRADE 6

Concept of Ratio

Ratios and Proportional Relationships	CCSSM 6.RP
Understand ratio concepts and use ratio reasoning to solve problems.	
1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."	

IMPORTANT UNDERLYING IDEAS

➢ ***Defining ratios.*** A ratio is a multiplicative comparison between two numbers. It tells how much/many of one quantity there is relative to how much/many of another. It is important that students discuss ratios in both discrete and continuous situations. For example, the picture of discrete objects below shows a ratio of 3:2 for light circles compared to dark circles; every time there are 3 light circles, there are 2 dark ones.



3:2 is called a part–part ratio because it compares two parts of a whole, in this case, a whole set of counters. There are also part–whole ratios that compare a part to the whole set. In this example, one part–whole ratio is 3:5, since there are 3 light circles within every group of 5 circles; another is 2:5, which compares the number of dark circles to the number in the full group of circles.

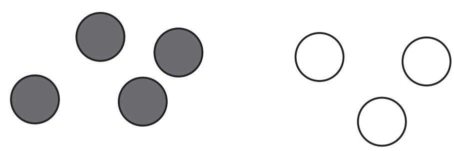
Students should also meet ratios involving continuous attributes, such as capacity. For example, they might consider which mixture would be sweeter: a drink made up of 1 cup of milk and 2 tbsp of chocolate syrup or one made up of $1\frac{1}{4}$ cups of milk and 3 tbsp of chocolate syrup. This is a multiplicative comparison;

we want students to realize that 1 cup of milk and 2 tbsp of chocolate syrup is the same ratio as $1\frac{1}{2}$ cups of milk and 3 tbsp of chocolate syrup. Since this drink has more milk for the same amount of syrup, it is more dilute and thus less sweet than $1\frac{1}{4}$ cups of milk and 3 tbsp of chocolate syrup.

At the Grade 6 level, students generally meet the colon notation typically used for ratios, for example, 1:3 to express a *1 to 3* ratio. There are several ways to effectively read such a ratio to students, such as 1 cup of milk *for every* 2 tbsp of chocolate syrup, or 1 cup *to* 2 tablespoons.

Students must realize that the order of the terms in a ratio matters. For example, 3 light circles:2 dark circles is not the same as 2 light circles:3 dark circles. However, any ratio can be read in two ways; 3 light circles:2 dark circles can also be read as 2 dark circles:3 light circles.

➢ ***Associating fractions with ratios and rates.*** Fractions can be associated with any ratio situation. For example, the ratio of dark circles to white ones below is 4:3.



Many fractions may be associated with this comparison:

- $\frac{4}{7}$ describes the fraction of all circles that are dark (a part–whole ratio).
- $\frac{3}{7}$ describes the fraction of all circles that are white (a part–whole ratio).
- $\frac{7}{4}$ describes how many times as many circles there are in total than dark circles (a whole–part ratio).
- $\frac{7}{3}$ describes how many times as many circles there are in total than white circles (a whole–part ratio).
- $\frac{4}{3}$ compares the number of dark circles to white ones. If the number of white ones is known, it can be multiplied by $\frac{4}{3}$ to determine the number of dark ones (a part–part ratio). Another way to express this relationship is to say that there are $\frac{4}{3}$ times as many dark circles as white ones.
- $\frac{3}{4}$ compares the number of white circles to dark ones. If the number of dark ones is known, it can be multiplied by $\frac{3}{4}$ to determine the number of white ones (a part–part ratio). Another way to express this relationship is to say that there are $\frac{3}{4}$ as many white circles as dark ones.

Recognizing these different viewpoints is an opportunity to apply the mathematical practice standard of looking for and making use of structure.

Rates, that is, ratios where the terms have different units, can also be related to fractions. If, for example, 2 boxes of an item cost \$12, then the equivalent fractions

$\frac{12}{2}$, $\frac{6}{1}$, and $\frac{18}{3}$ describe the per-box cost, but based on the amount spent for 2, 1, and 3 boxes, respectively. The equivalent fractions $\frac{2}{12}$, $\frac{1}{6}$, and $\frac{3}{18}$ describe the ratio of numbers of boxes to numbers of dollars spent, but in various forms. Part–part ratios are not as meaningful in a rate situation as they might be in other situations.

➢ ***The distinction between fractions and ratios.*** Clark, Berenson, and Cavey (2003) presented a very interesting discussion on how different people see fractions as a subset of ratios, ratios as a subset of fractions, ratios as identical to fractions, and ratios as totally different from fractions.

Although fractions and ratios are linked, as described in the previous section, there are distinctions. Both fractions and ratios compare two amounts. In the case of ratios or rates, it might be the terms of the ratio (or rate) that are compared and, in the case of fractions, there is a comparison of the numerator and the denominator. On the other hand, whereas a fraction is a single number associated with one point on a number line, a ratio or rate is generally not viewed as a single number.

Why this distinction is important is that "operating" with fractions is handled differently than operating with ratios. We add fractions by combining two parts of the same whole. For example, $\frac{2}{3} + \frac{1}{4}$ is the length of a segment that goes from 0 to $\frac{1}{4}$ appended to a segment that goes from 0 to $\frac{2}{3}$; both are related to the same unit, 1. We do not add numerators and denominators to determine the total length. On the other hand, it makes sense to say that if there are 2 circles for every 3 squares in one set and 1 circle for every 4 squares in another, then if the sets are combined, there are 3 circles for every 7 squares; in this case, we add first terms and second terms.

Good Questions to Ask

- ***Ask:*** Which of the following juice mixtures would taste more "orangey"?

  3 cups water:1 cup orange juice
  2 cups water:1 cup orange juice
  $2\frac{1}{2}$ cups water:$1\frac{1}{2}$ cups orange juice

- ***Ask:*** Draw two very different pictures that both show the ratio 3:5.
- ***Ask:*** How are fractions and ratios alike? How are they different?
- ***Ask:*** Draw any number of dark and light squares and write a number of different fractions as well as ratios associated with the comparison.
- ***Ask:*** Draw light and dark squares so that each of the fractions $\frac{3}{5}$ and $\frac{5}{8}$ could describe some aspect of the situation. What other fractions also describe aspects of the situation?
- ***Ask:*** Represent a ratio situation where the first quantity is much less than $\frac{1}{4}$ of the second one. What do you notice about the fractions that describe that situation?

Equivalent Ratios

Ratios and Proportional Relationships	CCSSM 6.RP
Understand ratio concepts and use ratio reasoning to solve problems.	
3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. a. Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.	

IMPORTANT UNDERLYING IDEAS

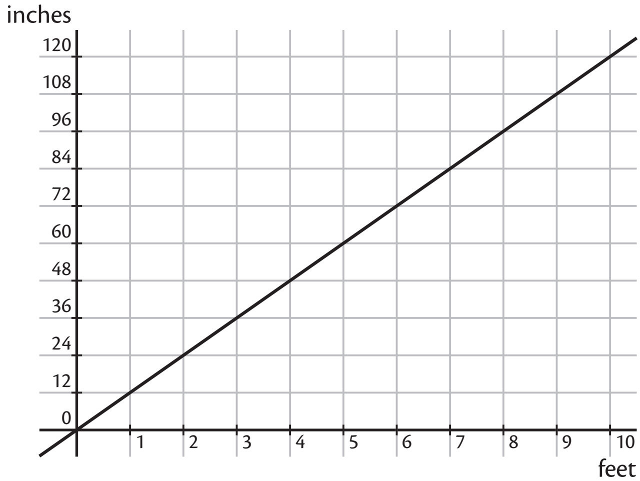
➢ ***The notion of equivalence.*** Just as a fraction can be renamed as an equivalent one by multiplying numerator and denominator by a non-zero amount, so can a ratio or rate. For example, the fraction $\frac{3}{5}$ is equivalent to the fraction $\frac{6}{10}$; similarly, when there are 3 circles for every 5 squares, there are 6 circles for every 10 squares, so the ratios 3:5 and 6:10 are equivalent. Or if 5 boxes cost \$3, then 10 boxes cost \$6, so the rates 5 for \$3 and 10 for \$6 are equivalent.

Understanding equivalence is fundamental to solving ratio, rate, or percent problems. Solving any such problem typically involves determining a rate or ratio that is equivalent to a given one. For example, determining the number of boys in a class if the ratio of boys:girls is 6:5 and there are 10 girls in the class is a question of renaming 6:5 as an equivalent ratio where the second term is 10 (instead of 5).

Determining the distance traveled in 8 hours at a speed of 48 mph is a question of determining an equivalent to the ratio 48:1 that takes the form □:8. Determining what the original price of an item is if 75% of the price is \$24 is a question of determining a ratio equivalent to 75:100 that takes the form 24:□.

The CCSSM standards speak specifically of ratios/rates dealing with measurements. For example, a student might recognize that since 1 ft = 12 inches, a table of values like the one shown below describes equivalent ratios; the values are graphed on the next page.

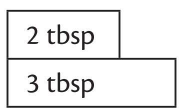
Feet	1	2	3	4	5	6	7	8
Inches	12	24	36	48	60	72	84	96



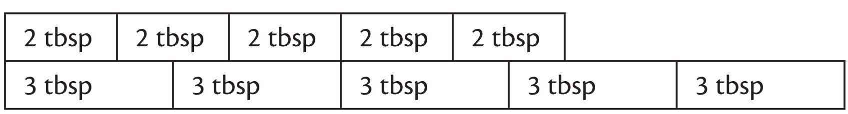
Use of the graph reinforces the linear nature of this relationship. It also helps a student, using interpolation or extrapolation, to determine equivalent ratios not included in the table.

Another tool for creating and modeling equivalent ratios is a tape diagram. For example, suppose a salad dressing recipe requires 3 tbsp of vinegar for every 2 tbsp of mustard. Students might be asked to solve the following problem: *If you put in 15 tbsp of vinegar, how much mustard should you put in?*

A possible model for solving the problem is started below. It shows the relationship between the vinegar and mustard amounts.



To get up to 15 tbsp of vinegar, the 3 tbsp needs to be repeated 5 times; the same is true for the 2 tbsp of mustard.



Now we can see many equivalent ratios for 2:3, including 4:6 (2 boxes on each line), 6:9 (3 boxes on each line), 8:12 (4 boxes on each line), and 10:15 (5 boxes on each line). The last equivalent ratio is the one that is most useful in solving the problem that was posed.

This approach is an example of applying the mathematical practice standard of using appropriate tools strategically.

Good Questions to Ask

- ***Ask:*** Draw a picture that shows why 3:5 and 6:10 describe the same ratio.
- ***Ask:*** How is changing 13 yards to feet an example of using equivalent ratios?
- ***Ask:*** Suppose 10 cookies cost $8.00. Why might it be useful to write the rate/ratio 10:8 as an equivalent one to figure out how much 12 cookies would cost? How might you use a tape diagram to do this?
- ***Ask:*** Make up a problem you might solve by changing the ratio 36:1 to an equivalent one.

Unit Rate

Ratios and Proportional Relationships	CCSSM 6.RP
Understand ratio concepts and use ratio reasoning to solve problems.	

2. Understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a{:}b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $\frac{3}{4}$ cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."
3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
   b. Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

IMPORTANT UNDERLYING IDEAS

➢ ***Choice of unit rate.*** Students should recognize that there is always more than one unit rate associated with a rate or ratio situation. For example, if you were describing a drive of 100 miles that took 2 hours, the unit rate could be described as either 50 miles/1 hour (or 50 mph) or as 1 mile per 0.02 hours (i.e., 0.02 hours/mile). And, in fact, since miles can be renamed as feet or yards and hours as minutes or seconds, there are other unit rates, as well, to describe the same situation. These include 88,000 yd/1 hour or 264,000 feet/1 hour or $\frac{5}{6}$ miles/1 minute or $1466\frac{2}{3}$ yards/1 minute, etc.

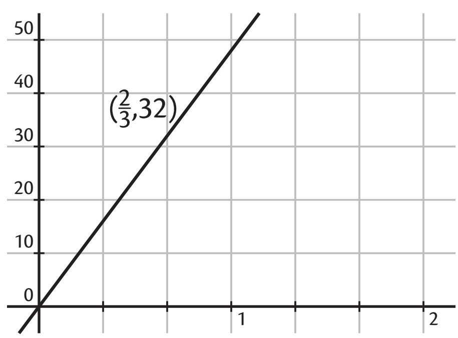
Although unit rates involve a coefficient of 1 (i.e., the b term in a:b is 1), there are circumstances where a single unit might consist of several subunits. For example, if 6 cookies cost \$15 and a student were asked to calculate the price of 4 cookies, he or she might use a 2-cookie unit (almost like a package of 2 cookies) to help figure out the answer. A 2-cookie pack would cost \$5 (since the ratio of 6:2 is the same as 3:1) and so 4 cookies, or two 2-cookie packs, would cost \$10.

➢ ***Calculating unit rates with equations.*** When given a rate, students can use equations to determine the unit rate. For example, if 2 quarts of juice costs \$4.89, the equation $2x = \$4.89$ can be solved to determine the unit rate.

If, instead, the unit rate is given, an equivalent ratio can be determined using an equation. For example, if 1 mile = 5280 feet, then to determine the number of feet in 80 miles, one could solve the equation $x = 5280 \times 80$.

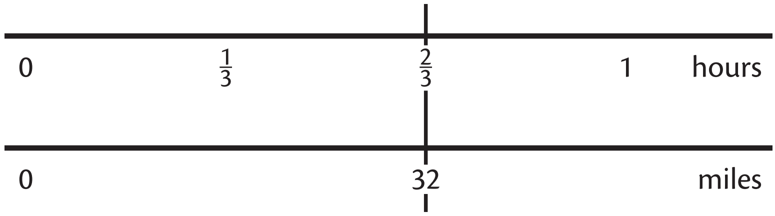
➢ ***Calculating unit rates with visuals.*** Often equations, or equivalent rates/ratios, are used to determine the price for one item when the price for many items or many units of that item is known. Similarly, the time required to go a certain distance at a given speed or the speed required to go a certain distance in a given time is often calculated by using equations or equivalent rates. But visuals can also be used. These include graphs and double number lines.

For example, suppose you know that someone drove 32 miles in 40 minutes (which is $\frac{2}{3}$ of an hour) and you want to know the speed in miles per hour. One strategy is to use a graph that goes through (0,0) since the person would drive 0 miles in 0 minutes. Connecting (0,0) to ($\frac{2}{3}$,32) gives the graph below:

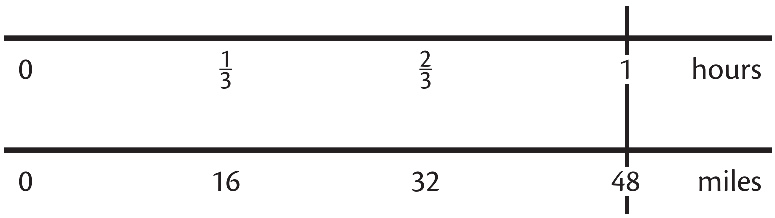


Looking at the value when $x = 1$ gives the unit rate of 48 miles per hour.

The double number line diagram relates $\frac{2}{3}$ of an hour to 32 miles in the following way:



Since 32 miles matches $\frac{2}{3}$ of an hour, then 16 miles matches $\frac{1}{3}$ of an hour and 48 miles matches 1 hour.



Good Questions to Ask

- ***Ask:*** Suppose 100 oz of detergent costs $11.99. Describe three unit rates: how much 1 oz costs, how much a unit of 10 oz costs, and how much detergent $1 buys.
- ***Ask:*** The equation $12x = 500$ was solved to do a measurement conversion. What do you think was being converted to what? Why do you think that?
- ***Ask:*** Draw a double number line that would help you figure out a car's speed in miles per hour if you knew it went 200 miles in $4\frac{1}{2}$ hours. Then draw a graph that would accomplish the same goal.
- ***Ask:*** Why is a unit rate often a very useful equivalent rate for a given rate?

Percent

Ratios and Proportional Relationships	**CCSSM 6.RP**
Understand ratio concepts and use ratio reasoning to solve problems.	
3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means $\frac{30}{100}$ times the quantity); solve the problems involving finding the whole, given a part and the percent.	

IMPORTANT UNDERLYING IDEAS

➢ ***What is important about percents?*** Most students are informally already familiar with percent when they arrive in 6th grade, but, at this level, work with percent is formalized and extended. First of all, students need to become more consciously aware that percents are ratios that relate one number in a multiplicative way to 100. 50% is the multiplicative relationship that compares 50 to 100; that is why 50% is $\frac{1}{2}$.

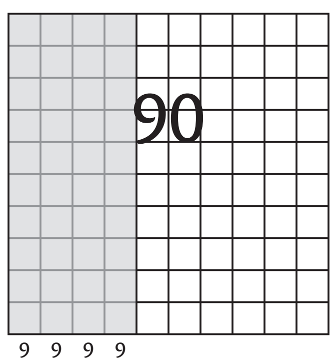
Another important thought about percents is that because they describe multiplicative relationships, they cannot be worth a lot at some times and not a lot at other times. For example, it is fair to say that 10% is never a lot of the whole, even though it may be a lot in absolute value (e.g., 10% of 10 million dollars is a lot, namely 1 million dollars, but that is still a small part of the whole 10 million dollars).

Students should realize that percents can represent values less than 1, but also values greater than 1. For example, even though 20% of something is not even $\frac{1}{4}$ of that thing, 200% of something is 2 of it; 200% = 2.

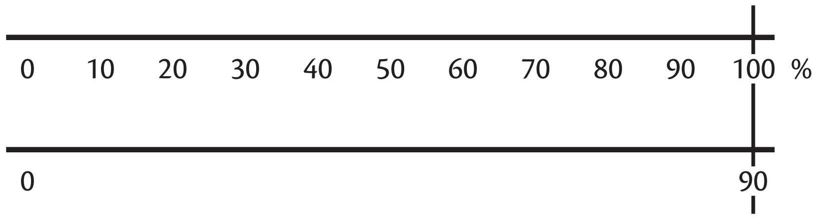
➢ ***Calculating percents using diagrams.*** Working with percents is no different from working with other ratios because percents *are* ratios. Sometimes the student knows the whole (100%) and wants to calculate a percent, but at other times he or she knows the percent and wants to calculate the whole (100%).

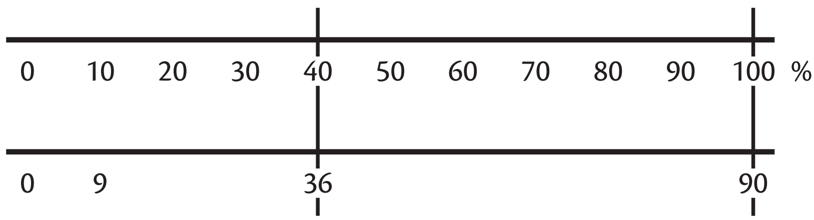
Two different types of diagrams are particularly useful when working with percent problems: one is a hundredth grid and the other is a double number line.

For example, suppose you need to determine 40% of 90. Of course, you might recognize that 40:100 is equivalent to 2:5, which is equivalent to 36:90, so the result is 36. But you might use a grid such as the one below. If the entire grid is worth 90, each column is worth 9, so 4 columns, or 40%, is worth 36.



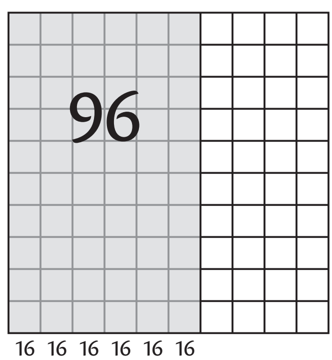
Alternatively, you might choose a double number line to solve the problem. In the diagram below, since 100% matches 90, then 10% must match 9 and 40% must match 36.



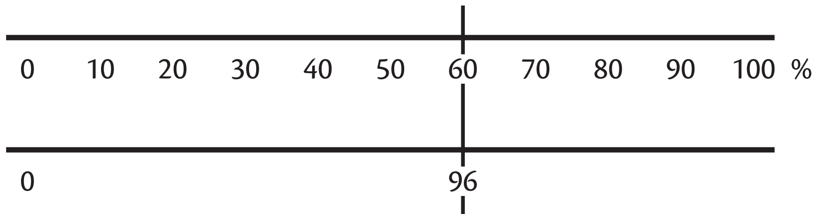


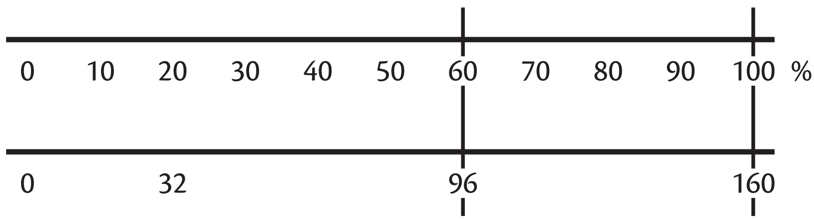
Similarly, if you know the percent, each of these types of diagrams could be used to determine the whole, or 100%. For example, suppose that the sale price of an item purchased for 40% off is \$96 and you want to know the original price. That means that you know that 60% of an amount is \$96 (if it is 40% off, then you are paying 60%) and you want to know the full amount.

In the hundredth grid picture, you know that 6 columns are worth 96, which means that each column is worth 16, and the entire grid is worth 160. The original price must have been \$160.



If a double number line is used, the value 96 matches the value 60% and the objective is to determine what matches 100%. Since 60% matches 96, $\frac{1}{3}$ of it, 20%, matches 32; then, 100%, which is $5 \times 20\%$, matches $5 \times 32 = 160$.





- ***Calculating percents using equations.*** As with other ratio situations, equations can be used to describe percent relationships. For example, to calculate 30% of 52, the student realizes that the ratio 30:100 is equivalent to x:52. This is solved using either of these equations:

$$100x = 30 \times 52 \quad \text{OR} \quad x = \frac{30}{100} \times 52$$

To calculate the whole when 30% is known to be 216, the student uses the equivalent ratios 30:100 and 216:x. He or she could solve any of these equations:

$$30x = 100 \times 216 \quad \text{OR} \quad \frac{30x}{100} = 216 \quad \text{OR} \quad x = 216 \div \frac{30}{100}$$

Looking at these equations, students begin to realize that calculating x% of a number involves multiplying it by $\frac{x}{100}$ (e.g., calculating 30% of 52 led to $x = \frac{30}{100} \times 52$) but calculating 100% when x% is known involves dividing by $\frac{x}{100}$ (e.g., calculating the whole when 30% was 216 led to the equation $x = 216 \div \frac{30}{100}$).

- ***Using fractions to calculate percents.*** Although students often prefer using percents to fractions, sometimes it makes more sense to use a fraction than a percent. Take, for example, the situation of calculating 25% of 484. It is much easier to calculate 484 ÷ 4 (thinking of 25% as $\frac{1}{4}$) than to multiply 484 by $\frac{25}{100}$.

Students might benefit from becoming familiar with some basic fraction/percent relationships. These would include 50% = $\frac{1}{2}$, 25% = $\frac{1}{4}$, $33\frac{1}{3}$% = $\frac{1}{3}$, 20% = $\frac{1}{5}$, and 10% = $\frac{1}{10}$.

Good Questions to Ask

- ***Ask:*** What does 300% mean? Is it a lot or not? [This is an example of the mathematical practice standard of constructing viable arguments and critiquing the arguments of others.]
- ***Ask:*** Choose a large number. Then draw a diagram that would show what 40% of that large number is. Next, choose a large number that you decide to be 40% of some other number. Draw a diagram that would show that other number.
- ***Ask:*** Some people think it's easier to calculate a percent of a given amount than calculating the amount if you know the percent. But some people don't. What do you think? Why?
- ***Ask:*** Suppose you had to solve this problem: $12\frac{1}{2}$% of a number is 23. What is the number? Would you use a diagram or an equation? Why?
- ***Ask:*** What percent problem might I be solving if I use the equation $48 \div 0.8 = x$?
- ***Ask:*** Why is there always more than one equation you could write to solve a percent problem?
- ***Ask:*** How might thinking of 40% as a fraction help you realize that there is no solution to this problem?

> Exactly 40% of the students in Maple School are going on a field trip. There are 432 students at the school. How many are going on the trip?

Converting Measurement Units

Ratios and Proportional Relationships	CCSSM 6.RP
Understand ratio concepts and use ratio reasoning to solve problems.	
3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations. d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.	

IMPORTANT UNDERLYING IDEAS

➢ ***Treating measurement unit conversion as a proportion.*** As was mentioned in an earlier section, one of the uses of ratio is in converting from one measurement unit to another. The conversion could be between units in the same system (Imperial or metric) or between units in different systems. In each of these unit conversion cases, if □ of one unit = ✶ of another unit, then the ratio of □:✶ can be used to convert other numbers of units in either direction.

For example, since 1 mile = 1760 yards, to determine the number of yards in $4\frac{1}{2}$ miles, the ratio $4\frac{1}{2}$:□ must be equivalent to 1:1760. Similarly, the number of miles in 5000 yards is calculated by creating a ratio of the form □:5000 that is equivalent to 1:1760. All of the strategies discussed in previous sections can be applied to create these equivalent ratios—using equations, diagrams, or tables of values.

This strategy for conversions works across systems as well. For example, since 1 inch = 2.54 cm, to calculate the number of centimeters in $3\frac{1}{8}$ inches, a ratio equivalent to 1:2.54 of the form $3\frac{1}{8}:x$ is created. Or, conversely, to calculate the number of inches in 10 cm, a ratio equivalent to 1:2.54 of the form x:10 is created.

Good Questions to Ask

- ***Ask:*** Why is knowing how many meters make a yard sufficient information to decide how many meters 32 yards make?
- ***Ask:*** How does knowing that 1 meter = 1.08 yards help you realize that the number of meters that 12 yards make must be close to 12?
- ***Ask:*** Suppose you drew a graph to help you convert 25 cm^2 into square inches. What shape would the graph have? Why? How could you use it to perform the conversion?

Relating Fraction Division to Proportional Reasoning

The Number System	CCSSM 6.NS
Apply and extend previous understandings of multiplication and division to divide fractions by fractions.	
1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $\frac{2}{3} \div \frac{3}{4}$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$ because $\frac{3}{4}$ of $\frac{8}{9}$ is $\frac{2}{3}$. (In general, $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.) How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{3}{4}$-cup servings are in $\frac{2}{3}$ of a cup of yogurt? How wide is a rectangular strip of land with length $\frac{3}{4}$ mi and area $\frac{1}{2}$ square mi?	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating fraction division to determining equivalent ratios.*** Because ratios are fundamentally multiplicative comparisons, it makes sense that fraction division relates to proportional thinking. For example, asking what to multiply $\frac{2}{3}$ by to get $\frac{4}{3}$ (or $\frac{4}{3} \div \frac{2}{3}$) is the same as asking for another form of the ratio $\frac{4}{3}:\frac{2}{3}$. But thinking about fraction division in terms of unit rates is a particularly useful connection.

Consider this unit rate problem: *You buy 100 oz. of detergent for \$12 and you want to know the price for 1 oz.* You would generally solve this problem by dividing 12 by 100, but you can think about this as determining an equivalent rate to 12:100 in the form □:1.

Similarly, determining $\frac{5}{6} \div \frac{1}{3}$ is the same as determining an equivalent ratio to $\frac{5}{6}:\frac{1}{3}$ in the form □:1. In this case the equivalent ratio is determined by multiplying both fractions by 3. Since $\frac{5}{6}:\frac{1}{3} = \frac{15}{6}:1$, the quotient is $\frac{15}{6}$.

In working with such problems, students would soon realize that were the question, for example, $\frac{5}{6} \div \frac{2}{3}$, the equivalent ratios would be $\frac{5}{6}:\frac{2}{3}$ and $(\frac{5}{6} \times \frac{3}{2})$:1. The divisor is inverted and multiplied by the dividend.

It would be useful to help students see that this proportional approach is useful for problems like this one: *You can mow $\frac{5}{6}$ of a lawn in $\frac{2}{3}$ of an hour. How many lawns can you mow in an hour?*

Good Questions to Ask

- ***Ask:*** Explain why calculating $\frac{3}{4} \div \frac{1}{2}$ is the same as creating a ratio of the form $x:1 = \frac{3}{4}:\frac{1}{2}$.
- ***Ask:*** Choose a value for y in the ratio $\frac{4}{5}:y$. What fraction division question might you be solving if you are seeking an equivalent ratio to $\frac{4}{5}:y$?
- ***Ask:*** How does thinking about proportions help you see why the solution to $3\frac{1}{2} \div \frac{2}{3}$ is the same as the solution to $1\frac{3}{4} \div \frac{1}{3}$?

Using Equations to Express Proportional Relationships

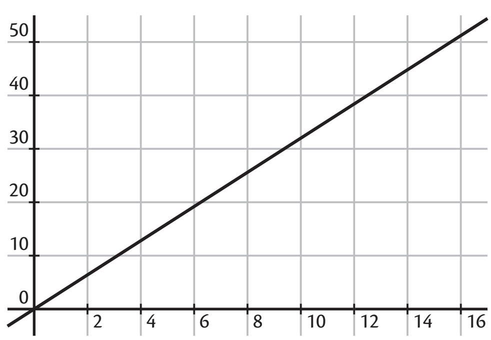
Expressions and Equations	CCSSM 6.EE
Represent and analyze quantitative relationships between dependent and independent variables.	
9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.	

IMPORTANT UNDERLYING IDEAS

➢ ***Direct variation relating two variables.*** When one variable is a direct multiple of another, the two variables are related proportionally. For example, if $y = 3.2x$, then the values for any x-value and a corresponding y-value form a ratio that is equivalent to the ratio for any other value of x and its corresponding y-value. That is because $y:x = 3.2$, not matter what the related y and x are.

The graph of such a relationship is always a line through the origin, since $0 = n \times 0$. The table of values always shows a multiplicative relationship. For example, the graph for $y = 3.2x$ might be built starting with this table of values:

x	0	1	2	3	4	5
y	0	3.2	6.4	9.6	12.8	16.0



These kinds of relationships arise in many measurement situations. For example, the circumference of a circle is related to the radius of that circle. The relationship is described by the equation $C = 2\pi r$, so C is a direct multiple of r. Another relationship might be between months and years. In this case $m = 12y$.

Good Questions to Ask

- ***Ask:*** How are the equations describing these problems similar and how are they different?
  - How many minutes are there in 8 weeks?
  - How many centimeters make 50 inches?
  - How much tax would you pay if you pay 8% tax on a \$35 purchase?
- ***Ask:*** Explain why the graph for $y = 8\frac{1}{2}x$ must go through the origin and why it is a straight line.
- ***Ask:*** Suppose that when the value of x is tripled, so is the value of y. What equation might describe the relationship between x and y? Are there other possibilities? How are they all alike?
- ***Ask:*** How is graphing the equation $y = \frac{3}{2}x$ helpful in determining equivalent ratios for 3:2?

Summary

At the 6th-grade level, students are formally introduced to the concept of ratio and ratio notation, although they had informally met the concept of ratio earlier. They learn about equivalent ratios, including unit rates, and about the relationship between fractions and ratios, and they begin to use percents. They also have the opportunity to learn about the relationship between fraction division and solving unit rate problems. Grade 6 students solve proportion problems, including those involving measurement conversions, using many different strategies, including graphs, equations, and visual models such as hundredth grids, double number lines, and tape diagrams.

Grade 7

Exploring Proportionality Numerically

Ratios and Proportional Relationships	CCSSM 7.RP
Analyze proportional relationships and use them to solve real-world and mathematical problems.	
1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas, and other quantities measured in like or different units. For example, if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, compute the unit rate as the complex fraction $\frac{\frac{1}{2}}{\frac{1}{4}}$ miles per hour, equivalently 2 miles per hour. 3. Use proportional relationships to solve multistep ratio and percent problems. *Examples:* simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.	

IMPORTANT UNDERLYING IDEAS

➢ ***Using unit rates.*** It is important for students to realize that any ratio can be described as a unit rate in two ways. For example, if a person jogs 2 km in $\frac{1}{2}$ hour, that is the same as $\frac{1}{2}$ hour for 2 km. The two unit rates are 4 km per 1 hour or $\frac{1}{4}$ hour per 1 km. Although one of those descriptions may seem more conventional, both are unit rates. Expressing rates in these ways is not different, for example, from comparing the price of two laundry detergents either in terms of how much 1 cup of each would cost or in terms of how many cups of each \$1 would buy.

Students should also understand that the unit rate is only one option for the description of a ratio. It is just as meaningful to say *2 km in $\frac{1}{2}$ hour* as it is to speak of *4 km in 1 hour*.

Students might connect unit rates to division of fractions or to equivalent ratios. For example, completing $\frac{2}{3}$ of a job in $\frac{1}{2}$ hour can be interpreted as the complex fraction $\frac{\frac{2}{3}}{\frac{1}{2}}$ or as $\frac{2}{3} \div \frac{1}{2}$, both of which produce the same result, $\frac{4}{3}$. Alternatively, the rate might be thought of as the ratio $\frac{2}{3}$:$\frac{1}{2}$, which is equivalent to $\frac{4}{3}$:1.

To better develop proportional thinking, students should have many opportunities to estimate unit rates. For example, if a car drives 11 miles in 15 minutes, students should be able to estimate the unit rate as either 44 miles in 1 hour (realizing that 15 minutes is $\frac{1}{4}$ of an hour) or about $\frac{3}{4}$ mile per minute.

➢ ***Solving proportions.*** Students should become aware of many ways to solve proportion problems. For example, students should recognize several ways of approaching a problem such as this one:

The ratio of adults to children at a festival is 2:5, or $\frac{2}{5}$. If there are 198 adults, how many children are there?

- Approach 1: If $\frac{2}{5} = \frac{198}{x}$, then since 2 was multiplied by 99 to get 198, 5 must be multiplied by 99 to figure out the number of children.
- Approach 2: If $\frac{2}{5} = \frac{198}{x}$, then since 5 is $2\frac{1}{2}$ twos, 198 must be multiplied by $2\frac{1}{2}$ to figure out the number of children.
- Approach 3: Divide 198 by $\frac{2}{5}$ since the number of children must be $\frac{198}{\frac{2}{5}}$.
- Approach 4: Cross-multiply. That is, $\frac{2}{5} = \frac{198}{x}$ only if $2x = 5 \times 198$. The equation is solved for x to determine the number of children.

The first approach focuses on creating an equivalent fraction by multiplying numerator and denominator by the same amount. The second focuses more on proportional thinking, that is, realizing that any fraction equivalent to $\frac{2}{5}$ is one where the numerator is $\frac{2}{5}$ of the denominator or the denominator is $2\frac{1}{2}$ times the numerator. The third approach essentially uses a unit rate: there is $\frac{2}{5}$ of an adult for every 1 child, or 1 adult for every $\frac{5}{2}$ children, so the total number of children is $\frac{5}{2} \times 198$ adults. The fourth approach focuses on the fact that two fractions with the same denominator are equal only if the numerators are equal. The fractions $\frac{2}{5}$ and $\frac{198}{x}$ are rewritten with denominators of $5x$ to produce $\frac{2x}{5x}$ and $\frac{198 \times 5}{5x}$; their numerators are equal only if $2x = 198 \times 5$.

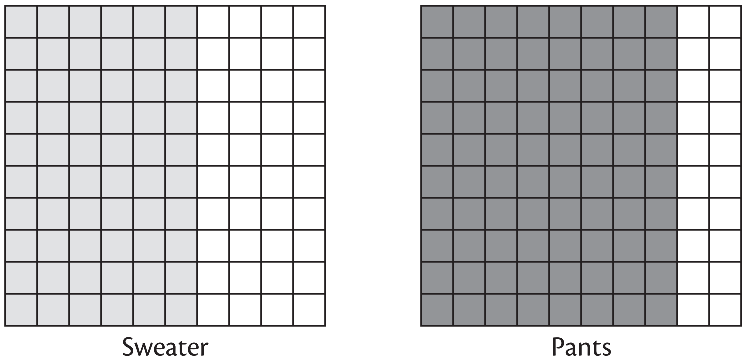
All of these approaches lead to correct solutions and are useful in different circumstances. Many teachers tend to emphasize Approaches 1 and 4, but the other two approaches focus somewhat more on proportional reasoning and should also be encouraged.

➢ ***Using percent models.*** Because percents are simply particular types of ratios—ones with a second term of 100—solving problems or proportions involving percents is no different from solving other types of proportions. However, because the second term is always 100, there are some special tools that might be useful in percent problems.

One of these tools is the 100-grid. Suppose, for example, students needed to solve a problem like this one:

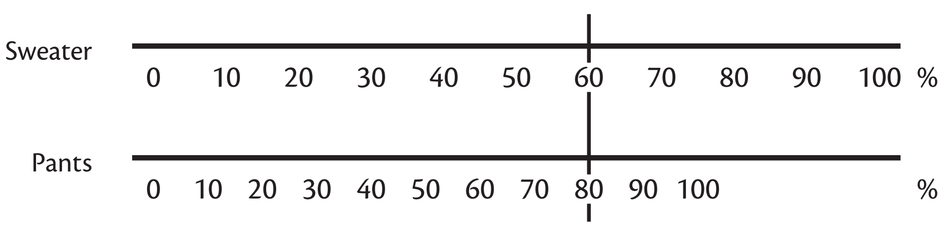
A sweater that is 40% off costs the same as a pair of pants that is 20% off. How do the two original costs relate?

The gray sections of the two grids represent the sale prices, and the full grids represent the original prices.



If the two gray sections are worth the same amount, 6 light gray columns (for the sweater) are worth the same as 8 dark gray columns (for the pants). Thus, each dark gray column is worth $\frac{3}{4}$ as much as a light gray column. That means that 10 columns in the dark gray grid (the original price of the pants) must be $\frac{3}{4}$ as much as 10 columns in the light gray grid (the original price of the sweater). So the original prices might be \$100 for the sweater and \$75 for the pants, or \$80 and \$60, or \$44 and \$33, etc.; it could be any pants price that is $\frac{3}{4}$ as much as the sweater price.

Another useful model in working with percents is the double number line. It is interesting if these lines are created on pieces of elastic so they can be stretched. To solve the problem described above, a student might build a model, matching the 60% on one elastic strip (the sweater strip) to the 80% on the other (the pants strip). This makes 100% on the pants strip (i.e., the original pants price) match 75% on the sweater strip, so the original price of the pants was 75% of the original sweater price.



➤ ***Interpreting percent increases and decreases.*** Work with percent increases and decreases is sometimes challenging for students. It is smoother if students think proportionally.

For example, consider this problem involving a percent increase: *A new housing development was built and the population of the local school increased by 50%. If the new school population is 639, what was the original school population?*

Students might realize that referring to an increase of 50% is another way of saying that the new value is 150% of the old value. They could then solve the proportion $\frac{150}{100} = \frac{639}{x}$ in some fashion.

Or, students might realize that talking about an increase of 50% of a whole is another way of saying $1\frac{1}{2}$ of the whole. Since $1\frac{1}{2} = \frac{3}{2}$, then if 3 halves is 639, 1 half is 213 (639 ÷ 3) and 2 halves is 426. This is proportional thinking.

Proportional reasoning can also be applied to problems involving percent decrease. Students might figure out why a decrease in a price of 50% followed by an increase of 50% does *not* result in the original price: because the second 50% is 50% of a smaller amount, so it is less than the first 50%. For example, suppose a price started at $200 and was reduced to $100, a 50% decrease. If the reduced price was then increased by 50%, the new price would be only $150 (not the original $200). This result relates to the fundamental proportional notion that if the second unit is half of the first, it would take twice as many of them to make the same amount. In this case, an increase of 100%, not 50%, would be required to get back to the original amount.

Good Questions to Ask

- ***Ask:*** If you know that you can sort out 3 bookshelves in a little less than $\frac{1}{4}$ of an hour, estimate how long it would take you to sort out 21 bookshelves. Explain your thinking.
- ***Ask:*** Describe a situation involving a rate. Now use two different unit rates to describe the same relationship.
- ***Ask:*** How do you know there is a mistake in this problem?

  80% of the students in Samantha's grade signed up to go on a field trip. If 98 students signed up, how many students are in her grade?

- ***Ask:*** Describe a picture you could draw to solve this problem. Then solve it.

  A TV is for sale at 50% off, and it costs the same as a computer that is on sale for 70% off. How were the original two prices related?

- ***Ask:*** A businessman sold 20% of his inventory. By what percent does he need to increase his new inventory to get back to his original amount? How do you know?
- ***Ask:*** What single multiplication can you do to solve the proportion $\frac{6}{9} = \frac{x}{82}$?

Exploring Proportionality Algebraically

Ratios and Proportional Relationships	CCSSM 7.RP
Analyze proportional relationships and use them to solve real-world and mathematical problems.	

2. Recognize and represent proportional relationships between quantities.
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as $t = pn$.
   d. Explain what a point (x,y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0,0) and $(1,r)$ where r is the unit rate.

IMPORTANT UNDERLYING IDEAS

➢ ***Recognizing proportionality using tables of values.*** Students in earlier grades have met situations where unit rates are involved, for example, the number of minutes in a given number of hours (where the unit rate is 1 hour = 60 minutes), the number of inches for a given number of centimeters (where the unit rate is 1 inch = 2.54 cm), or the cost of different numbers of boxes of a particular cereal (where the unit rate is the cost of one box). They have also had experience creating tables of values describing the relationship between the two variables involved. For example, if a box of cereal costs $2.97, students might have created a table like the one below to show the cost of different numbers of boxes of that cereal.

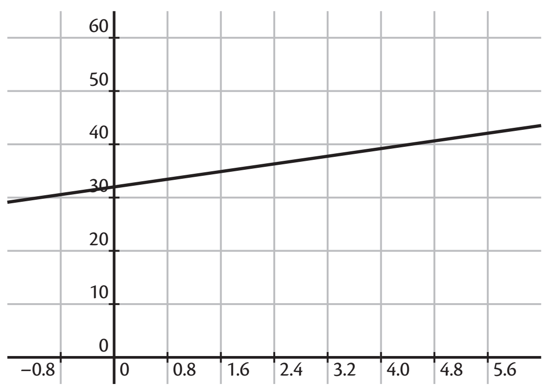
Number of boxes	Cost
0	$0
1	$2.97
2	$5.94
3	$8.91
4	$11.88
5	$14.85

When a table is created so that the value of the independent variable (in this case, the number of boxes) increases by 1, the dependent variable (in this case, the cost) increases by that unit rate. This makes sense because one more item results in one more unit. In 7th grade, we introduce the term *proportional* and indicate that variables are proportional when this kind of relationship exists. Students should notice that when the independent variable is 0, so is the dependent variable. If there are no boxes, there is no cost; if there are no hours, there are no minutes; if there are no centimeters, there are no inches.

Another way for students to test whether or not two variables are proportional is more indirect, but it is useful. If doubling the independent variable always doubles the dependent variable, the variables are also proportional.

➢ ***Recognizing proportionality using graphs.*** If students use a table of values as described in the preceding section and graph each (independent variable, dependent variable) combination as an ordered pair, they will observe that the graph is always a straight line that goes through the point (0,0). Conversely, if they see such a graph, they should be able to create the related table of values.

If a linear graph does not go through (0,0), the two variables represented by x and y are not proportional. For example, consider the case of Celsius temperature compared to Fahrenheit temperature (where 0°C matches 32°F):



Even though the two temperatures are not proportional, it is true that the Celsius temperature is proportional to a variable that represents 32° less than the Fahrenheit temperature.

The table below shows that °F is not proportional to °C, but °F – 32° is proportional to °C, since the values go up by a unit rate (specifically 1.8), and if the Celsius temperature is 0°, so is the °F – 32° temperature.

° Celsius	° Fahrenheit	° Fahrenheit – 32°
0	32	0
1	33.8	1.8
2	35.6	3.6
3	37.4	5.4
4	39.2	7.2

Because a unit rate is always involved in proportional situations, students might observe that the y-value associated with the x-value of 1 is that unit rate; the graph always goes through $(1,r)$ where r is the unit rate. For example, if we graph the number of seconds in different numbers of hours, the graph goes through (0,0), since 0 hours results in 0 seconds, but also through (1,3600), since there are 3600 seconds in 1 hour, the unit rate. Students should notice that the increase in the y-coordinate for each increase of 1 in the x-coordinate anywhere in the graph is the unit rate.

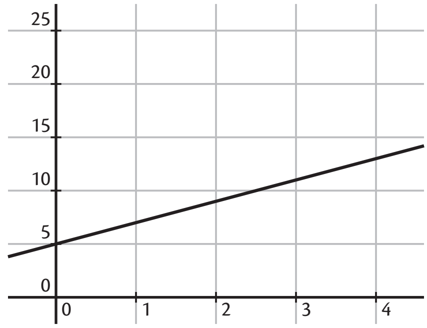
➢ ***Recognizing proportionality using equations.*** Students, looking at a table of values showing a proportional relationship, should notice that the y-coordinate is always a multiple of the x-coordinate. The multiplier is the unit rate. For example, if we relate the number of people to the number of eyes those people have, the number of eyes is always double the number of people. The relationship can be described in an equation as $e = 2p$ in this case. In the case of the cost for different numbers of boxes of cereal described earlier, the cost is \$2.97 multiplied by the number of boxes. The equation would be $c = 2.97n$.

Students should notice that all of these equations are of the form $y = rx$, where x is the value of the independent variable, r is the unit rate, and y is the value of the associated dependent variable.

Good Questions to Ask

- ***Ask:*** Why are the following statements true if $y = 3x$?
  - y is proportional to x.
  - If x is doubled, y is doubled.
  - If x is tripled, y is tripled.
- ***Ask:*** What might variables x and y represent if $y = 4x$? Describe why it makes sense that those variables are proportional.

- ***Ask:*** Why can't the variables represented by x and y in the graph below be proportional? Why are the variables x and $(y - 5)$ proportional?



- ***Ask:*** Describe several pairs of proportional variables. Prove that these pairs are really proportional.

Describing Fractions as Decimals

The Number System	CCSSM 7.NS
Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.	
2. Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers. d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.	

IMPORTANT UNDERLYING IDEAS

➢ ***Converting a rational number to a decimal.*** Students in earlier grades met the notion that the bar of a fraction can be interpreted as a division sign and learned why that is. However, it makes sense at this level for students to look at the relationships they see in the fraction and in the decimal representation for the same amount.

For example, using division, a student could convert $\frac{3}{11}$ to 0.272727. . . . It is important for students to make sense of this result. Since 3 is between $\frac{1}{4}$ and $\frac{1}{3}$ of 11, 0.27 is between $\frac{1}{4}$ of 1 (0.25) and $\frac{1}{3}$ of 1 (0.333. . .). It seems that $\frac{3}{11}$ must be closer to $\frac{1}{4}$ than to $\frac{1}{3}$ since 0.272727. . . is closer to 0.25 than to 0.333. . . , and that makes sense, since $\frac{3}{11}$ is closer to $\frac{3}{12}$ $(\frac{1}{4})$ than to $\frac{3}{9}$ $(\frac{1}{3})$.

Good Questions to Ask

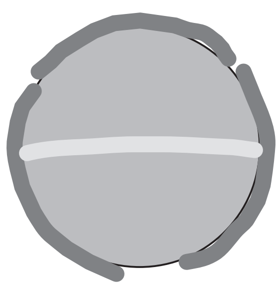
- ***Ask:*** Explain why it makes sense that $\frac{3}{14}$ is 0.2142857142857. . . without actually calculating the value.
- ***Ask:*** Use the relationship between 7 and 8 to decide which is a better estimate of the decimal for $\frac{7}{8}$: 0.78, 0.92, or 0.875. Why?
- ***Ask:*** Use the relationship between 5 and 9 to estimate a decimal representation for $\frac{5}{9}$. Then calculate the decimal. How good was your prediction?

Circle Measurement Ratios

Geometry	CCSSM 7.G
Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.	
4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.	

IMPORTANT UNDERLYING IDEAS

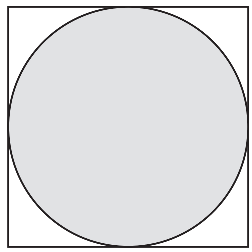
➢ ***Interpreting π.*** Inherent in work with circle measurements is an understanding of the number π, which is the ratio of the circumference of a circle to its diameter. Although π is a number, students should think of it as that ratio. If it takes more copies of π to describe the circumference of a circle, that is because the circle has a larger diameter (or radius). Most specifically, students should realize that if they were to cut a string the size of the diameter of a circle and wrap it around the circle's circumference, the string would fit around the circle slightly more than 3 times.



Students might be encouraged to contrast this relationship with the ratio of the perimeter of a square to its side length, which is 4, rather than slightly more than 3.

➢ ***Determining ratios of circle measures.*** Not only should students realize that the circumference of a circle is about 3 times its diameter (no matter the size of the circle); they should also understand why the diameter of a circle is about $\frac{1}{3}$ of its circumference and its radius is about $\frac{1}{6}$ of its circumference.

Students might be led to see that the area of a circle is about $\frac{3}{4}$ of the area of a square with a side length the same as the diameter of the circle, again no matter what the size of the circle.



Students will discover that although many ratios between circle measures are fixed proportions, not all are. For example, using a circle of radius 10", the ratio of the number of square inches in the area to the number of inches in the circumference is 5 ($100\pi/20\pi$); but using a circle of radius 5", the ratio of the number of square inches in the area to the number of inches in the circumference is 2.5.

On other occasions, students might compare the ratios of circumferences or areas of two different circles. For example, they might learn that if a circle's diameter is tripled, so is its circumference, but not its area; the area is multiplied by 9.

Good Questions to Ask

- Tell students that a certain circle was measured and the ratio of the two measurements was 1:6. ***Ask:*** What measurements might have been taken? Why did you choose those measurements?
- ***Ask:*** What picture could you draw to show that the radius of a circle is less than $\frac{1}{6}$ of its circumference?
- Tell students that the circumference of one circle is $3\frac{1}{2}$ times as great as the circumference of another. ***Ask:*** How do the radii of the two circles compare? Their areas?
- ***Ask:*** The number of square inches in the area of a circle is 18 times the number of inches in its circumference. What else do you know about the relationship between the measures of the circle?

Volume and Surface Area Ratios

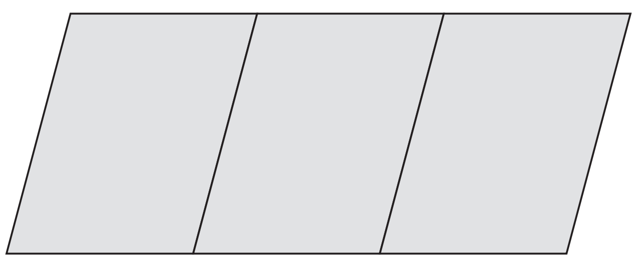
Geometry	CCSSM 7.G
Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.	
6. Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.	

IMPORTANT UNDERLYING IDEAS

➢ ***Exploring area relationships.*** Students can explore many different kinds of situations involving proportionality when considering areas of two-dimensional objects. These include the following:

- Recognizing that multiplying the base or height of a triangle or parallelogram by any factor multiplies its area by the same factor
- Recognizing that if the base or height of a triangle or parallelogram is multiplied by any factor, the other dimension must be divided by that factor to maintain the same area
- Recognizing that the ratio of the areas of a triangle and a parallelogram with the same base and height is always 1:2

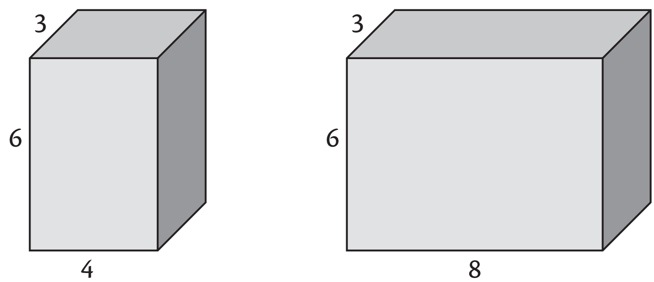
These issues can be explored in a number of ways. For example, to show that if the base of a parallelogram is tripled, so is its area, algebra can be used: $(3b)h = 3(bh)$. These relationships can also be explored numerically, by using many specific cases, or geometrically, by using a visual like the one below, for example, to show what happens when the base of a parallelogram is tripled.



➢ ***Exploring volume and surface area relationships.*** Students can explore many different kinds of situations involving proportionality when considering volumes and surface areas of three-dimensional objects. Among these are the following:

- Recognizing that multiplying the height of a prism or the area of its base by any factor multiplies its volume by the same factor
- Recognizing that if the area of the base or the height of a prism is multiplied by any factor, the other dimension must be divided by that factor to maintain the same volume
- Recognizing that multiplying the height of a prism or the area of its base by any factor does not necessarily multiply its surface area by that factor

For example, consider the two prisms shown below:



Although the area of the base was doubled from 12 square units in the prism on the left to 24 square units in the prism on the right, the surface area increased only from 108 square units to 180 (not 216) square units.

Again, these issues can be explored numerically, as above, or algebraically, for example, by realizing that if the width is doubled, the new surface area is $2l(2w) + 2lh + 2(2w)h$, which is not the same as $2(2lw + 2lh + 2wh)$.

Good Questions to Ask

- Tell students that a triangle was enlarged so that the new area is 20 times the old area. ***Ask:*** How might the base and the height of the original triangle have been altered?
- ***Ask:*** Describe different ways to start with a prism with a particular volume and create a new one with 24 times the volume of the original.
- ***Ask:*** Which dimension of a 4 × 3 × 9 prism would you need to triple to have the most effect on surface area? How does the new surface area relate to the old one? Why did you select the dimension that you did for tripling?
- ***Ask:*** One triangle is 4 times as tall as a parallelogram with the same base. How do their areas relate?

Predicting Likelihood

Statistics and Probability	CCSSM 7.SP
Investigate chance processes and develop, use, and evaluate probability models.	
5. Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $\frac{1}{2}$ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.	

IMPORTANT UNDERLYING IDEAS

➤ ***Solving probability problems using proportional reasoning.*** Most work in probability at the 7th-grade level is, by definition, an application of proportional reasoning. If, for example, we predict that 1 out of 4 rolls of two dice will yield two evens, we use that ratio of 1:4 to predict an actual number of likely occurrences of two evens when rolling a pair of dice a specific number of times. For example, we predict 20 occurrences of two evens if we roll the dice 80 times, but 25 occurrences if we roll the dice 100 times.

Although no really new proportional reasoning concepts emerge here, it is important for students to realize that the reason that a probability is a fraction and not a number like 5 or 8 is because the probability is the statement of a proportion and, thus, a fraction.

Added to the mix, though, when working with probability is randomness. So even though we use proportional thinking to predict that if we flip a coin 100 times, there will be 50 heads, we may not actually get 50 heads. This is often very disconcerting for students. They might even start to wonder if it makes any sense to predict, if the result will not be a certainty.

Good Questions to Ask

- ***Ask:*** Provide a reasonable estimate of the theoretical probability of an event that occurred 29 times in 53 tries.
- ***Ask:*** Why does it make sense that the maximum possible probability is 1 instead of a large number?
- ***Ask:*** Describe a situation where one possible result in an experiment is likely to be $\frac{2}{3}$ of another possible result.

Summary

Because ratio and proportion have been a formal part of the math curriculum for over a year by the time students reach 7th grade, there are many opportunities at this level to encourage students to refine their proportional reasoning. Such opportunities include number work where they solve ratio, rate, and percent problems and write fractions as decimals; work in algebra where they examine what proportionality looks like in tables, graphs, and equations; work in measurement, particularly with circle measurements; and work in probability.

GRADE 8

Using Exponent Situations to Build on Proportional Reasoning

Expressions and Equations	CCSSM 8.EE
Work with radicals and integer exponents.	
1. Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = \frac{1}{3^3} = \frac{1}{27}$.	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating exponent laws to proportional thinking.*** Students learn the exponent laws based on the meaning of exponents, but it is useful to relate what they learn to proportional thinking as well. In particular, the law that states that $\frac{a^b}{a^c} = a^{b-c}$ relates closely to proportional thinking. For example, students should realize that the ratio of $5^3:5^2$ is the same as the ratio of $5^1:1$ since in both cases the first term is 5 times the second term.

Good Questions to Ask

- ***Ask:*** The ratio of two powers is the same as the ratio $4^3:1$. What might the two powers be?
- ***Ask:*** The ratio of $a^b:c^d = a^e$. What could a, b, c, d, and e be? [This is an example of the mathematical practice standard of reasoning abstractly and quantitatively.]
- ***Ask:*** How can you use proportional thinking to help someone see why $4^8 = 4^{10}/4^2$?

Connecting Proportional Relationships to Linear Equations

Expressions and Equations	CCSSM 8.EE
Understand the connections between proportional relationships, lines, and linear equations.	
5. Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance–time graph to a distance–time equation to determine which of two moving objects has greater speed.	

IMPORTANT UNDERLYING IDEAS

➢ ***What are proportional relationships?*** In earlier grades, students have met the concept of proportional relationships. They have seen that the equation $y = mx$ (or $y = rx$) leads to a table of values that includes the coordinates (0,0) and $(1,m)$ (or $(1,r)$), where m (or r) is the ratio relating x and y, and that the graph is a line.

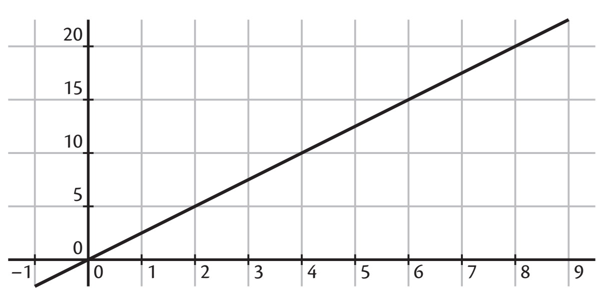
It is important that students understand that when two variables are related proportionally, each y-value is the same multiple of its corresponding x-value, no matter what the value of x is. The multiple need not be an integer or even a rational number. That is because if $y = mx$, then $y:x = m$, the ratio.

The table for the equation $y = 2\frac{1}{2}x$ might look like this:

x	0	1	2	3	4
y	0	$2\frac{1}{2}$	5	$7\frac{1}{2}$	10

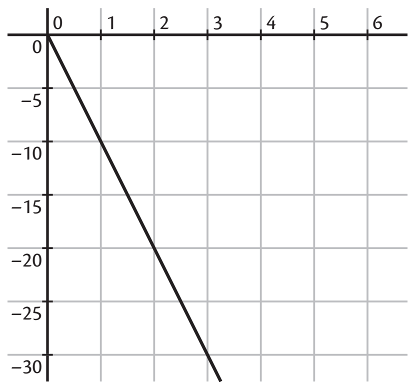
Students should notice that there is a constant change of $2\frac{1}{2}$ in y for every change of 1 in x. The ratios $2\frac{1}{2}:1$, 5:2, $7\frac{1}{2}:3$, and 10:4 are all equivalent ratios.

The graph for this equation is a line through the origin. The slope of the line, defined to be the ratio of (change in y):(change in x), is the multiple y is of x.



Often the relationship between x and y leads to an increase in y as x increases, but not always. For example, if someone borrows \$10 from a friend every week, the value the borrower possesses is negatively related to the number of weeks. In this case, the graph is a line where the values of y decrease as the values of x increase.

Weeks (x)	0	1	2	3	4	5	6
Value (y)	0	−10	−20	−30	−40	−50	−60



➢ ***Relating slope to unit rate.*** As discussed earlier, the change in y-values is a constant multiple of the change in x-values for proportional relationships. For example, in the first graph above, where $y = 2\frac{1}{2}x$, a change of 1 in x leads to a change of $2\frac{1}{2}$ in y. That is because $2\frac{1}{2}(x + 1) - 2\frac{1}{2}x = 2\frac{1}{2}$ no matter what the value of x is. Similarly, a change of 2 in x leads to a change in y of $2 \times 2\frac{1}{2} = 5$, since $2\frac{1}{2}(x + 2) - 2\frac{1}{2}x = 5$. No matter which two values (x_1,y_1) and (x_2,y_2) are selected, the ratio of the change in y values to the change in x values is $(2\frac{1}{2}x_2 - 2\frac{1}{2}x_1):(x_2 - x_1) = 2\frac{1}{2}$, the value of y when $x = 1$, or the unit rate. The same happens even if the slope is negative.

➢ ***Comparing proportional relationships.*** All proportional relationships have the form $y = mx$, so in some ways they are all similar. The differences, of course, are related to the value of m. High positive or low negative values of m lead to steeper lines than lower positive values or higher negative values of m, assuming the same scale on the axes. All of the lines pass through the origin.

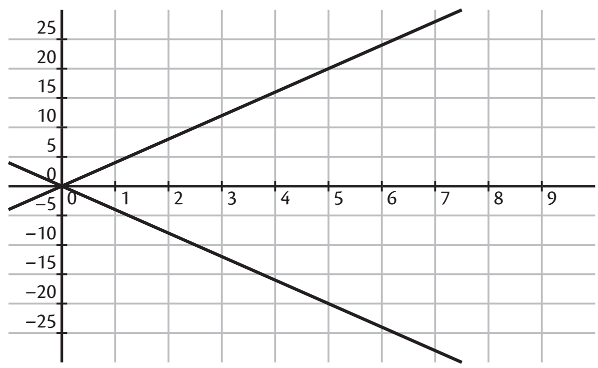
Good Questions to Ask

- ***Ask:*** Describe a real-life situation where two values are related proportionally. Remember that this means that one variable is a constant multiple of the other.

- ***Ask:*** How does the table of values below show a unit rate (maybe in more than one way)? What does that unit rate tell you about the graph of the line for this relationship?

x	0	1	2	3	4	5	6	7
y	0	−4	−8	−12	−16	−20	−24	−28

- ***Ask:*** One proportional relationship leads to much greater y-values for the same x-value than another. What could the two equations be? How would their graphs compare?
- ***Ask:*** What might the equations be for the two lines graphed below? Why did you choose those equations? [This is an example of the mathematical practice standard of constructing viable arguments.]



Using Proportional Reasoning in Solving Equations

Expressions and Equations	**CCSSM 8.EE**
Analyze and solve linear equations and pairs of simultaneous linear equations.	
7. Solve linear equations in one variable. a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers). b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.	

IMPORTANT UNDERLYING IDEAS

➢ ***Multiplying and dividing both sides of an equation as proportional thinking.*** Even when students are in earlier grades, they learn that they can multiply both sides of an equation or divide both sides of an equation by the same amount without affecting the equality. The notion is that if A = B, then 2 copies of A = 2 copies of B or *n* copies of A = *n* copies of B. Another way to think of this is that if A = B, then the ratio A:B = 1:1, so the ratio of 2A:2B or 3A:3B, etc. is equivalent.

Similarly, if both equations are divided by a constant, the ratios remain equivalent.

Good Questions to Ask

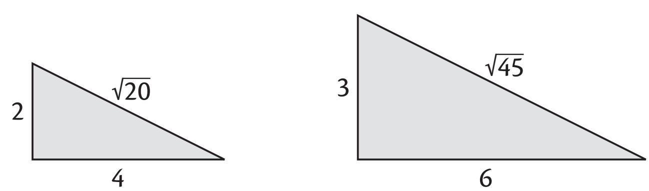
- ***Ask:*** Suppose $2x = x + 3$. What do you know about the ratio $2x:(x + 3)$?
- ***Ask:*** Use the concept of equal ratios to explain why, if $3x - 12 = 2\frac{1}{2}$, then $6x - 24 = 5$.
- ***Ask:*** If $2x = 3y$, what do you know about possible pairs of solutions?

Similarity

Geometry	CCSSM 8.G
Understand congruence and similarity using physical models, transparencies, or geometry software.	
3. Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating the concept of similarity to proportionality.*** Students learn that two shapes are similar if the linear dimensions of one are proportional to those of the other. For example, the triangles below are similar because any pair of matching length dimensions is related by the same ratio as any other pair's. In this case, the ratios of 2:3, 4:6 and $\sqrt{20}:\sqrt{45}$ are equal. So, too, are the ratios of corresponding side lengths within the triangles: 2:4 = 3:6, just as $4:\sqrt{20} = 6:\sqrt{45}$, or $2:\sqrt{20} = 3:\sqrt{45}$.

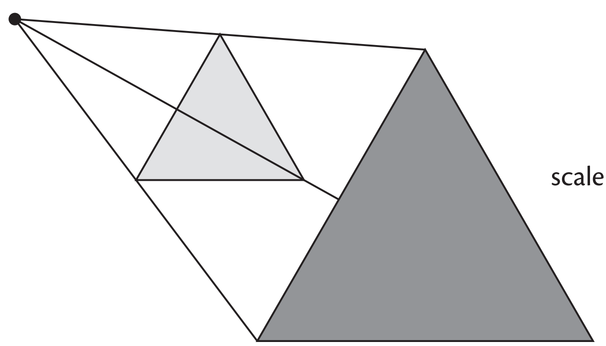


Students need to realize that the corresponding angle measures are also proportional, but the proportionality factor is 1; the angles are equal. It turns out that the ratio of the areas is also predictable; it is the square of the factor that relates the side lengths.

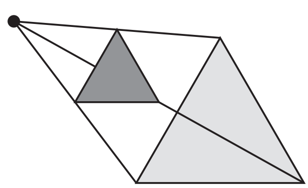
Although it is usually polygons that students consider when studying similarity, they might examine other shapes, such as circles. In fact, all circles are similar: any two circles' circumferences, diameters, or radii are proportional, using the ratio of the two radii as the scaling factor.

➢ ***Using dilations to build similarity.*** Students learn to perform dilations in 8th grade. A dilation requires a dilation center as well as a scale factor, either positive or negative. The transformation is performed by attaching each vertex of the shape being dilated to the dilation center and extending or contracting the lengths of the segments connecting the dilation center to the vertices by the given scale factor.

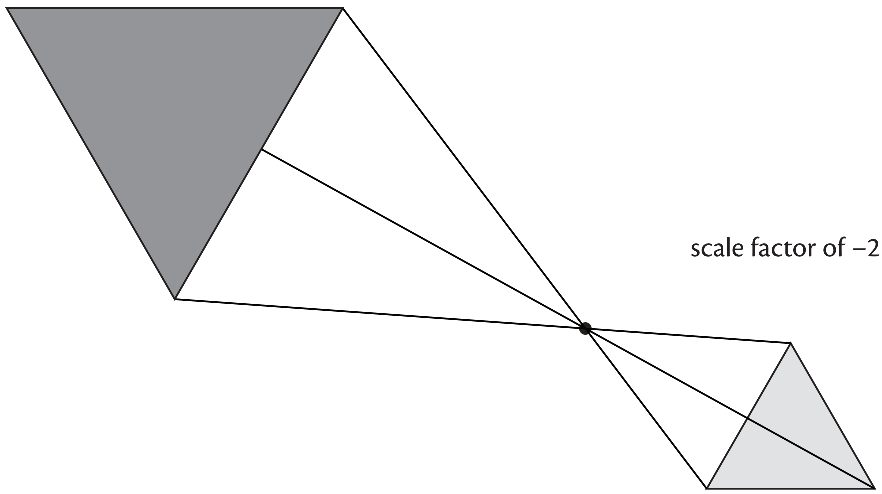
Below and on the next page are dilations using scale factors of 2, of $\frac{1}{2}$, and of –2. In each case, the original shape is colored light gray and the final shape dark gray. A dilation using a positive fraction less than 1 reduces the size of a shape. A dilation using a negative number is performed by creating segment lengths on the opposite side of the dilation center.



scale factor of 2



scale factor of $\frac{1}{2}$



Good Questions to Ask

- ***Ask:*** A dilation is performed to enlarge a figure's side lengths using a scale factor of 4.5. Draw a picture showing such a dilation, marking the dilation center, as well as the side lengths of the original and final figure.
- ***Ask:*** A right triangle has side lengths of 3-4-5. A similar right triangle has one side length of 10. What could the three side lengths of the similar triangle be? Think of all the possibilities. [This is an example of the mathematical practice standard of reasoning abstractly and quantitatively.]
- ***Ask:*** You are testing whether two rectangles are similar. Explain why you need to know both the length and width of each. If you do, what options do you have for testing similarity?
- ***Ask:*** How could you fill in the blank to make this statement true?

Any _____ is similar to any other _____.

Volume Ratios

Geometry	CCSSM 8.G
Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.	
9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.	

IMPORTANT UNDERLYING IDEAS

➢ ***Relating volumes to heights and base areas proportionally.*** When using volume formulas, students should become aware that there can be a proportional relationship between the heights of certain sets of cones or cylinders and their volumes. For any group of cones or cylinders with the same base, the volume is directly proportional to the height; as the height is multiplied by a value n, so is the volume. Similarly, for any group of cones or cylinders with the same height, the volume is directly proportional to the area of the base; as the area of the base is multiplied by a value n, so is the volume.

Students can look at this concept either algebraically, numerically using tables of values, or graphically. For example, consider cylinders or cones with a circular base of radius 10. The equations relating volume to height for these graphs are

$$V = 100\pi h \text{ (for a cylinder) or } V = \frac{100\pi h}{3} \text{ (for a cone).}$$

In each case, it is clear that V is a constant multiple of h. Or consider cylinders or cones with a height of 10. The equations relating volume to height for these graphs are

$$V = 10A \text{ (for a cylinder) or } V = \frac{10A}{3} \text{ (for a cone).}$$

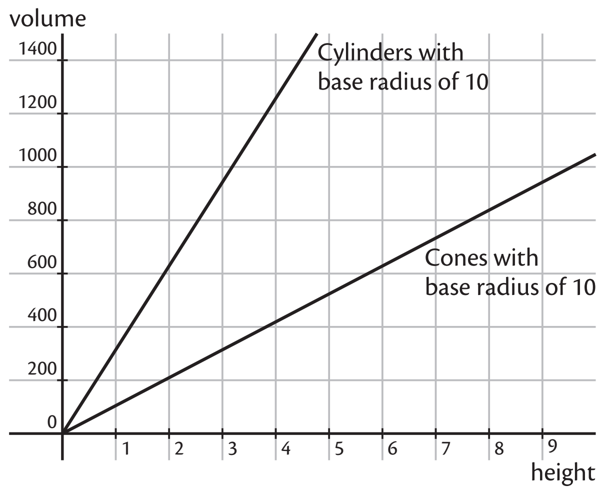
Although the radius of the base and volume are not directly proportional, the area of the base and volume are.

Looking at tables of values for these four situations, students would see that V is a constant multiple of either h or A, depending on the situation.

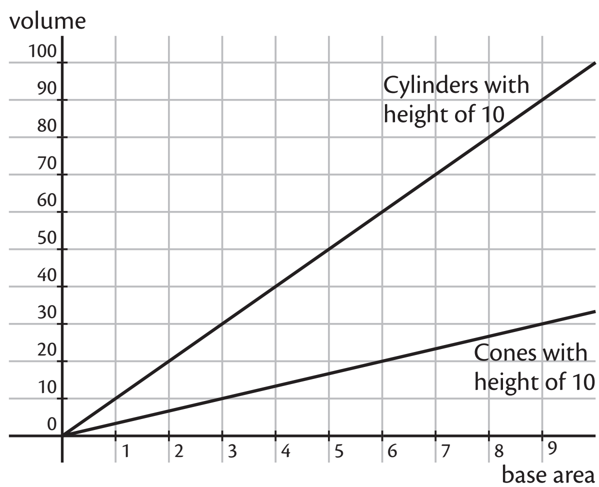
Cylinders with a radius of 10:					
Height (h)	0	1	2	3	4
Volume (V)	0	100π	200π	300π	400π

Cones with a radius of 10:					
Height (h)	0	1	2	3	4
Volume (V)	0	$\frac{100\pi}{3}$	$\frac{200\pi}{3}$	$\frac{300\pi}{3}$	$\frac{400\pi}{3}$
Cylinders with a height of 10:					
Area of base (A)	0	1	2	3	4
Volume (V)	0	10	20	30	40
Cones with a height of 10:					
Area of base (A)	0	1	2	3	4
Volume (V)	0	$\frac{10}{3}$	$\frac{20}{3}$	$\frac{30}{3}$	$\frac{40}{3}$

The graphs of the equations relating volume to height when the base has a radius of 10 cm are shown here:



The graphs of the equations relating volume to base area when the height is 10 cm are shown here:



Good Questions to Ask

- ***Ask:*** Why is the volume of a cone not proportional to the radius of the base? What is it proportional to?
- ***Ask:*** The equation $y = 20x$ relates the volume of a cylinder to another measurement of the cylinder. What might you know about the cylinders? What might be related to the volume? [This is an example of the mathematical practice standard of constructing viable arguments.]
- ***Ask:*** How are the volumes of a cylinder and cone with the same height and same base related? Are those volumes proportional? Explain.

Data Relationships

Statistics and Probability	CCSSM 8.SP
Investigate patterns of association in bivariate data.	
2. Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line. 3. Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.	

IMPORTANT UNDERLYING IDEAS

➢ ***Lines of good fit.*** Often when information relating one variable to another is available at only certain data values, we try to determine a linear relationship that would include or be close to those data points. In this way, values as yet unknown can be predicted.

For example, if, for a particular experiment, several of the data points were as shown in the chart below, we might decide that the y values are pretty close to what they would be were $y = 3x - 7$. Then we could use that equation to predict values for y that relate to values of x other than the given ones.

x	1	2	3	4	5
y	–4	0	3	5	7

In this particular case, y is not proportional to x, but the variable $q = y - 7$ is proportional to x. The slope of the line tells us the multiple that q is of x (i.e., the proportionality constant), and the intercept of the line tells us how y must be changed to obtain the variable that actually is proportional to x.

Good Questions to Ask

- ***Ask:*** Suppose the line $y = 2x + 8$ is a line of good fit for a set of data. Is y proportional to x? Why or why not? What variable is proportional to x?
- ***Ask:*** A line of good fit has a very steep slope. What does that tell you about the proportionality constant relating y to x if they are proportional?
- ***Ask:*** A line of good fit has a negative slope. What does that tell you about the proportionality constant relating y to x if they are proportional?

Summary

There are many opportunities in 8th grade to encourage students to extend and apply their proportional reasoning. These includes work with exponent laws, work with solving and graphing linear equations (including lines of good fit), work with similar shapes and dilations, and work with measurements of certain groups of shapes.

CONCLUSION

I HOPE that this resource provides some valuable mathematical background not only for classroom teachers in the middle grades, as they conduct formal instruction in proportional reasoning, but also for teachers in earlier grades, as they work to build an appropriate foundation in proportional reasoning for their students. The focus in this work is less on specific teaching procedures than on ways to help students make sense of proportionality concepts.

The good questions provided are only examples of questions that might be posed in the classroom setting. What is significant about these questions is that most of them are designed to foster thinking and evoke the mathematical practice standards that are critical in today's math classroom.

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Index

INDEX OF SUBJECTS AND CITED AUTHORS

INDEX OF COMMON CORE STANDARDS IN MATHEMATICS

STANDARDS FOR MATHEMATICAL CONTENT

Counting and Cardinality

Expressions and Equations

Geometry

Measurement and Data

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MARIAN SMALL is the former dean of education at the University of New Brunswick. She speaks regularly about asking better questions in K–12 mathematics.

She has been an author on many mathematics text series at both the elementary and the secondary levels. She has served on the author team for the National Council of Teachers of Mathematics (NCTM) Navigation series (pre-K–2), as the NCTM representative on the Mathcounts question writing committee for middle school mathematics competitions throughout the United States, and as a member of the editorial panel for the NCTM 2011 yearbook on motivation and disposition.

Dr. Small is probably best known for her books *Good Questions: Great Ways to Differentiate Mathematics Instruction* and *More Good Questions: Great Ways to Differentiate Secondary Mathematics Instruction* (with Amy Lin). *Eyes on Math: A Visual Approach to Teaching Math Concepts* was published in 2013, as was *Uncomplicating Fractions to Meet Common Core Standards in Math, K–7*. In 2014, she authored *Uncomplicating Algebra to Meet Common Core Standards in Math, K–8*. She is also author of the first and second editions of a text for university preservice teachers and practicing teachers, *Making Math Meaningful to Canadian Students: K–8*, as well as the professional resources *Big Ideas from Dr. Small: Grades 4–8*; *Big Ideas from Dr. Small: Grades K–3*; and *Leaps and Bounds toward Math Understanding: Grades 3–4*, *Grades 5–6*, and *Grades 7–8*, all published by Nelson Education Ltd.

She led the research resulting in the creation of maps describing student mathematical development in each of the five NCTM mathematical strands for the K–8 levels and has created the associated professional development program, PRIME.