STUDENT-CENTERED MATHEMATICS SERIES

Teaching Student-Centered Mathematics

THIRD EDITION

VOLUME III
GRADES 6–8

Developmentally Appropriate Instruction for Grades 6–8

Pearson

John A. Van de Walle • Jennifer M. Bay-Williams
LouAnn H. Lovin • Karen S. Karp

Volume III
Student-Centered Mathematics Series

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Developmentally Appropriate Instruction for Grades 6–8

Third Edition

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About the Authors

The late **John A. Van de Walle** was a professor emeritus at Virginia Commonwealth University. He was a mathematics education consultant who regularly gave professional development workshops for K–8 teachers in the United States and Canada. He visited and taught in elementary school classrooms and worked with teachers to implement student-centered math lessons. He coauthored the Scott Foresman-Addison Wesley Mathematics K–6 series and contributed to the Pearson School mathematics program, enVisionMATH. In addition, he wrote numerous chapters and articles for the National Council of Teachers of Mathematics (NCTM) books and journals and was very active in NCTM, including serving on the Board of Directors, as the chair of the Educational Materials Committee, and as a frequent speaker at national and regional meetings.

Jennifer M. Bay-Williams is a professor of mathematics education at the University of Louisville (Kentucky). Jennifer frequently offers professional development about effective mathematics teaching to K–12 teachers and leaders. She has coauthored numerous books, including *On the Money: Math Activities to Build Financial Literacy*; *Mathematics Coaching: Resources and Tools for Coaches and Leaders, K–12; Developing Essential Understanding of Addition and Subtraction for Teaching Mathematics in Pre-K–Grade 2*; *Math and Literature: Grades 6–8;* and *Navigating through Connections in Grades 6–8*. Additionally, she has written dozens of articles on teaching and learning in NCTM journals. Jennifer serves on the NCTM Board of Directors, and has served on the TODOS: Equity for All Board, and president of the Association of Mathematics Teacher Educators (AMTE). Jennifer taught elementary, middle, and high school in Missouri and in Peru, and continues to work in classrooms at all levels with students and with teachers.

LouAnn H. Lovin is a professor of mathematics education at James Madison University (Virginia). She coauthored the first edition of the *Teaching Student-Centered Mathematics* Professional Development Series with John A. Van de Walle as well as *Teaching Mathematics Meaningfully: Solutions for Reaching Struggling Learners* (2nd ed.) with David Allsopp and Sarah Vaningen. LouAnn taught mathematics to middle and high school students before transitioning to preK–grade 8. For almost twenty years, she has worked in preK through grade 8 classrooms and engaged with teachers in professional development as they implement a student-centered approach to teaching mathematics. She has published articles in *Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and *Teaching Exceptional Children* and has served on NCTM's Educational Materials Committee. LouAnn's research on teachers' mathematical knowledge for teaching has focused most recently on the developmental nature of prospective teachers' fraction knowledge.

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Preface

All students can learn mathematics with understanding! We believe that teachers can and must create learning environments in which students have this experience. Effective mathematics instruction involves posing worthwhile tasks that will engage students in the mathematics they are expected to learn. Then, by allowing students to interact with and productively struggle with the mathematics using *their* ideas and *their* strategies—a student-centered approach—students will develop a robust understanding of the mathematics. As they learn to see the connections among mathematical topics and to their world, students will value mathematics and feel empowered to use it. The title of this book, *Teaching Student-Centered Mathematics: Developmentally Appropriate Instruction for Grades 6–8*, reflects this vision. Part 1 of this book is dedicated to addressing how to build a student-centered environment in which students can become mathematically proficient and Part 2 elaborates on how that environment can be realized across all content in the Grades 6–8 mathematics curriculum.

What Are Our Goals for the Student-Centered Mathematics Series?

Creating a classroom in which students design their solution pathways, engage in productive struggle, and connect mathematical ideas is complex. Questions arise, such as, "How do I get students to wrestle with problems if they just want me to show them how to do it? What kinds of tasks lend themselves to this type of engagement? Where can I learn the mathematics content I need in order to be able to teach in this way?" With these and other questions firmly in mind, we have three main objectives for the third edition of this series:

1. Illustrate what it means to teach mathematics using a student-centered, problem-based approach.
2. Serve as a go-to reference for all of the mathematics content suggested for grades 6-8 as recommended in the *Common Core State Standards for Mathematics* (CCSSO, 2010) and in other standards used by various states, as well as research-based strategies that depict how students best learn this content.
3. Present a practical resource of robust, problem-based activities and tasks that can engage students in the mathematics that is important for them to learn.

These are also goals of *Elementary and Middle School Mathematics: Teaching Developmentally*, a comprehensive resource for teachers in grades K–8, which has been widely used in universities and in schools. There is overlap between the comprehensive K–8 book and this Student-Centered Math Series; however, we have adapted the Student-Centered Math Series to be more useful for a practicing classroom teacher by addressing the content for specific grade bands in more depth (with more activities!), removing content aimed at preservice teachers, and adding additional information more appropriate for practicing teachers. We hope you will find that this is a valuable resource for teaching and learning mathematics!

What's New to the Third Edition of the Student-Centered Mathematics Series?

The most significant change to the third edition is its availability as an Enhanced Pearson eText. Teachers can now take advantage of eText technology, easily accessing downloadable resources to support many of the math activities offered in the text and linking to videos that demonstrate how to teach certain math concepts. Another big change is that the third edition appears in four-color so pedagogical features are more easily found and studied. We are hopeful too that the addition of color helps to enhance and clarify the ideas we have intended to convey. We have also included some new features that we briefly describe below. (More detailed information about the new features can be found in the following section.) We then highlight the most substantial changes we have made to specific chapters to reflect the changing landscape of mathematics education.

What Are the New eText Features?

Each volume in the Student-Centered Mathematics Series is also available as an Enhanced Pearson eText* with the following point-of-use features:

- *Downloadable Activity Pages and Blackline Masters.* Hyperlinks provide access to ready-to-use teaching resources including Activity Pages and Blackline Masters to support students' engagement in a large number of math activities.
- *Videos.* Links to videos allow teachers to observe an interview with a student, watch an idea play out in a classroom, or listen to a more in-depth description of an important math concept.
- *Activities Correspond to CCSS-M.* The numerous problem-based tasks presented in activity boxes are now linked to the appropriate *Common Core State Standards for Mathematics*.
- *Immediate Access to Expanded Lessons.* A custom basket located on the navigation bar links teachers to full and Expanded Lessons and include the Blackline Masters or Activity Pages if needed to execute each lesson. Expanded Lessons are referenced at point-of-use in numbered math activities throughout the eText.

What's New in Part 1?

Part 1 consists of seven chapters that focus on important "hot" topics that address ideas for creating a classroom environment in which all students can succeed. These chapters are, by design, shorter in length than the content chapters in Part 2, but are full of effective strategies and ideas. The intent is that these chapters can be used in professional development workshops, book study, or professional learning community (PLC) discussions. Changes to Part 1 chapters include:

Chapter 1: Setting a Vision for Learning High-Quality Mathematics. Changes to this chapter include a new table that relates CCSS-M's mathematical practices (CCSSO, 2010) to NCTM's process standards (2000), clarification about the difference between modeling mathematics and modeling with mathematics, and an additional emphasis on the characteristics of productive classrooms that promote student understanding.

Chapter 2: Teaching Mathematics through Problem Solving. The eight mathematics teaching practices from *Principles to Actions* (NCTM, 2014) have been added! Several new sections were added: evaluating and adapting tasks to increase their potential for learning, growth

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*These features are only available in the Pearson eText, available exclusively from www.pearsonhighered.com/etextbooks or by ordering the Pearson eText plus Vol III Book Package (ISBN: 0134090691) or the Pearson eText Access Code Card (ISBN: 0134556399).

versus fixed mindsets (connected to productive struggle and learning from mistakes); and effective aspects of questioning. Finally, more detail pertaining to the three-phases (before, during, and after) is provided.

Chapter 3: Creating Assessments for Learning. Supported by the recent position statement from professional organizations (NCSM and AMTE) about assessment for learning (AFL), this chapter was revised to be more explicit about how to collect evidence from students on their progress, interpret that evidence, make informed decisions about the next instructional steps and provide actionable feedback to students. There is also an expanded section on using writing to learn mathematics.

Chapter 4: Differentiating Instruction. This chapter was revised to better highlight differentiated tasks for whole-classroom instruction. We also added new team-building activities to enhance your students' interactions with each other when working in groups.

Chapter 5: Teaching Culturally and Linguistically Diverse Students. In this chapter, significant revisions were made to reflect research in the field (twenty-two new references). Among these changes were increased attention to Culturally Responsive Mathematics Instruction (CRMI) and an expanded section on nurturing students mathematical identities.

Chapter 6: Planning, Teaching, and Assessing Students with Exceptionalities. Several new tools were added to this chapter including a printable set of cards, each with a Strategy for Making Math Accessible for learners who struggle. This tool can be used when planning core instruction modifications or interventions for students with special needs. There is also a Mathematics Integration Plan Template to support planning for gifted students or students with a high interest in exploring mathematical topics in relation to other subject areas or perspectives.

Chapter 7: Collaborating with Families and Other Stakeholders. This chapter was significantly revised in order to focus on advocacy across stakeholders. This included increased attention to communicating about CCSS Mathematics. Finally, the homework section was expanded, including new activities and games for families.

What's New to Part 2?

In addition to the changes listed above that are true across all three volumes, there are many changes specific to Volume III, Part 2 and the specific content for middle school learners. Across all content chapters, more activities and many activity pages were added, in particular to address hard-to-teach topics. The list below highlights significant new content in each chapter:

Chapter 8: Fraction Concepts and Computation. Major changes were added to this robust chapter. While much of this content is originally taught in elementary grades, many middle school students need ongoing support in fraction sense and fraction operations. The now thirty activities can serve as warm-up tasks, review, enrichment, or remediation support. All four operations have increased attention. Fraction multiplication, especially, has been broken into a learning trajectory of problem types.

Chapter 9: Decimal Concepts and Computation. This chapter includes more activities, new videos, new activity pages, expanded attention to common errors, which collectively provides a much stronger focus on decimal fluency as part of rational number fluency.

Chapter 10: The Number System. The order of operations has been completely revised to reflect current research. The integer sections have been altered to more strongly focus on positive and negative numbers (including more decimals and fractions), and more number line ideas were added.

Chapter 11: Ratios and Proportional Relationships. The idea of covariation, central to ratio and proportional reasoning, receives much more attention in this edition. Each of the strategies for solving proportional story problems has been revised, including more attention to rate and scale strategies, addition of tape diagrams as a strategy, and elaboration on double number line. These strategies, described in the CCSS-M, are organized from most intuitive, beginning strategies, to more abstract strategies, laying out a teaching progression for solving proportional problems.

Chapter 12: Algebraic Thinking: Expressions, Equations, and Functions. This critical chapter had major revisions. It is organized around three strands of algebraic thinking. We added a new section on integrating algebra across the curriculum. There is more attention and activities on covariation, inequalities, and connections between arithmetic and algebra.

Chapter 13: Developing Geometry Concepts. This chapter has much more about transformations and more on learning progressions and seven new activities on important middle-school CCSS-M geometry concepts.

Chapter 14: Exploring Measurement Concepts. In this chapter we increased attention to angle measures (new discussion and activities). We also revised discussion about area as it applies to the middle-level learner, in particular the area of circles has been enhanced and expanded. Finally, we expanded the section about spheres.

Chapter 15: Working with Data and Doing Statistics. Doing statistics begins with asking statistical questions! So, we expanded this section to include more ideas and strategies. We added more about dot plots and box plots and increased attention to critical ideas in statistics: shape of data, variability, and distribution.

Chapter 16: Investigating Concepts of Probability. In this chapter, there were many changes to the activities (for a net gain of 5), figures were changed to better support the text, and more attention was given to common misconceptions about chance. We also increased attention to the models emphasized in CCSS-M (dot plots, area representations, tree diagrams).

What Special Features Appear in Pearson's Student-Centered Mathematics Series?

Features Found in Parts 1 and 2

- *Teaching Tips.* These brief tips identify practical take-away ideas that can support the teaching and learning of specific chapter content being addressed. These might be an instructional suggestion, a particular point about language use, a common student misconception, or a suggestion about a resource.
- *Stop and Reflect.* Reflective thinking is the key to effective learning. This is true not only for our students but also for ourselves as we continue to learn more about effective mathematics teaching. Keep your eye out for these sections that ask you pause to solve a problem or reflect on some aspect of what you have read. These Stop and Reflect sections do not signal every important idea, but we have tried to place them where it seemed natural and helpful for you to slow down a bit and think deeply about an idea.

In addition, every chapter in Part 1 ends with a Stop and Reflect section. Use these for discussions in professional learning communities or for reflection on your own.

- ***New!*** *Downloadable Resources including Activity Pages and Expanded Lessons.* Many activities that previously required cards or recording sheets now include these as ready-to-use, downloadable pages. You will also find a variety of downloadable resources that support teaching activities such as formative assessment and team-building activities. You can access these downloadable pages by clicking the blue text in the eText at point of use.
- ***New!*** *Videos.* The book now includes a collection of videos that are positioned right when you need them—when a child's misconception during a diagnostic interview will reinforce a point, when a strategy needs a more in-depth description or when it helps to see a teacher carry out an idea in a classroom of children. When accessing the e-book, you can click on the link to see an idea in action. There is also a video of John Van de Walle sharing some of his insights on how to teach a mathematics topic through the perspective of a student-centered, problem-based approach.

Additional Features Found in Part 2

- ***New!*** *Appendix with the Mathematics Teaching Practices from NCTM Principles to Actions* (NCTM, 2014) eight teaching practices are provided in Appendix C. These describe the actions that teachers do to support student thinking and provide guidance on how to enact student-centered mathematics.
- ***New!*** *Blackline Masters Hyperlinked in Chapters.* Blackline Masters are used in some of the activities and Expanded Lessons. Look for the call outs for the hyperlinks embedded in the activities that alert you to the corresponding print-friendly PDF of the Blackline Master. In Appendix E, you will find a list of the Blackline Masters and a thumbnail version of each.
- ***New!*** *Activities Correspond to the CCSS-M.* Numerous problem-based tasks are presented in activity boxes that are connected to the appropriate *Common Core State Standards for Mathematics*. Additional ideas are described directly in the text or in the illustrations. They are designed to engage your children in *doing* mathematics (as described in Chapter 2).
- ***Expanded!*** Many activities include adaptation and accommodation suggestions for English language learners and children with special needs, as well as ways to incorporate technology. For easy reference, these activities have icons for each of these considerations. In Appendix D, you will find *Activities at a Glance*. This table lists all the named and numbered activities with a short statement about the mathematical content goal for each, the CCSS-M standard(s) and the page number where it can be located.

It is important that you see these activities as an integral part of the text that surrounds them. The activities are inserted as examples to support the development of the mathematics being discussed and how your children can be supported in learning that content. Therefore, we hope that you will not use any activity for instruction without reading carefully the full text in which it is embedded.

- ***New!*** *Downloadable Expanded Lessons.* In each chapter, one or more activities have been expanded it into a complete lesson plan, following the *before*, *during*, *after* structure described in Chapter 2 and are available by clicking on the link in the eText. These Expanded Lessons provide a model for converting an activity description into a full lesson that can engage children in developing a robust understanding of the related concept. In this new edition, all of the Expanded Lessons are now aligned with CCSSO grade-level recommendations and include adaptation suggestions for English language learners and children with disabilities. Many use the new Activity Pages or Blackline Masters.

- *Literature Connections.* Each Part 2 chapter concludes with early adolescent literature (fiction, non-fiction, poems, etc.) that provide engaging contexts for launching into mathematics.
- *Formative Assessment Notes.* Assessment should be an integral part of instruction. We embed ideas for what to listen and look for (assess) in different areas of content development. Formative Assessment Notes describe tasks, tools and strategies you might use to formatively assess your children's developing knowledge and understanding. These Formative Assessment Notes can also help improve your understanding about how to help your children through targeted instruction.
- *Technology Notes.* These notes provide practical information about how technology can be used to help your children learn the content in that section. Descriptions include open-source software, interactive applets, and other Web-based resources—all of which are free.
- *Standards for Mathematical Practice Notes.* Connections to the eight Standards of Mathematical Practice from the *Common Core State Standards* are highlighted in the margins. The location of the note indicates an example of the identified practice in the nearby text.
- *Common Core State Standards Appendices.* The *Common Core State Standards* outline eight Standards for Mathematical Practice (Appendix A) that help children develop and demonstrate a deep understanding of and capacity to do mathematics. We initially describe these practices in *Chapter 1* and highlight examples of the mathematical practices throughout the content chapters in Part 2 through margin notes. We used the *Common Core State Standards for Mathematics* (CCSSO, 2010) as a guide to determine the content emphasis in each volume of the series. Appendix B provides a list of the critical content areas for each grade level discussed in this volume.
- *Big Ideas.* Much of the research and literature espousing a developmental approach suggests that teachers plan their instruction around "big ideas" rather than isolated skills or concepts. At the beginning of each chapter, you will find a list of the key mathematical ideas associated with the chapter. These lists of learning targets can provide a snapshot of the mathematics you are teaching.

Acknowledgments

We would like to begin by acknowledging *you:* the reader, the teacher, the leader, and the advocate for your students. The strong commitment of teachers and teacher leaders to always strive to improve how we teach mathematics is the reason this book was written in the first place. And, because of ongoing input and feedback, we endeavored to revise this edition to meet your changing needs. We have received thoughtful input from many teachers and reviewers, and all of it has informed the development of this substantially revised third edition!

In preparing the third edition we benefited from the thoughtful input of the following educators who offered comments on the second edition or on the manuscript for the third: Shawn D. Broderick, Keene State College; Mia Carpenter, St. Louis Public Schools; Maureen Fitzsimmons, Mooresville Middle School. Their comments helped push our thinking on many important topics. Many specific suggestions offered by these reviewers found their way into the pages of this book. We offer our sincere appreciation to these esteemed educators for their valued suggestions and constructive feedback.

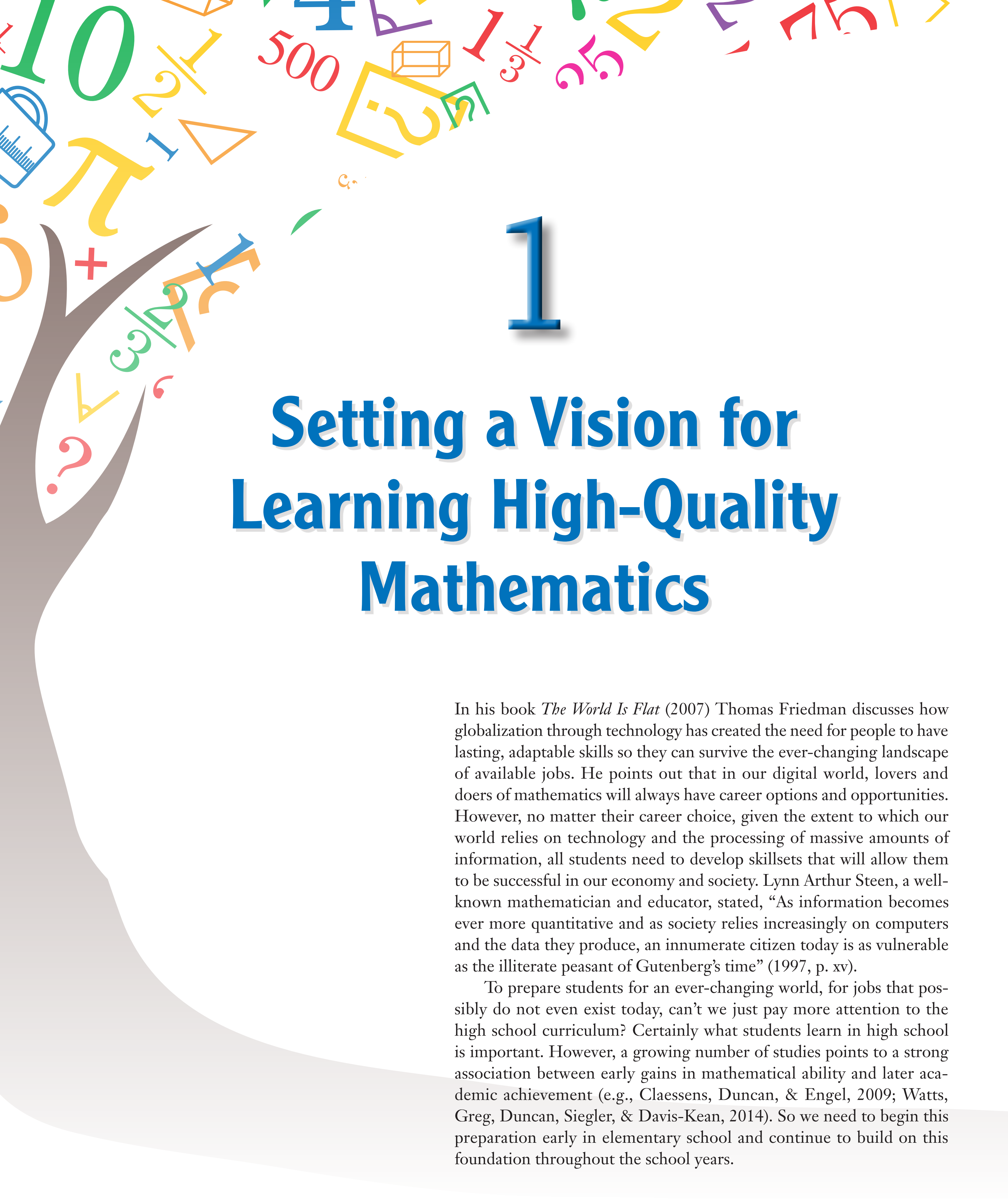
As we reviewed standards, research, and teaching articles; visited classrooms; and collected students' work samples, we were continually reminded of the amazing commitment to effective mathematics teaching and learning. From the mathematics educators and mathematicians working on standards documents, to the teachers who facilitate discussions about mathematics in preK–grade 8 classrooms and then share the results with others, we are grateful for the broad and heartfelt commitment to mathematics education for *all* students on the part of so many educators—particularly the teachers with whom we have worked in recent years.

We also want to acknowledge the strong support of our editorial team throughout the process, from the first discussions about what the third edition might include, through the tedious editing at later stages in the development. Without their support, the final product would not be the quality resource we hope you find it to be. Specifically, we thank Meredith Fossil for helping us envision our work, Linda Bishop for seeing this vision through, and both of them for their words of encouragement and wisdom. Working on three volumes of a book simultaneously is quite an undertaking! We are also truly grateful for Miryam Chandler at Pearson and Jason Hammond and the team at SPi Global who helped us wade through the permissions process and the production and editing of our new edition and eText.

Even with the support of so many, researching and writing takes time. Simple words cannot express the gratitude we have to our families for their support, patience, and contributions to the production of these books. Briefly we recognize them by name here: Jennifer thanks her husband, Mitch, and her children, MacKenna and Nicolas. LouAnn thanks her husband, Ramsey, and her two sons, Nathan and Jacob. Karen thanks her husband, Bob Ronau, and her children and grandchildren, Matthew, Tammy, Josh, Misty, Matt, Christine, Jeff, Pamela, Jessica, Zane, Madeline, Jack, and Emma.

The origin of this book began many years ago with the development of *Elementary and Middle School Mathematics: Teaching Developmentally* by John A. Van de Walle. What began as a methods book for preservice teachers spread enthusiastically throughout the teaching community because it offered content support, activities, and up-to-date best practices for teaching mathematics. The three-volume series was developed as a way to focus on and expand the specific grade-level topics. John was adamant that *all* children can learn to reason and make sense of mathematics. We acknowledge his commitment and his significant contributions to the field of mathematics education. His ideas and enduring vision continue to inspire the work you see in this new edition.

The response to the second edition has been amazing. We hope the third edition will be received with as much interest and enthusiasm as the second and continue to be a valuable support to your mathematics teaching and your children's learning.



1

Setting a Vision for Learning High-Quality Mathematics

In his book *The World Is Flat* (2007) Thomas Friedman discusses how globalization through technology has created the need for people to have lasting, adaptable skills so they can survive the ever-changing landscape of available jobs. He points out that in our digital world, lovers and doers of mathematics will always have career options and opportunities. However, no matter their career choice, given the extent to which our world relies on technology and the processing of massive amounts of information, all students need to develop skillsets that will allow them to be successful in our economy and society. Lynn Arthur Steen, a well-known mathematician and educator, stated, “As information becomes ever more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg’s time” (1997, p. xv).

To prepare students for an ever-changing world, for jobs that possibly do not even exist today, can’t we just pay more attention to the high school curriculum? Certainly what students learn in high school is important. However, a growing number of studies points to a strong association between early gains in mathematical ability and later academic achievement (e.g., Claessens, Duncan, & Engel, 2009; Watts, Greg, Duncan, Siegler, & Davis-Kean, 2014). So we need to begin this preparation early in elementary school and continue to build on this foundation throughout the school years.

Understanding and Doing Mathematics

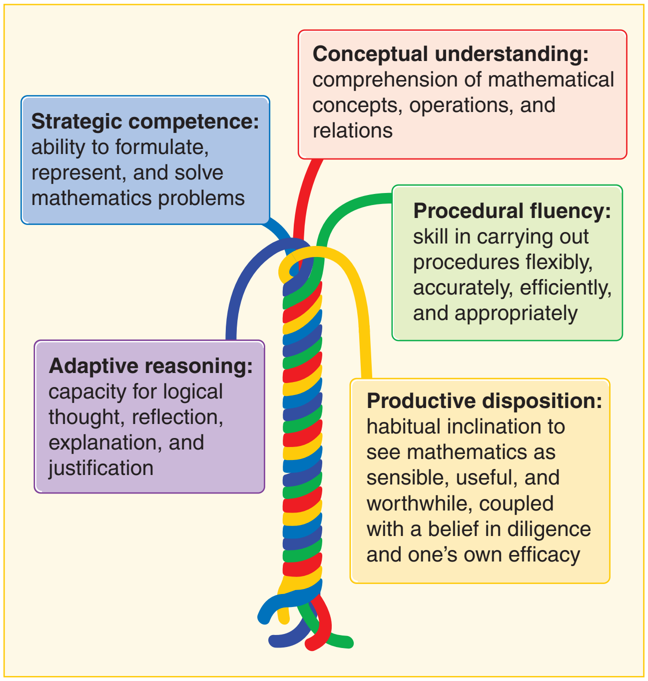
The changing world influences what should be taught in mathematics classrooms, even in grades 6–8. The dialogue on the best ways to prepare students to be successful in this changing world has involved mathematics educators, researchers, teachers, policymakers, and elected officials and has considered the many National Council of Teachers of Mathematics (NCTM) standards documents, international assessments, and research.

One of the influential documents that added to this dialogue is *Adding It Up* (National Research Council (NRC), 2001). Based on a review of the research on how students learn mathematics, this document identified the following five strands of mathematical proficiency that are seen as indicators of someone who understands (and can do) mathematics.

- *Conceptual understanding:* Having a robust web of connections and relationships within and between ideas, interpretations, and representations of mathematical concepts.
- *Procedural fluency:* Being able to flexibly choose and accurately and efficiently perform an appropriate strategy for a particular problem. For more on fluency, go to the NCTM website and search for the 2014 Annual Meeting Webcast “President’s Session–Fluency . . . It’s More Than Fast and Accurate.”
- *Strategic competence:* Being able to make sense of, represent, and determine solutions to mathematical problems.
- *Adaptive reasoning:* Being able to think about, explain, and justify one’s ideas using mathematically sensible reasons coupled with the ability to shift strategies when needed.
- *Productive disposition:* Having an ingrained awareness that mathematics makes sense and is useful, valuable, and rewarding along with the belief that one is capable of being successful in learning and doing mathematics through hard work and perseverance.

Figure 1.1

Interrelated and intertwined strands of mathematical proficiency.



Source: From “The Strands of Mathematical Proficiency” in *Adding It Up: Helping Children Learn Mathematics*. Edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell. Published by proceedings of the National Academy of Sciences, © 2001.

Figure 1.1 illustrates the interrelated and interwoven nature of the strands of mathematical proficiency: As one strand develops it builds on and builds up other strands, resulting in a strengthened whole. As an example, consider the ineffective practice of teaching procedures in the absence of conceptual understanding. Often this approach yields a lack of retention and increased errors, rigid approaches, and inefficient strategy use (**watch a related video at https://www.youtube.com/watch?v=FVKtQwARe6c**). When students are in classrooms where these strands of proficiency are allowed to develop together, they are able to build a stronger understanding of both mathematical concepts and procedures.

Numerous other reports and standards were developed as part of the effort to improve mathematics teaching and learning and prepare students for the ever-changing world. Among these are the *Common Core State Standards* (CCSS-M) (CCSSO, 2010) and other state standards that recognize the need for coherent and rigorous standards that promote college- and career-readiness.

In particular, the CCSS-M articulates an overview of critical areas for each grade from K-8 to provide a coherent curriculum built around big mathematical ideas. At this time, more than 40 states; Washington, D.C.; four territories; and

Department of Defense Schools have adopted the CCSS-M. A few states chose to not adopt these standards from the start and others are still deciding their level of participation or re-evaluating their own standards against CCSS-M. Nonetheless, this represents the largest shift of mathematics content in the United States in more than 100 years.

This effort to develop standards that promote college- and career-readiness has resulted in attention to the *processes* of *doing* mathematics, not just the *content*. Notably, NCTM (2000) identifies the process standards of problem solving, reasoning and proof, representation, communication, and connections as ways in which students acquire and use mathematical knowledge. Students engaged in the process of *problem solving* build mathematical knowledge and understanding by grappling with and solving genuine problems as opposed to completing routine exercises. They use *reasoning and proof* to make sense of mathematical tasks and concepts and to develop, justify, and evaluate mathematical arguments and solutions. Students create and use *representations* (e.g., diagrams, graphs, symbols, and manipulatives) to reason through problems. They also engage in *communication* as they explain their ideas and reasoning verbally, in writing, and through representations. Students develop and use *connections* between mathematical ideas as they learn new mathematical concepts and procedures. They also build *connections* between mathematics and other disciplines by applying mathematics to real-world situations. The process standards should not be regarded as separate content or strands in the mathematics curriculum; rather they are integral components of all mathematics teaching and learning. By engaging in these processes, students *learn* mathematics by *doing* mathematics.

The CCSS-M also includes the Standards for Mathematical Practice (CCSSO, 2010), which are ways in which students can develop and demonstrate a deep understanding of and capacity to do mathematics (see Appendix A). Whether your state has adopted the CCSS, the eight Standards of Mathematical Practice are worthy of attention. These mathematical practices are based on the underlying frameworks of the NCTM process standards and the components of mathematical proficiency identified by the National Research Council's document *Adding It Up* (NRC, 2001). Like the NCTM process standards, these practices are not separate, but integral to all mathematics teaching and learning. Here we provide a brief discussion about each mathematical practice.

1. *Make sense of problems and persevere in solving them.* To make sense of problems, students need to learn how to analyze the given information, parameters, and relationships in a problem so that they can understand the situation and identify possible ways to solve it. One way to help students analyze problems is to have them create proportional drawings to make sense of the quantities and relationships involved. Once students learn various strategies for making sense of problems, encourage them to remain committed to solving them. As they learn to monitor and assess their progress and change course as needed, they will solve the problems they set out to solve!

2. *Reason abstractly and quantitatively.* This practice involves students reasoning with quantities and their relationships in problem situations. You can support students' development of this practice by helping them create representations that correspond to the meanings of the quantities and the units involved. Also, a significant aspect of this practice is to be able to represent and manipulate the situation symbolically. Encourage students to find connections between the abstract symbols and the representation that illustrates the quantities, their relationships, and related operations. For example, suppose students are generalizing a situation in which they are trying to determine what their earnings at the local ice cream store will be if they earn \$8.00 for each hour they work but spend \$2.50 every two hours for their own ice cream cones. They may represent the relationship symbolically as $y = 8x - 2.5(\frac{x}{2})$ or as $y = (8 - 1.25)x$ or as $y = 6.75x$.

Ultimately, students should be able to reason how these equations are equal and relate the equations to the situation.

3. *Construct viable arguments and critique the reasoning of others.* This practice emphasizes the importance of students using mathematical reasoning as the basis for justifying their ideas and solutions, including being able to recognize and use counterexamples. Encourage students to examine each others' arguments to determine whether they make sense and to identify ways to clarify or improve the arguments. This practice emphasizes that mathematics is based on reasoning and should be examined in a community—not carried out in isolation. Tips for supporting students as they learn to justify their ideas can be found in Chapter 2.
4. *Model with mathematics.* This practice encourages students to use the mathematics they know to describe, explain, and solve problems from a real-world context, and to be able to represent them symbolically (i.e., the equation serves as a model of the situation). The equations given above for the ice cream shop are models for describing the students' earnings. The equation (model) can then be used to predict and find earnings for any number of hours worked. Be sure to encourage students to determine whether their mathematical results make sense in the context of the given situation. Note that this practice is different from *modeling mathematics*, which involves using representations to illustrate mathematical ideas (e.g., a graph to show a linear function).
5. *Use appropriate tools strategically.* Students should become familiar with a variety of visuals and tools that can be used to solve a problem and they should learn to choose which ones are most appropriate for a given situation. For example, suppose students have used the following tools to investigate probability: coins, spinners, number cubes, and computerized simulations. If students are asked to create a simulation with two outcomes, one outcome twice as likely as the second outcome, they should consider which of these tools can best support their simulation. If the number cubes are six-sided cubes, the students might define one outcome as rolling a 1, 2, 3, or 4 and the second outcome as rolling a 5 or 6. However, a coin, because it is two-sided, would not be an appropriate tool for this particular investigation because the outcomes could not be modified to model the given situation.

Teaching Tip

Research suggests that students, in particular girls, may tend to continue to use the same tools because they feel comfortable with the tools and are afraid to take risks (Ambrose, 2002). Look for students who tend to use the same tool or strategy every time they work on tasks. Encourage all students to take risks and try new tools and strategies.

6. *Attend to precision.* In communicating ideas to others, it is imperative that students learn to be explicit about their reasoning. For example, they need to be clear about the meanings of the operations and symbols they use, to indicate the units involved in a problem, and to clearly label the diagrams they provide in their explanations. As students share their ideas, make this expectation clear and ask clarifying questions that help make the details of their reasoning more apparent. Teachers can further encourage students' attention to precision by introducing, highlighting, and encouraging the use of accurate mathematical terminology in explanations and diagrams.
7. *Look for and make use of structure.* Students who look for and recognize a pattern or structure can experience a shift in their perspective or understanding. Therefore, set the expectation that students will look for patterns and structure, and help them reflect on their significance. For example, when students begin to write rational numbers in decimal form, they learn that the decimal either terminates or repeats. Look for opportunities to help students notice that the denominator of any simplified rational number whose decimal form terminates can be rewritten as a power of ten—which provides insight into why the decimal terminates.

8. *Look for and express regularity in repeated reasoning.* Encourage students to step back and reflect on any regularity that occurs to help them develop a general idea or method to identify shortcuts. For example, as students begin adding signed numbers, they will encounter situations such as $5 - 3$ and $5 + (-3)$. Over time, help them reflect on the results of these situations. Eventually they should be able to express that subtracting a positive number is equivalent to adding the opposite (or negative) of the number.

Note that the process standards are embedded in the mathematical practices. Table 1.1 shows one way to think about the relationship between the process standards and the eight mathematical practices. For example, ways to engage students in reasoning and proof involve the mathematical practices of reasoning abstractly and quantitatively, constructing viable arguments and critiquing others' reasoning, modeling with mathematics, and looking for and making use of structure. Bleiler, Baxter, Stephens, and Barlow (2015) provide additional ideas to help teachers further their understanding of the eight mathematical practices. Students who learn to use these processes and practices of *doing* mathematics have a greater chance of becoming mathematically proficient.

How Do Students Learn?

As noted in NCTM's (2014) *Principles to Actions* document, research in mathematics education as well as in cognitive science supports the notion that learning is an active process in which each student, through personal experiences, interactions with others, and reflective thought, develops his or her own mathematical knowledge. The active nature required in developing the NCTM Process Standards and the CCSS-M Mathematical Practices is what makes them such powerful places for students to learn as they engage in doing mathematics.

Two research-based theories about learning, constructivism and sociocultural theory, provide us with specific insights into the active nature of the learning process. Although one theory focuses on the individual learner and the other emphasizes the social and cultural aspects of the classroom, these theories are not competing and are actually compatible (Norton & D'Ambrosio, 2008).

Table 1.1. Connections between NCTM's process standards and CCSS-M's mathematical practices.

	MP1	**MP2**	**MP3**	**MP4**	**MP5**	**MP6**	**MP7**	**MP8**
Problem Solving	🟪	🟪		🟪	🟪		🟪	🟪
Reasoning and Proof		🟪	🟪	🟪			🟪	
Communication			🟪			🟪		
Representation	🟪		🟪	🟪	🟪			
Connections				🟪			🟪	🟪

MP1: *Make sense of problems and persevere in solving them.*
MP2: *Reason abstractly and quantitatively.*
MP3: *Construct viable arguments and critique the reasoning of others.*
MP4: *Model with mathematics.*
MP5: *Use appropriate tools strategically.*
MP6: *Attend to precision.*
MP7: *Look for and make use of structure.*
MP8: *Look for and express regularity in repeated reasoning.*

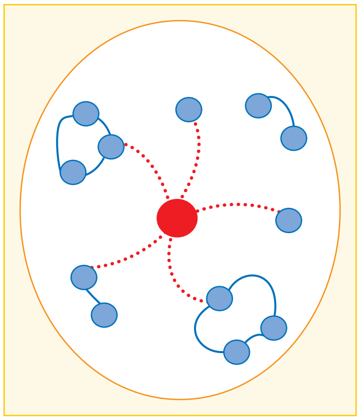
Constructivism

At the heart of constructivism is the notion that learners are not blank slates but rather creators (constructors) of their own learning (Piaget, 1976; von Glasersfeld, 1995). All people, all of the time, construct or give meaning to things they experience or think about. Whether you are listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain uses your existing knowledge to make sense of the new information.

Through reflective thought, people connect existing ideas to new information and in this way modify their existing knowledge to incorporate new ideas. Making these connections can happen in either of two ways—*assimilation* or *accommodation*. Assimilation occurs when a new concept "fits" with prior knowledge and the new information expands an existing mental network. Accommodation takes place when the new concept does not "fit" with the existing network, thus creating a cognitive conflict or state of confusion that causes what learning theorists call *disequilibrium*. As an example, consider what happens when students start learning about variables. They begin in elementary school by using variables as unknowns, as in $2 + ? = 5$ or $4 \times ? = 24$, in which their goal is to determine what the question mark represents. Consequently, some students come to see a variable as a placeholder or missing number. So, when students encounter equations such as $y = 4x + 5$, the variable does not represent a single missing number (assimilation), but rather many values. Students must adapt their mental image of what a variable means (accommodation). It is through the struggle to resolve the disequilibrium that the brain modifies or replaces the existing knowledge so that the new concept fits and makes sense, resulting in a revision of thought and a deepening of the learner's understanding.

Figure 1.2

How someone constructs a new idea.



For an illustration of what it means to construct an idea, consider Figure 1.2. The blue and red dots represent ideas, and the lines joining the ideas represent the logical connections or relationships that develop between ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas (blue dots) are used in the construction are connected to the new idea (red dot) because those are the ideas that give meaning to the new idea. The more existing ideas that are used to give meaning to the new one, the more connections will be made.

Each student's unique collection of ideas is connected in different ways. Some ideas are well understood and well formed (i.e., connected), and others are less so as they evolve. Students' experiences help them develop connections and ideas about whatever they are learning.

Sociocultural Theory

Like constructivism, sociocultural theory not only positions the learner as actively engaged in seeking meaning during the learning process, but it also suggests that the learner can be assisted by working with others who are "more knowledgeable." Sociocultural theory proposes that learners have their own zone of proximal development, which is a range of knowledge that may be out of reach for the individuals to learn on their own but is accessible if learners have the support of peers or more knowledgeable others (Vygotsky, 1978). For example, when students are learning about experimental probability, they do not necessarily recognize the significance of sample size. Teachers often have students collect data to explore the probability of two events, such as flipping two coins. Students may think that HH, TT, and HT (H = Heads; T = Tails) are equally likely and therefore each has a probability of $\frac{1}{3}$. A more knowledgeable person (a peer or teacher) will know that if students explore a large number of trials, the data will suggest that "HT" (i.e., one H and one T) actually has a 50 percent probability, and that creating a list of possible outcomes will help to demonstrate why the probabilities are $\frac{1}{4}$, $\frac{1}{4}$, *and* $\frac{1}{2}$, respectively. The more knowledgeable person can draw students' attention to this critical idea of how possible outcomes connect to probability.

The most effective learning for any given student occurs when classroom activities are within his or her zone of proximal development. Targeting that zone helps teachers provide students with the right amount of challenge while avoiding boredom on the one hand and anxiety on the other when the challenge is beyond the student's current capability. Consequently, classroom discussions based on students' own ideas and solutions are absolutely crucial to their learning (Wood & Turner-Vorbeck, 2001).

Teaching for Understanding

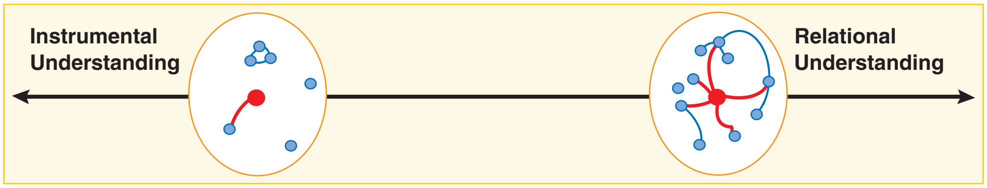
Teachers generally agree that teaching for understanding is a good thing. But what do we mean by *understanding*? Understanding is being able to flexibly think about and use a topic or concept. It goes beyond knowing; it is more than a collection of information, facts, or data. It is more than being able to follow steps in a procedure. One hallmark of mathematical understanding is a student's ability to justify why a given mathematical claim or answer is true or why a mathematical rule makes sense (CCSSO, 2010). Although students might *know* their basic multiplication facts and be able to give you quick answers to questions about these facts, they might not *understand* multiplication. They might not be able to justify how they know an answer is correct or provide an example of when it would make sense to use a particular basic fact. These tasks go beyond simply knowing mathematical facts and procedures. Understanding must be a primary goal for all of the mathematics you teach.

Understanding exists along a continuum (Figure 1.3) from an instrumental understanding—doing or knowing something without meaning—to a relational understanding—knowing what to do and why. These two terms were introduced by Richard Skemp in 1978 and continue to provide an important distinction about what is important for students to know about mathematics. Instrumental understanding, at the left end of the continuum, shows that ideas (e.g., concepts and procedures) are learned, but in isolation (or nearly so) to other ideas. Here you find ideas that have been memorized. Due to their isolation, poorly understood ideas are easily forgotten and are unlikely to be useful for constructing new ideas. At the right end of the continuum is relational understanding. Relational understanding means that each new concept or procedure (red dot) is not only learned, but is also connected to many existing ideas (blue dots), so there is a rich set of connections.

The common notion of quickly "covering the material" and moving on is problematic when trying to help children develop relational understanding. Relational understanding is an end goal—that is, it is developed over time by incorporating active learning through the process standards and mathematical practices and striving toward mathematical proficiency. Therefore, relational understanding must be a goal for both daily and long-term instruction.

Figure 1.3

Continuum of understanding.



Teaching for Relational Understanding

To explore the notion of understanding further, let's look into a learner-centered sixth-grade classroom. In learner-centered classrooms, teachers begin *where the students are*—with *the students'* ideas. Students are allowed to solve problems or to approach tasks in ways that make sense to them. They develop a robust understanding of mathematics because they are the ones who explain, provide evidence or justification, find or create examples, generalize, analyze, make predictions, apply concepts, represent ideas in different ways, and articulate connections or relationships between the given topic and other ideas.

In this particular sixth-grade classroom, the students are going to explore division of fractions by fractions. They have recently investigated multiplication of fractions and division of fractions by whole numbers and have used real contexts, manipulatives, and diagrams to make sense of the operations. Their work with multiplication of fractions has emphasized identifying the whole that is being used, an important idea in working with fractions and fraction computation. They also have revisited the different meanings of division: sharing or partitive division, and repeated subtraction or measurement division. The students have had previous experiences dividing fractions by whole numbers and have not been taught the standard algorithm for division of fractions.

The teacher sets the following instructional objectives for the students:

1. Solve word problems involving the division of fractions by fractions by using diagrams.
2. Compute and interpret the quotients of fractions.

The lesson begins with a task that is designed to set the stage for the main part of the lesson. As is often the case, this class begins with a story problem to provide context and relevance to the mathematics. The teacher displays this problem on the board:

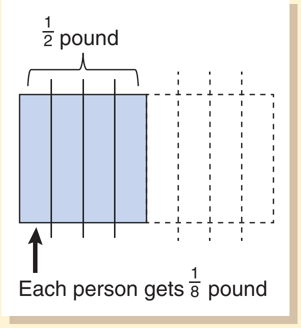
How much chocolate will each person get if 4 people share $\frac{1}{2}$ pound of chocolate equally?

Stop and Reflect

Which operation is being described in this situation? Before reading further, try to solve this problem without using the standard algorithm. As a hint, can you act it out or use a diagram to help you reason about the quantities and their relationship?

Figure 1.4

One solution for $\frac{1}{2} \div 4$.



Before students start working on the task, the teacher asks them to think about what operation is being used in the situation. After some wait time, some of the students explain that because the chocolate is being shared or divided equally among four people, fair sharing or partitive division is being used. Students are then given a few minutes to work on the problem with a partner, share their ideas with another group, and prepare to share their ideas and answers with the class. The following two ideas were most prominent among the strategies used:

- We cut the half of a pound into four equal pieces so that each person would have an equal share. But then we needed to figure out what those pieces were, so we extended our drawing to make the whole pound. Since there are eight equal pieces in the whole pound, one piece would be one-eighth. So each person would get one-eighth pound of chocolate (Figure 1.4).
- We pretended we had one pound of chocolate. Then each person would get one-fourth of a pound. But each person really gets only half of that since we started with one-half

pound. We knew that one-fourth is the same as two-eighths. So one-eighth is half of one-fourth or two-eighths. So each person gets one-eighth pound of chocolate.

As students share their ideas, the teacher highlights the notion of sharing that is going on in each of the solutions as well as the attention given to the whole or the pound of chocolate.

Because the teacher wants to extend students' thinking to division of fractions by fractions, she poses the following problem, which involves a fractional amount of money.

Dan paid $2\frac{1}{4}$ dollars for a $\frac{3}{5}$-pound box of candy. How much money is that per pound?

She uses money because she has found that students can easily think about partitioning money in a variety of ways. As is the norm in the class, students are told that they should be prepared to explain their reasoning with words and numbers as well as a diagram to support their explanation.

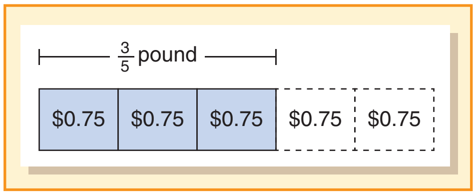
Stop and Reflect

Before reading further, how many different ways can you solve the problem $2\frac{1}{4} \div \frac{3}{5}$?

The students work in pairs for about 15 minutes as the teacher circulates. The teacher listens to different students talk about the task and offers a hint to a few who are stuck. For example, she asks, "Did Dan buy an entire pound of candy? What ways can you partition the whole (the box of candy that Dan bought) to help you think about this problem?" Soon, the teacher begins a discussion by having specific students share their ideas and answers. As the students report, the teacher records their ideas on the board. Sometimes, the teacher asks questions to help clarify ideas for others. She makes no evaluative comments, although she asks the students who are listening if they understand or have any questions to ask the presenters. The following solution strategies are common in classes where students are regularly asked to generate their own approaches. Figure 1.5 shows a sketch for the first method shared below.

Figure 1.5

One solution for $2\frac{1}{4}$ dollars ÷ $\frac{3}{5}$.



Group 1: Since we had $\frac{3}{5}$ pound, we drew three rectangles. We know that in $2\frac{1}{4}$ dollars we had nine fourths, so we put three-fourths of a dollar in each rectangle. Then, the whole pound would be two more three-fourths, or a total of $3\frac{3}{4}$ dollars or $3.75.

Teacher: What do the three rectangles represent in the problem?

Group 1: Each of the rectangles is $\frac{1}{5}$ pound. Since we had $\frac{3}{5}$ pound, we had to show three rectangles.

Teacher: And how did you get $\frac{3}{4}$ dollars?

Group 1: We knew each rectangle or $\frac{1}{5}$ pound was seventy-five cents or three-fourths of a dollar. Since there are five $\frac{1}{5}$ pounds in one pound, we multiplied $3\frac{3}{4}$ by 5.

Group 2: We used a circle and divided it into five parts but used only three of them since we only had $\frac{3}{5}$ pound of candy. We knew we could not put $1 in each part because we did not have $3, so we started with $0.50 (one-half dollar). That left $0.75 (or three-fourths of a dollar) to share, and we knew

75 divided by 3 is 25. So that meant there was \$0.50 plus \$0.25 or \$0.75 in each section. To figure out how much the whole pound would be, we just added \$0.75 to \$2.25 to get \$3.00 and then added \$0.75 to \$3.00 to get \$3.75 or $3\frac{3}{4}$ dollars.

Stop and Reflect

The invert-and-multiply algorithm plays out in this kind of division problem (partitive or fair sharing). Can you find in the explanations above where the calculation $2\frac{1}{4} \times \frac{5}{3}$ occurs? (Hint: Think of $\frac{5}{3}$ as $\frac{1}{3} \times 5$.)

This vignette illustrates that when students are encouraged to solve a problem in their own way (using their own particular set of blue dots or ideas), they are able to make sense of their solution strategies and explain their reasoning. This is evidence of their development of mathematical proficiency.

During the discussions in classes such as this one, ideas continue to grow. The students may hear and immediately understand a clever strategy that they could have used but that did not occur to them. Others may begin to create new ideas that build from thinking about their classmates' strategies. Some in the class may hear excellent ideas from their peers that do not make sense to them. These students are simply not ready or do not have the prerequisite concepts (blue dots) to understand these new ideas. On subsequent days there will be similar opportunities for all students to grow at their own pace based on what they already understand.

Teaching for Instrumental Understanding

In contrast to the lesson just described, in which students are developing concepts (understanding of fraction division) and procedures (ability to flexibly divide) and building relationships between these ideas, let's consider how a lesson with the same basic objective (fraction division) might look if the focus is on instrumental understanding.

In this classroom, the teacher distributes coins to all students. The teacher reads to the class the same problem that was used in the first classroom about a $\frac{3}{5}$-pound box of candy costing \$2.25 or $2\frac{1}{4}$ dollars. She explains that this means that they will be dividing $2\frac{1}{4}$ by $\frac{3}{5}$ and that this is the same as flipping the second fraction and multiplying so that the problem becomes $2\frac{1}{4} \times \frac{5}{3}$. The teacher directs the students to count out $2\frac{1}{4}$ dollars from the coins (nine quarters). The discussion continues:

T: What is $2\frac{1}{4}$ divided by 3? Use your coins to help you find this amount.

S: [Students take some time to partition their coins] \$0.75.
T: So $2\frac{1}{4}$ dollars divided by 3 is \$0.75 or $\frac{3}{4}$ of a dollar. This means that each $\frac{1}{5}$ pound is worth \$0.75 or $\frac{3}{4}$ of a dollar. Can you see with our manipulatives that $\frac{3}{5}$ pound of candy costs $2\frac{1}{4}$ dollars or \$2.25 because we have three groups of \$0.75?

T: [pause] Now we need to find out how much one pound costs, and that's where we multiply the \$0.75 or $\frac{3}{4}$ of a dollar by 5. *One*-fifth of a pound is \$0.75, but there are *five*-fifths in a whole pound, so we need to multiply \$0.75 by 5 to get \$3.75.

Next, the students are given three other similar problems to solve with manipulatives. They work in pairs and record their answers on their papers. The teacher helps anyone having difficulty by guiding them to use and connect manipulatives to the steps in the standard algorithm.

In this lesson, the teacher and students are using manipulatives to illustrate the invert-and-multiply algorithm for fraction division. After engaging in several lessons similar to this one, most students are likely to remember, and possibly understand, how to divide fractions with the standard algorithm. It is important to note that this lesson on the standard algorithm, in combination with other lessons that reinforce other approaches, *can* build relational understanding, as it adds to students' repertoire of strategies. But if this lesson represents the sole approach to fraction division, then students are more likely to develop an instrumental understanding of mathematics because the lesson provides few opportunities to build connections between mathematical concepts. For example, students are not provided opportunities to apply other strategies that may help them build connections between subtraction and division, multiplication and division, or the sharing (partitive) and repeated subtraction (measurement) meanings of division. Building these connections between mathematical ideas and across representations is a fundamental characteristic of relational understanding.

Stop and Reflect

Before reading further, what similarities and differences did you notice between the two classrooms? How do you think these differences might affect the learning that takes place?

The Importance of Students' Ideas

Let's take a minute to compare these two classrooms. By examining them more closely, you can see several important differences. These differences affect what is learned and who learns. Let's consider the first difference: Who determines the procedure to use?

In the first classroom, the students think about the meaning of division in the situation and the relationships between the numbers involved. Using this information, they generate a diagram of the situation to help them make sense of and solve the problem. So they *choose* a strategy that is based on *their* ideas, using what they know about subtraction, multiplication, and division. The students in the first classroom are being taught mathematics for understanding—*relational* understanding—and are developing the kinds of mathematical proficiency described earlier.

In the second classroom, the teacher provides one strategy for how to divide fractions—the standard algorithm. Although the standard algorithm is a valid strategy, the entire focus of the lesson is on the steps and procedures that the teacher has outlined. The teacher solicits no ideas from individual students about how to partition the numbers and instead is only able to find out who has or has not been able to follow directions.

When students have more choice in determining which strategies to use, as in the first classroom, there are more opportunities for learners to interact with each other and with the teacher as they share ideas and results, compare and evaluate strategies, challenge results, determine the validity of answers, and negotiate ideas. As a result, they can learn more content and make more connections. In addition, if teachers do not seek out and value students' ideas, students may come to believe that mathematics is a body of rules and procedures that are learned by waiting for the teacher to tell them what to do. This view of mathematics—and what is involved in learning it—is inconsistent with mathematics as a discipline and with the learning theories described previously.

A second difference between the two classrooms is the learning goals. Both teachers might write "understand fraction division" as the objective for the day. However, what is captured in the word "understand" is very different in each setting. In the first classroom, the teacher's goal is for students to connect fractions and division to what they already know. In the second classroom, understanding is connected to being able to carry out the standard algorithm. The learning goals, and more specifically how the teacher interprets the meaning behind the learning goals, affect what students learn.

These lessons also differ in terms of how accessible they are—and this, in turn, affects *who* learns the mathematics. The first lesson is differentiated in that it meets students where they are in their current understanding. When a task is presented as "solve this in your own way," it has multiple entry points, meaning it can be approached in a variety of ways. Consequently, students with different prior knowledge or learning strategies can figure out a way to solve the problem. This makes the task accessible to more learners. Then, as students observe strategies that are more efficient than their own, they develop new and better ways to solve the problem.

In the second classroom, everyone has to do the problem in the same way. Students do not have the opportunity to apply their own ideas or to see that there are numerous ways to solve the problem. This may deprive students who need to continue working on the development of basic ideas of fractions or division, as well as students who could easily find one or more ways to do the problem if only they were asked to do so. The students in the second classroom are also likely to use the same method to divide all fractions instead of looking for more efficient ways to divide based on the meanings of division and relationships between the numbers. For example, they are likely to divide $\frac{2}{3}$ by 2 using the standard algorithm instead of thinking that $\frac{2}{3} \div 2$ means that you divide $\frac{2}{3}$ into two equal parts, each of which is $\frac{1}{3}$. Recall in the discussion of learning theory the importance of building on prior knowledge and learning from others. In the first classroom, student-generated strategies, multiple approaches, and discussion about the problem represent the kinds of strategies that enhance learning for a range of learners.

Students in both classrooms will eventually succeed at dividing fractions, but what they learn about fractions and division—and about doing mathematics—is quite different. Understanding and doing mathematics involves generating strategies for solving problems, applying those approaches, seeing if they lead to solutions, and checking to see whether answers make sense. These activities were all present in the first classroom, but not in the second. Consequently, students in the first classroom, in addition to successfully dividing fractions, will develop richer mathematical understanding, become more flexible thinkers and better problem solvers, remain more engaged in learning, and develop more positive attitudes toward learning mathematics.

For more information about relational and instrumental understanding as well as the short-term and the long-term effects of teaching with each type of understanding as your goal, **watch this video: https://www.youtube.com/watch?v=TW_RQXWiCFU.**

Mathematics Classrooms That Promote Understanding

An important part of helping students develop relational understanding and mathematical proficiency is to ensure the students are the ones doing the thinking, talking, and the mathematics, however, there are other factors that play a role. Based on an extensive review of the literature, Schoenfeld and Floden (2014) identified the following five dimensions of "productive classrooms—classrooms that produce powerful mathematical thinkers" (p. 2). We address each of these dimensions throughout the book, but in particular, in Chapters 1–7.

1. *The Mathematics:* We have discussed in this chapter the importance of coherent and rigorous standards that promote college- and career-readiness. This includes developing procedural proficiency in conjunction with conceptual understanding as well as developing productive habits of minds through the mathematical processes and practices.
2. *Cognitive Demand:* Productive struggle is an important part of the process of learning and doing mathematics. We discuss in Chapter 2 the importance of engaging students in appropriate levels of challenge that allow them to productively struggle. Activities throughout Part 2 of the book are designed to be high cognitive demand tasks.
3. *Access to Mathematical Content:* All students must be actively engaged in learning core mathematical ideas. To make this happen for some students, purposeful and appropriate levels of support must be in place. These ideas are addressed in Chapters 4–6.
4. *Agency, Authority, and Identity:* Learning is enhanced when students are engaged with others who are working on the same ideas. Encouraging student-to-student dialogue that involves making conjectures, explaining, justifying, and building on each other's ideas can help students think of themselves as capable of making sense of and doers of mathematics. This goal requires that the teacher create a classroom culture in which students can learn from one another, a topic that is addressed in Chapter 2.
5. *Uses of Assessment:* Eliciting students' ideas and reasoning to inform subsequent instruction has the potential to contribute to everyone's learning, especially when common misunderstandings and mistakes are capitalized on and explored. We discuss assessment that supports instruction and learning in Chapter 3.

Teaching Tip

Listen carefully to students as they talk about what they are thinking and doing as they engage in a mathematical task. If they respond in an unexpected way, try to avoid imposing your ideas onto their ideas. Ask clarifying questions to try to make sense of the sense your students are making.

As with most complex phenomena, the dimensions that promote a productive classroom are interrelated. For example, the results of assessment (dimension 5) can help identify an appropriate mathematical task (dimension 1) that can, in turn, affect the potential for cognitive demand (dimension 2). The degree of appropriate support provided (dimension 3) in conjunction with all of the above can influence how a student perceives him or herself as a doer of mathematics (dimension 4). Students have the best chance of becoming powerful mathematical thinkers and doers in classrooms where these five dimensions are implemented well.

Three of the most common types of teaching are direct instruction, facilitative methods (also called a *constructivist approach*), and coaching (Wiggins & McTighe, 2005). With direct instruction, the teacher usually demonstrates or models, lectures, and asks questions that are convergent or closed-ended in nature. With facilitative methods, the teacher might use investigations and inquiry, cooperative learning, discussion, and questions that are more open-ended. In coaching, the teacher provides students with guided practice and feedback that highlights ways to improve their performances. You might wonder, if the goal is to teach mathematics for relational understanding, which type of instructional approach is most appropriate. Unfortunately, there is no definitive answer because there are times when it is appropriate to engage in each of these types of teaching. Your approach depends on your instructional goals, the learners, and the situation.

Constructivism, a theory of learning (not a theory of teaching), explains that students learn by developing and modifying ideas and by making connections between these ideas—and each type of instruction can support students' learning when used at the appropriate time.

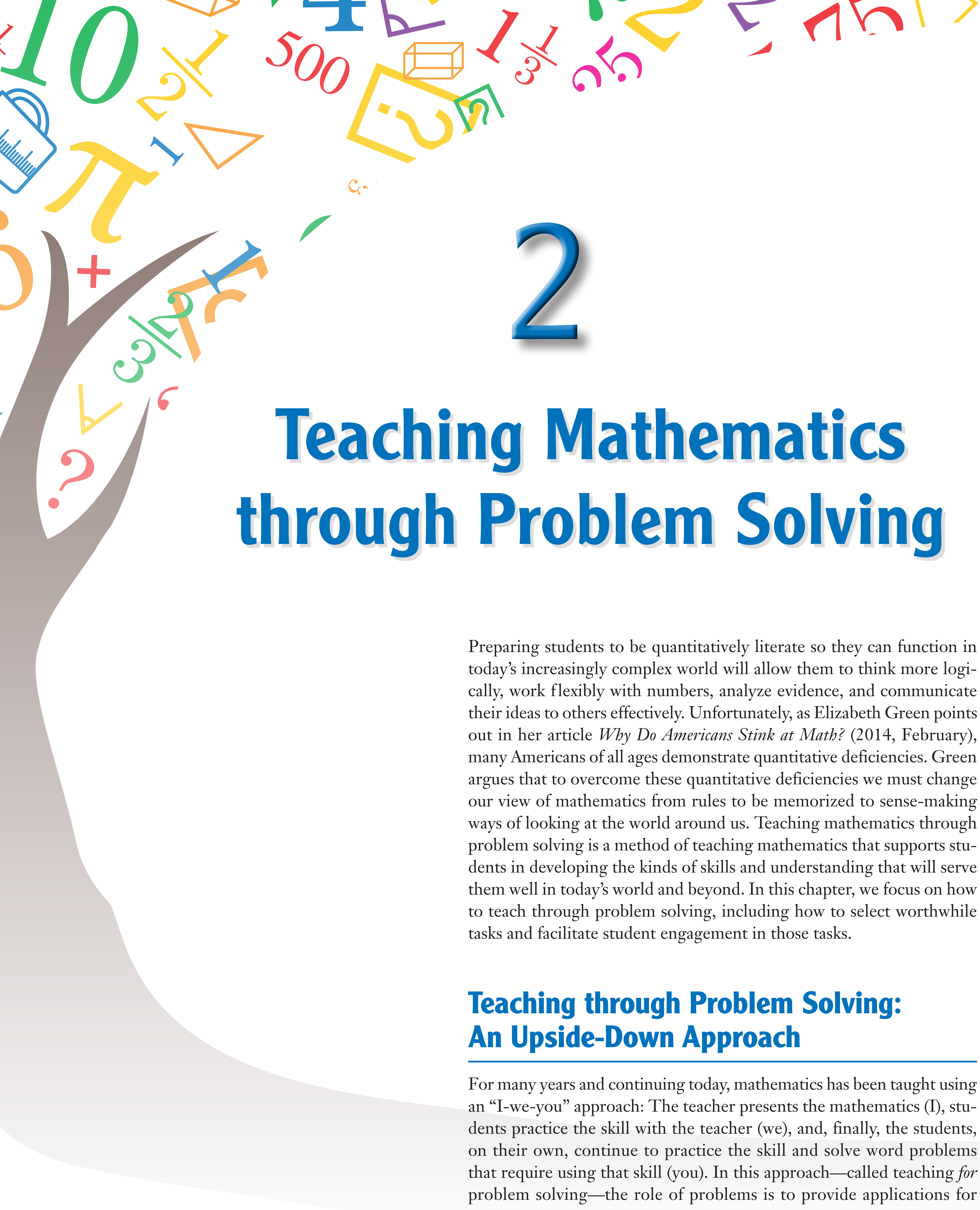
The instructional approach chosen should depend on the ideas and relationships students have already constructed. Assessment can shed light on what and how our students understand, and, in turn, can help us determine which teaching approach may be the most appropriate at a given time. Sometimes students need time to investigate a situation so they can become aware of the different ideas at play and how those ideas relate to one another (facilitative). Sometimes they need to practice a skill and receive feedback on their performance to become more accurate (coaching). Sometimes they are ready to make connections by listening to a lecture (direct instruction). The key to teaching for understanding, no matter which type of teaching you use, is to maintain the expectation for students to reflect on and productively struggle with the situation at hand. In other words, regardless of instructional design, the teacher should not be doing the thinking, reasoning, and connection building—it must be the students who are engaged in these activities.

Most people go into teaching because they want to help students learn. It is hard to think of allowing—much less planning for—the students in your classroom to struggle. Not to show them a solution when they are experiencing difficulty seems almost counterintuitive. If our goal is relational understanding, however, the struggle is part of the learning and teaching becomes less about the teacher and more about what the students are doing and thinking.

Keep in mind that you too are a learner. Some ideas in this book may make more sense to you than others. Some ideas may even create dissonance for you. Embrace this feeling of disequilibrium and unease as an opportunity to learn—to revise your perspectives on mathematics and on the teaching and learning of mathematics as you deepen your understanding so that you can, in turn, help your students deepen theirs.

Stop and Reflect

Look back at the chapter and identify any ideas that challenge your current thinking about mathematics or about teaching and learning mathematics or that simply make you uncomfortable. Try to determine why these ideas challenge or raise questions for you. Write these ideas down and revisit them later as you read and reflect further.



2

Teaching Mathematics through Problem Solving

Preparing students to be quantitatively literate so they can function in today's increasingly complex world will allow them to think more logically, work flexibly with numbers, analyze evidence, and communicate their ideas to others effectively. Unfortunately, as Elizabeth Green points out in her article *Why Do Americans Stink at Math?* (2014, February), many Americans of all ages demonstrate quantitative deficiencies. Green argues that to overcome these quantitative deficiencies we must change our view of mathematics from rules to be memorized to sense-making ways of looking at the world around us. Teaching mathematics through problem solving is a method of teaching mathematics that supports students in developing the kinds of skills and understanding that will serve them well in today's world and beyond. In this chapter, we focus on how to teach through problem solving, including how to select worthwhile tasks and facilitate student engagement in those tasks.

Teaching through Problem Solving: An Upside-Down Approach

For many years and continuing today, mathematics has been taught using an "I-we-you" approach: The teacher presents the mathematics (I), students practice the skill with the teacher (we), and, finally, the students, on their own, continue to practice the skill and solve word problems that require using that skill (you). In this approach—called teaching *for* problem solving—the role of problems is to provide applications for newly learned skills. For example, students learn how to solve for “x” in a linear equation and once that is mastered, solve word problems by translating the situation to a linear equation and solving the equation. Unfortunately, this “do-as-I-show-you” approach to mathematics teaching has not been successful in helping many students understand mathematics concepts (e.g., Pesek & Kirshner, 2002; Philipp & Vincent, 2003). Here are a few reasons why:

- It requires that all students have the necessary knowledge to understand the teacher’s explanations—which is rarely, if ever, the case.
- It communicates that there is only one way to think about and solve the problem, which is a misrepresentation of mathematics and genuine problems.
- It positions the student as a passive learner who is dependent on the teacher to show them ideas, rather than as an independent thinker who is capable and responsible for solving the problem.
- It decreases the likelihood that a student will attempt a novel problem without explicit instructions on how to solve it.

In the past we assumed that walking students through a procedure or showing a step-by-step method for solving a particular type of problem was the most helpful approach to learning. However, this approach can actually make students worse at solving problems and doing mathematics, not better.

Teaching mathematics *through* problem solving means that students work with problems to learn new mathematics and to extend their current understanding. This is sometimes called learning through *inquiry*. With this approach, problem solving is completely interwoven with learning. As students do mathematics—make sense of cognitively demanding tasks, provide evidence or justification for strategies and solutions, find examples and connections, and receive and provide feedback about ideas—they are simultaneously engaged in the activities of problem solving and learning. Students learn mathematics through real contexts, problems, situations, and models that allow them to build meaning for the concepts (Hiebert et al., 1997). Teaching *through* problem solving acknowledges what we now know about learning and doing mathematics (see Chapter 1). Our understanding is always changing, incomplete, situated in context, and interconnected. What we learn becomes part of our expanding and evolving network of ideas—a network without endpoints.

So teaching *through* problem solving might be described as “upside down” from the traditional approach of teaching *for* problem solving because the problem or task is presented at the beginning of a lesson and related knowledge or skills emerge from exploring the problem. An example of teaching through problem solving might have students explore the following situation before they are taught how to set up proportions and solve for the unknown.

Tatyana has a coupon for 4 pizzas for $10. If the restaurant will give her the same rate for multiple pizzas, how much will 18 pizzas cost?

The teacher would explain to the class that there is more than one way to solve this problem and that they are to find as many different solution paths as they can. Students might use manipulatives, create a drawing, make an organized list, or solve through a series of operations.

Stop and Reflect

Find a way to determine how much money 18 pizzas will cost without using the standard algorithm of setting up proportions and solving for the unknown.

Through this context and exploration, students could be introduced to ratio tables or a double number line as a way to organize their ideas. The pizza problem generates opportunities for students to improve their multiplicative reasoning as they find ways to relate numbers whose multiplicative relationship is not readily apparent. Using the standard algorithm of setting up proportions and solving for the unknown reduces the potential for students to develop multiplicative and proportional reasoning and should be introduced later and as only one way to solve these kinds of problems.

Teaching *through* problem solving positions students to engage with mathematics to learn important mathematical concepts. With this approach, students:

- Ask questions
- Determine solution paths
- Use mathematical tools
- Make conjectures
- Seek out patterns
- Communicate findings
- Make connections to other content
- Make generalizations
- Reflect on results

Hopefully these student behaviors sound familiar. This list reflects the CCSS Standards for Mathematical Practice and the NCTM process standards as well as components of being mathematically proficient that were discussed in Chapter 1.

Mathematics Teaching Practices for Teaching *through* Problem Solving

Teaching *through* problem solving requires a paradigm shift, which means that teachers are doing more than just tweaking a few things about their teaching; they are changing their philosophy about how they think students learn best and how they can best help them learn. At first glance, it may seem that the teacher's role is less demanding because the students are doing the mathematics, but the teacher's role is actually more demanding in such classrooms.

Classrooms where students are engaging with and making sense of mathematics through inquiry do not happen by accident—they happen because the teacher uses practices and establishes expectations that encourage risk taking, reasoning, the generation and sharing of ideas, and so forth. Table 2.1 lists eight research-informed, high-leverage teaching practices, identified in NCTM's (2014) *Principles to Actions*, that support students to develop a robust understanding of mathematics. These teaching practices are designed to address issues of access and equity so that all students can succeed in learning mathematics. We will refer back to these teaching practices throughout this chapter as we consider how to teach mathematics *through* problem solving.

Using Worthwhile Tasks

When teachers teach mathematics *through* problem solving, the teacher needs to use worthwhile or rich tasks that promote reasoning and problem solving. Not surprisingly, this is one of the eight mathematics teaching practices identified in *Principles to Actions* (NCTM, 2014) (see practice #2 in Table 2.1). As a teacher you need to know what constitutes a worthwhile task and where to find, adapt, or create such tasks. A worthwhile task is problematic

Table 2.1. Eight mathematical teaching practices that support student learning.

Teaching Practice	What Is the Teacher Doing to Enact the Practice?
1. Establish mathematics goals to focus learning	• Articulates clear learning goals that identify the mathematics students will learn in a lesson or lessons. • Identifies how the learning goals relate to a mathematics learning progression. • Helps students understand how the work they are doing relates to the learning goals. • Uses the articulated goals to inform instructional decisions involved in planning and implementation.

2. Implement tasks that promote reasoning and problem solving	• Selects tasks that:
	  • Have maximum potential to build and extend students' current mathematical understanding.
	  • Have multiple entry points.
	  • Require a high level of cognitive demand.
	• Supports students to make sense of and solve tasks using multiple strategies and representations, without doing the thinking for the students.

3. Use and connect mathematical representations	• Supports students to use and make connections between various representations. • Introduces representations when appropriate. • Expects students to use various representations to support their reasoning and explanations. • Allows students to choose which representations to use in their work. • Helps students attend to the essential features of a mathematical idea represented in a variety of ways.
4. Facilitate meaningful mathematical discourse	• Facilitates productive discussions among students by focusing on reasoning and justification. • Strategically selects and sequences students' strategies for whole class discussion. • Makes explicit connections between students' strategies and ideas.

5. Pose purposeful questions	• Asks questions that
	  • Probe students' thinking and that require explanation and justification.
	  • Build on students' ideas and avoids funneling (i.e., directing to one right answer or idea).
	  • Make students' ideas and the mathematics more visible so learners can examine the ideas more closely.
	• Provides appropriate amounts of wait time to allow students to organize their thoughts.

6. Build procedural fluency from conceptual understanding	• Encourages students to make sense of, use, and explain their own reasoning and strategies to solve tasks. • Makes explicit connections between strategies produced by students and conventional strategies and procedures.
7. Support productive struggle in learning mathematics	• Helps students see mistakes, misconceptions, naïve conceptions, and struggles as opportunities for learning. • Anticipates potential difficulties and prepares questions that will help scaffold and support students' thinking. • Allows students time to struggle with problems. • Praises students for their efforts and perseverance in problem solving.
8. Elicit and use evidence of student thinking	• Decides what will count as evidence of students' understanding. • Gathers evidence of students' understanding at key points during lesson. • Interprets students' thinking to gauge understanding and progress toward learning goals. • Decides during the lesson how to respond to students to probe, scaffold, and extend their thinking. • Uses evidence of students' learning to guide subsequent instruction.

Source: Based on *Principles to Actions: Ensuring Mathematical Success For All* (NCTM), © 2014.

as this video demonstrates (**https://www.youtube.com/watch?v=XI3-52B0V6s**). It poses a question for which (1) the students have no prescribed or memorized rules or methods, and (2) there is no perception that there is a specific "correct" solution method (Hiebert et al., 1997). Because the process or solution method is not obvious, justification is central to the task.

A worthwhile task can take many forms. It might be open-ended or clearly defined; it may involve problem solving or problem posing; it may include words or be purely symbolic; it may develop concepts, procedures, or both; it may take only a few minutes to solve or weeks to investigate; or, it may involve a real-life scenario or be abstract. What makes the task worthwhile is that it is problematic as it engages students in figuring out how to solve it. Here are some tasks to try.

CONCEPT: COMPARING RATIOS AND PROPORTIONAL REASONING

Jack and Jill were at the same spot at the bottom of a hill, hoping to fetch a pail of water. They both begin walking up the hill, Jack walking 5 yards every 25 seconds and Jill walking 3 yards every 10 seconds. Assuming a constant walking rate, who will get to the pail of water first?

CONCEPT: EQUALITY

$$64 \div 16 = 32 \div b$$

Find a number for b so that the equation is true. Is there more than one number that will make the equation true? Why or why not? Can you find more than one way to find a number for b so that the equation is true?

PROCEDURE: DIVIDING TWO FRACTIONS

Solve this problem in two different ways: $\frac{3}{4} \div \frac{1}{2} =$ _____.
For each way, explain how you solved it.

Note that a task can be problematic at first and then become routine as a student's knowledge and experience grows. For example, the third example above could be a rich task to explore in grade 6, but would be "routine" for older or more mathematically advanced students. If the task immediately triggers an approach that the student goes directly to use, the task may be appropriate practice, but it is not a task that is likely to provide the students with new mathematical insights or knowledge.

In thinking about the variety of worthwhile tasks that you can pose, it is important to incorporate tasks that develop both procedural fluency and conceptual understanding. Remember that *procedural fluency* and *conceptual understanding* are two of the five intertwined strands of mathematical proficiency discussed in Chapter 1. Both are equally valuable in students' development toward proficiency—so much so that they are included in one of the teaching practices from *Principles to Action* (NCTM. 2014): *build procedural fluency from conceptual understanding* (see Table 2.1). Using worthwhile tasks where students are able to use a variety of methods and strategies that make sense to them, are expected to explain and justify their approaches, and are encouraged to look for connections among strategies is precisely how students build procedural fluency from conceptual understanding.

Level of Cognitive Demand

Engaging students in productive struggle is one of the teaching practices identified in *Principles to Actions* (NCTM, 2014) (see Table 2.1). It is crucial to student's learning mathematics with understanding (Hiebert & Grouws, 2007). Posing worthwhile tasks sets the stage for this productive struggle because such tasks are cognitively demanding, meaning they require higher-level thinking. High-level, cognitively demanding tasks challenge students to make connections, analyze information, and draw conclusions (Smith & Stein, 1998). On the other hand, low-level cognitively demanding tasks (also called *routine problems* or *lower-level tasks*) are straightforward and involve stating facts, following known procedures, and solving routine problems. As an example of different levels of tasks, consider the degree of reasoning required if you ask students to find the average of five given numbers versus if you ask them to find five numbers whose average is 35. The first task only requires students to find the average of five numbers. The second task requires them to use number sense and their understanding of average to generate five numbers whose average is the given average. As a consequence of working on this second task, students have potential opportunities to think about and use number relationships while they work on their computational skills for finding averages. The second task also has more potential to enhance students' understanding of the meaning of "average," not just how to find it.

Table 2.2 shows a well-known framework that is useful in determining whether a task has the potential to challenge students (Smith & Stein, 1998). As you read through the

Table 2.2. Levels of cognitive demand.

Low-Level Cognitive Demand Tasks	High-Level Cognitive Demand Tasks
Memorization • Involve either memorizing or producing previously learned facts, rules, formulas, or definitions • Are routine, in that they involve exact reproduction of previously learned procedures • Have no connection to related concepts	**Procedures with Connections** • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas • Suggest general procedures that have close connections to underlying conceptual ideas • Are usually represented in multiple ways (e.g., visuals, manipulatives, symbols, problem situations) • Require that students engage with the conceptual ideas that underlie the procedures in order to successfully complete the task
Procedures without Connections • Use procedures specifically called for • Are straightforward, with little ambiguity about what needs to be done and how to do it • Have no connection to related concepts • Are focused on producing correct answers rather than developing mathematical understanding • Require no explanations or explanations that focus on the procedure only	***Doing* Mathematics** • Require complex and nonalgorithmic thinking (i.e., nonroutine—without a predictable, known approach) • Require students to explore and to understand the nature of mathematical concepts, processes, or relationships • Demand self-monitoring or self-regulation of students' own cognitive processes • Require students to access relevant knowledge in working through the task • Require students to analyze the task and actively examine task constraints • Require considerable cognitive effort

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descriptors for low-level and high-level cognitively demanding tasks, you will notice that the low-level tasks are straightforward and routine, meaning that they do not engage students in productive struggle. Although there are appropriate times to use low-level cognitively demanding tasks, a heavy or sole emphasis on tasks of this type will not lead to a relational understanding of mathematics. When students know that struggle is an expected part of the process of doing mathematics, they embrace the struggle and feel success when they reach a solution (Carter, 2008).

Multiple Entry and Exit Points

Your students will likely have a wide range of experiences in mathematics, so it is important to use problems that have multiple entry points, meaning that the problems have varying degrees of challenge or can be approached in a variety of ways. Having multiple entry points can help accommodate the diversity of learners in your classroom because students are encouraged to use a strategy that makes sense to them instead of using a predetermined strategy that they may or may not be ready to use successfully. Having a choice of strategies can also lower the anxiety of students, particularly English Language Learners (Murrey, 2008). Some students may initially use less efficient approaches, such as guess and check or counting, but they will develop more advanced strategies through effective questioning by the teacher and by reflecting on other students' approaches. For example, for the task of finding five different numbers whose average is 35, one student may use a guess-and-check approach, using five random numbers to see if their average is 35, while another student may use a more systematic approach, such as starting with five 35s and then moving part of one 35 to another 35 until he or she has five different numbers. Still another student may reason that if the five numbers were all 35s, then their sum would be 5 × 35, or 175, so partitioning 175 into five different parts would result in five numbers whose average is 35. Asking students to compare these approaches, and in particular to identify advantages and disadvantages of each, can help students move toward more advanced strategies.

Tasks should also have multiple exit points, or various ways that students can express solutions that reveal a range of mathematical sophistication. For example, students might draw a diagram, write an equation, use manipulatives, or draw a graph to demonstrate their understanding. Even though students might initially select an inefficient or less sophisticated approach, as ideas are exchanged during and after the problem is solved, students will have opportunities to understand and try other approaches. As they discuss ideas, draw visuals, use manipulatives, or act out a problem, defend their strategy, critique the reasoning of others, and write about their reasoning, they engage in higher-level thinking. As an added bonus, the teacher is also able to gather useful formative assessment data about students' mathematical understanding.

Consider the opportunities for multiple entry and exit points in the following tasks.

TASK 1:

If there are 1 red tile, 2 yellow tiles, and 1 blue tile in a bag, what is the probability of pulling out a red tile?

TASK 2:

The probability of an event is $\frac{1}{4}$. Describe what the event (situation) might be. Explain how you might use dice, a spinner, or some other tool to simulate the situation.

Stop and Reflect

To what degree do these tasks offer opportunities for multiple entry and exit points?

In the first task, students will gain some experience in thinking about the probability of one outcome in a given situation, but they will miss any opportunity to think deeply about what a probability of $\frac{1}{4}$ means and how it can represent various situations. The second task offers more opportunity for students to engage with the task in a variety of ways, which also offers the teacher more information about each student's level of understanding. For example, do students select examples with more than four objects (e.g., drawing hearts from a deck of cards)? Do students create tree diagrams to help them reason through the situation? Do students create outcomes that are all equally likely, or do they create some outcomes that are more likely than others? Can students think about compound events that result in a probability of $\frac{1}{4}$? Clearly, the second task offers many more opportunities for all students to engage in the task in a variety of ways.

Teaching Tip

Before giving a selected task to your class, anticipate several possible student responses to the task, including possible misconceptions, and think about how you might address these responses. Anticipating the responses gives you time to consider how you will respond to various ideas, and it also helps you to quickly recognize different strategies and misconceptions when students are working on the task.

Relevant and Well-Designed Contexts

One of the most powerful ways to teach mathematics through problem solving is to begin the lesson with a problem that can get students excited about learning mathematics. Compare the following sixth-grade introductory tasks on ratios. Which one do you think would be more exciting to students?

> *Classroom A:* "Today we are going to explore ratios and see how ratios can be used to compare amounts."
>
> *Classroom B:* "In a minute, I am going to read to you a passage from *Harry Potter* about how big Hagrid is. We are going to use ratios to compare our heights and widths to Hagrid's height and width."

Familiar and interesting contexts increase students' engagement. Your goal as a teacher is to design problems using familiar and interesting contexts that provide specific parameters, constraints, or structure that will support the development of the mathematical ideas you want students to learn. Note that in solving a worthwhile problem the problematic or engaging aspect of the problem must be a result of the mathematics that the students are to learn. Any context or external constraints used should not overshadow but highlight the mathematics to be learned.

In Classroom B above, a popular fantasy novel was used to engage students. Young adolescent literature is a rich source for generating high cognitive demand tasks with multiple entry and exit points. Books like *Harry Potter and the Sorcerer's Stone* (Rowling, 1998) offer several possibilities for tasks that can be launched from the story. For example, in addition to the task already posed, students in grades 6–8 can create a scatter plot of their widths and heights and see where Hagrid's data would be plotted on the graph. Measurement, number, and algebra content are all embedded in this task.

Evaluating and Adapting Tasks

In a great many places you will find suggestions for tasks that *someone* believes are effective for teaching a particular mathematics concept or skill. Unfortunately, many of these readily available tasks fall short of being worthwhile or rich tasks. Table 2.3 provides some reflective questions you can use to evaluate whether a task you are considering has the maximum potential to help your students learn relevant mathematics. These questions are meant to help you consider to what extent the task meets these criteria, so all the boxes do not need to be checked off. A task could rate very high on the number of problem-solving strategies, but miss the mark for having a relevant context for your students, and therefore you would decide to change the context to something more interesting. Or, the task is complete with worthwhile features and problem-solving strategies but it does not match the mathematical goals for the lesson. You may choose to alter the task to focus on the relevant mathematics or save it for when it is a better match.

You will find many problems in student textbooks, on the Internet, at workshops you attend, and in articles you read that don't quite meet the mark of a worthwhile task. Boaler (2016) offers the following six suggestions for adapting tasks to increase their potential for learning:

1. Allow multiple ways: Modify the task so students can use multiple methods, strategies, and representations to solve.
2. Make it an exploration: Change the task so that students must do more than complete a procedure and change it so there is potential to learn by doing the task. For example, rather than ask students to multiply -3×4, ask them for various numbers they could multiply to get -12.
3. Postpone teaching a solution method: This way students have the opportunity to use their intuition to think about the situation before learning about conventional methods. For

Table 2.3. Reflective questions to use in selecting worthwhile tasks.

Task Evaluation and Selection Guide	
Task potential	Try it and ask . . . ☐ What is problematic about the task? ☐ Is the mathematics interesting? ☐ What mathematical goals does the task address (and are they aligned to what you are seeking)? ☐ What strategies might students use? ☐ What key concepts and/or misconceptions might this task elicit?
Problem-solving strategies	Will the task elicit more than one problem-solving strategy ☐ Visualize ☐ Look for patterns ☐ Predict and check for reasonableness ☐ Formulate conjectures and justify claims ☐ Create a list, table, or chart ☐ Simplify or change the problem ☐ Write an equation
Worthwhile features	To what extent does the task have these key features: ☐ High cognitive demand ☐ Multiple entry and exit points ☐ Relevant contexts
Assessment	In what ways does the task provide opportunities for you to gain insights into student understanding through: ☐ Using tools or models to represent mathematics ☐ Student reflection, justification, and explanation ☐ Multiple ways to demonstrate understanding

example, prior to teaching students conventional methods for finding the probability of an event, ask them to devise ways to get a sense of the likelihood of the given event happening.

4. Add a visual requirement: Visualization can be a powerful tool for enhancing understanding. You can require students to use color coding to show connections or relationships. Or require that they use a visual representation to justify their solution.
5. Increase the number of entry points: To lower the floor, simply ask students to write down everything they know about the problem or the given concept. To raise the ceiling, once students have completed a task, have them write their own related questions. Challenge them to write questions that are more difficult than the original question and to justify why they are more difficult.
6. Reason and convince: Require students to not only provide their reasoning but to be convincing in their mathematical argument and to require others to be as well. Ask learners to be skeptics and to ask clarifying questions of others. You will need to model the expectations for being a skeptic by asking students follow-up questions when they have not been convincing enough.

Additional strategies for modifying tasks to offer differentiated challenges for students can be found in the chapter on differentiating instruction (Chapter 4).

What Do I Do When a Task Doesn't Work?

Sometimes students may not know what to do with a problem you pose, no matter how many hints and suggestions you offer. Do not give in to the temptation to "tell them." When you sense that a task is not moving forward, don't spend days just hoping that something wonderful may happen. You may need to regroup and offer students a simpler but related task that gets them prepared for the one that proved too difficult. If that does not work, set it aside for the moment. Ask yourself why it didn't work well. Did the students have the prior knowledge they needed? Was the task too advanced? Consider what might be a way to step back or step forward in the content in order to support and challenge the class. Nonetheless, trust that teaching mathematics *through* problem solving offers students the productive struggle that will allow them to develop understanding and become mathematically proficient.

Orchestrating Classroom Discourse

Participating in discussions about mathematics contributes to students' understanding in a multitude of ways. Discussions improve students' ability to reason logically as they learn to share their ideas and listen to the ideas of others. Misconceptions and naïve conceptions are also more likely to be revealed in discussions, providing opportunities for teachers to explicitly address them. As students realize they can learn from each other, they are more motivated and interested in what their classmates have to say.

Learning how to orchestrate an effective classroom discussion is quite complex and requires attention to a number of elements. The goal of productive discourse is to keep the cognitive demand high while students are learning and formalizing mathematical concepts (Breyfogle & Williams, 2008/2009; Kilic et al., 2010; Smith, Hughes, Engle, & Stein, 2009). The purpose is not for students to state their answers and get validation from the teacher. The aspects involved

Teaching Tip

What are misconceptions and naïve conceptions? Consider a misconception as a student understanding that is not mathematically accurate, for example, thinking the function $y = 3x$ describes the same relationship as $y = 3^x$. A naïve conception is a partial and typically less powerful, but mathematically accurate understanding-for example, understanding that a proportion shows two equivalent ratios, but not understanding the between and within relationships among the values. In either case, these conceptions are important to diagnose so that a deep understanding can be developed.

in orchestrating classroom discourse are so important, they directly involve three out of the eight teaching practices from *Principles to Actions* (NCTM, 2014): Facilitate meaningful mathematical discourse; Pose purposeful questions; and Elicit and use evidence of student thinking.

Classroom Discussions

The value of student talk throughout a mathematics lesson cannot be overemphasized. As students describe and evaluate solutions to tasks, share approaches, and make conjectures, learning will occur in ways that are otherwise unlikely to take place. Questions such as those that ask students whether they would do it differently next time, which strategy made sense to them (and why), and what caused problems for them (and how they overcame them) are critical in developing mathematically proficient students. As they listen to other students' ideas, they come to see the varied approaches in how problems can be solved and see mathematics as something that they can do.

Smith and Stein (2011) identified five teacher actions for orchestrating productive mathematics discussions: anticipating, monitoring, selecting, sequencing, and connecting. The first action, *anticipating* responses to the selected worthwhile task, takes place before the lesson even begins. As students work on the task, the teacher *monitors*, observing students' strategies and asking questions such as:

- How did you decide what to do? Did you use more than one strategy?
- What did you do that helped you make sense of the problem?
- Did you find any numbers or information you didn't need? How did you know that the information was not important?
- Did you try something that didn't work? How did you figure out it was not going to work?

These and similar questions are meant to help students reflect on their strategies and help the teacher determine which strategies to *select* for public discussion in the next part of the lesson. Having selected a range of strategies to be shared, the teacher strategically *sequences* the presentations so that particular mathematical ideas can be emphasized. During the presentations, the teacher generates questions and ideas that *connect* strategies and mathematical concepts. These tend to be questions that are specific to the task, but some general questions include:

- How did [Kathy] represent her solution? What mathematical terms, symbols, or tools did she use? How is this like/different from [Colin's] strategy?
- Was there something in the task that reminded you of another problem we've done?
- What might you do the same or differently the next time you encounter a similar problem?

Notice these questions focus on the problem-solving process as well as the answer, and what worked as well as what didn't work. A balanced discussion helps students learn how to do mathematics.

Because of the important benefits of talking about mathematical ideas, you also need to make sure that everyone participates in the classroom discussion. Finding ways to encourage students to share their ideas and to engage with others about their ideas is essential to productive discussions. You may need to explicitly discuss with students why discussions are important and what it means to actively listen and respond to others' ideas. Wagganer (2015) shares some helpful ideas to explicitly teach students how to engage in active listening. For example, students can demonstrate they are listening by making eye contact with the speaker and through nonverbal cues (e.g., nodding); letting the speaker finish before sharing questions or ideas; and responding appropriately and respectfully by asking questions or summarizing the speaker's ideas. Also, Table 2.4 identifies five "talk moves" that help get everyone talking about mathematics (Chapin, O'Connor, & Anderson, 2009).

Considerable research into how mathematical communities develop and operate provide additional insight into promoting effective classroom discourse (Chapin, O'Conner, & Anderson, 2009; Rasmussen, Yackel, & King, 2003; Stephan & Whitenack, 2003; Wood, Williams, & McNeal, 2006; Yackel & Cobb, 1996). This collection of research offers the following recommendations:

- Encourage student–student dialogue rather than student–teacher conversations that exclude the rest of the class. "Juanita, can you answer Lora's question?" "Devon, can you explain that so that LaToya and Kevin can understand what you are saying?" When students have differing solutions, have them work these ideas out as a class. "George, I noticed that you got a different answer than Tara. What do you think about her explanation?"
- Encourage students to ask questions. "Pete, did you understand how they did that? Do you want to ask Antonio a question?"
- Ask follow-up questions whether the answer is right or wrong. Your role is to understand students' thinking, not to lead students to the correct answer. So follow up with probes to learn more about their answers. Sometimes you will find that what you assumed they were thinking is not correct. And if you only follow up on wrong answers, students quickly figure this out and get nervous when you ask them to explain their thinking.
- Call on students in such a way that, over time, everyone is able to participate. Use time when students are working in small groups to identify interesting solutions that you will

Table 2.4. Productive talk moves for supporting classroom discussions.

Talk Moves	What It Means and Why	Example Teacher Prompts
1. Revoicing	This move involves restating the statement as a question in order to clarify, apply appropriate language, and involve more students. It is an important strategy to reinforce language and enhance comprehension for ELLs.	"You used the hundreds chart and counted on?" "So, first you recorded your measurements in a table?"
2. Rephrasing	Asking students to restate someone else's ideas in their own words will ensure that ideas are stated in a variety of ways and encourage students to listen to each other.	'Who can share what Ricardo just said, but using your own words?"
3. Reasoning	Rather than restate, as in talk move 2, this move asks the student what they think of the idea proposed by another student.	"Do you agree or disagree with Johanna? Why?"
4. Elaborating	This is a request for students to challenge, add on, elaborate, or give an example. It is intended to get more participation from students, deepen student understanding, and provide extensions.	"Can you give an example?" "Do you see a connection between Julio's idea and Rhonda's idea?" "What if . . . "
5. Waiting	Ironically, one "talk move" is to not talk. Quiet time should not feel uncomfortable, but should feel like thinking time. If it gets awkward, ask students to pair-share, and then try again.	"This question is important Let's take some time to think about it"

Source: Based on *Classroom Discussions: Using Math Talk to Help Students Learn, Grades K–6*, by Suzanne H. Chapin, Catherine O'Connor, Nancy Canavan Anderson. Published by Math Solutions, © 2009.

highlight during the sharing time. Be intentional about the order in which the solutions are shared; for example, select two that you would like to compare and have students present them back-to-back. All students should be prepared to share their strategies.

- Demonstrate to students that it is okay to be confused and that asking clarifying questions is appropriate. This confusion, or disequilibrium, just means they are engaged in doing real mathematics and indicates that they are learning. Make a point to tell them this!
- Move students to more conceptually based explanations when appropriate. For example, if a student says that he knows $3y = 8x + 2$ has a greater rate of change than $y = 2x - 1$, you can ask him (or another student) to explain why this is so.
- Be sure all students are involved in the discussion. ELLs, in particular, need more than vocabulary support; they need support with mathematical discussions (Moschkovich, 1998). For example, use sentence starters or examples to help students know what kinds of responses you are hoping to hear and to reduce the language demands. Sentence starters can also be helpful for students with disabilities because it adds structure. Have students practice their explanations with a peer. Invite students to use illustrations and actual objects, when appropriate, to support their explanations. These strategies benefit all students, not just students in the class who struggle with language.
- Pay attention to whether you are taking over students' thinking. Jacobs, Martin, Ambrose, and Philipp (2014) identify warning signs of such behavior—for example, interrupting a student's strategy or explanation, manipulating the tools instead of allowing the student to do so, and asking a string of closed questions (i.e., funneling). Taking over students' thinking sends the message that you do not believe they are capable and can inhibit the discourse you are trying to encourage.

Aspects of Questioning

Questions are important in learning about students' thinking, challenging conclusions, and extending the inquiry to help generalize patterns. Questioning is very complex and something that effective teachers continue to improve on throughout their careers. Here are some major considerations related to questioning that influence students' learning.

1. *The "level" of the question asked.* There are numerous models that identify different levels of questions. For example, the Levels of Cognitive Demand in Table 2.2 include two low-level demand categories and two high-level demand categories. Also, Bloom's Taxonomy Revised includes six levels (knowledge, comprehension, application, analysis, synthesis, evaluation), with each level meant to be more cognitively demanding than the previous (Anderson & Krathwohl, 2001). However, you can still ask a low-level cognitively demanding "create" question on Bloom's Taxonomy Revised. For example, brainstorming ideas would be lower-level than say, designing a model, but both could be categorized as "create." Check out Simpson, Mokalled, Ellenburg, and Che (2015) who share a tool that can help analyze the depth of knowledge that can occur across the categories in Bloom's Taxonomy Revised. Regardless of the taxonomy or specific categories, it is critical to ask higher-level, cognitively demanding questions to support students in developing a robust understanding of mathematics.
2. *The type of understanding that is targeted.* Both procedural and conceptual understanding are important, and questions must target both. If questions are limited to procedural

ideas, such as "How did you solve this?" or "What are the steps?" then students will think about procedures, but not about related concepts. Questions focused on conceptual knowledge include, "Will this rule always work?" "Why are common denominators not necessary when multiplying fractions?" "How does the equation you wrote connect to the graph?" and "How is your strategy like Caroline's?"

3. *The pattern of questioning.* Some patterns of questioning teachers use do not lead to classroom discussions that encourage all students to think (Herbel-Eisenmann & Breyfogle, 2005). One such common pattern of questioning goes like this: teacher asks a question, student answers the question, teacher confirms or challenges answer (called "initiation-response-feedback"or "IRF" pattern). Another ineffective pattern is "funneling," when a teacher continues to probe students in ways to get them to a particular answer. This is different than a "focusing" pattern, which uses probing questions to negotiate a classroom discussion and help students understand the mathematics. The talk moves described previously are intended to help facilitate a focusing discussion.
4. *Who is doing the thinking.* Make sure your questions engage all students! When you ask a great question, and only one student responds, then students will quickly figure out they don't need to think about the answer and all your effort to ask a great question is wasted. Instead, use strategies to be sure everyone is accountable to think about the questions you pose. For example, ask students to "talk to a partner" about the question. Or have students record their ideas on a whiteboard or index card.
5. *How you respond to an answer.* When you confirm a correct solution rather than use one of the talk moves, you lose an opportunity to engage students in meaningful discussions about mathematics and so limit the learning opportunities. Use students' solutions to find out if others think the conclusions made are correct, whether they can justify why, and if there are other strategies or solutions to the problem and how they are connected.

How Much to Tell and Not to Tell

One of the most perplexing dilemmas for teachers is how much information or direction to provide to students during mathematical inquiry. On one hand, telling can diminish what is learned and lower the level of cognitive demand in a lesson by eliminating the productive struggle that is key to conceptual understanding (Hiebert & Grouws, 2007). On the other hand, telling too little can sometimes leave students floundering. Following are suggestions about three things that you need to tell students (Hiebert et al., 1997):

- *Mathematical conventions:* Symbols, such as $\sqrt{}$ and x^3, and notations, such as $(1, -2)$, are conventions. Terminology is also a convention. As a rule of thumb, symbolism and terminology should be introduced *after* concepts have been developed, and then specifically as a means of expressing or labeling ideas.
- *Alternative methods:* If an important strategy does not emerge naturally from students, then you should propose the strategy, being careful to identify it as "another" way, not the only or the preferred way.
- *Clarification or formalization of students' methods:* You should help students clarify or interpret their ideas and point out related concepts. A student may divide $\frac{5}{6}$ by $\frac{1}{3}$ by thinking about how many one-thirds can be measured out or subtracted from $\frac{5}{6}$. However, in her explanation, she may describe it as "how many times does $\frac{1}{3}$ go into $\frac{5}{6}$" Ask the student to use a visual diagram to support her explanation to show what she means by the phrase "go into." Emphasize the idea of the one-thirds being measured out or subtracted from $\frac{5}{6}$. Also, this strategy can be related to measurement division with whole numbers, such as thinking of $12 \div 3$ as how many 3s can be measured out or subtracted from 12.

> Drawing everyone's attention to this connection can help other students see the connection while also building the confidence of the student who originally proposed the strategy. (Hiebert et al., 1997).

The key is that you can share information as long as it does not solve the problem, remove the need for students to reflect on what they are doing, or prevent them from developing solution methods that make sense to them (Hiebert et al., 1997).

Leveraging Mistakes and Misconceptions to Enhance Learning

Students inevitably will make mistakes and exhibit misconceptions and naïve conceptions—especially when we pose challenging tasks in our classrooms. You may not want to highlight a student's mistake or misconception because you are concerned that it might embarrass the student or confuse the struggling learners in your classroom. How we choose to treat mistakes and misconceptions in the classroom can have a tremendous effect on students' perceptions about learning and themselves as learners.

When mistakes, misconceptions, and naïve conceptions are perceived and used (explicitly or implicitly) to judge how "smart" someone is, students want to hide their mistakes as well as their lack of understanding. Students can develop a fixed mindset in which they believe their intelligence is set and cannot be further developed through effort (Dweck, 2006), which means they are very unlikely to persevere in solving difficult problems that require productive struggle (Boaler, 2013). To them, difficult tasks are not perceived as opportunities to learn and improve, but rather as spotlights that highlight their inadequacies (Boaler, 2016; Dweck, 2006). On the other hand, students who adopt a growth mindset appreciate, take on, and persist with challenges because they perceive these as opportunities to learn. They also view mistakes as a chance to reconsider, revise, and improve their understanding. An online TED talk of Carol Dweck describing mindsets (titled "The power of believing you can improve") is worth watching.

We want students to embrace a growth mindset—to see mistakes, misconceptions, and struggles as opportunities for learning. In fact, this is a critical part of the teaching practice *support productive struggle in learning mathematics* (see Table 2.1). Publicly valuing a mistake or misconception in class and having students think about why it is a mistake or misconception reinforces the important message that we all make mistakes and have misconceptions and can improve our understanding by examining them more closely. You can even design lessons around tasks that elicit common misconceptions or mistakes (Bray, 2013; Lim, 2014). For example, Lim (2014) shared a task where students choose the graph that best represents a given scenario from four distance-vs-time graphs. Because of the graphs used, this task elicits a common student misconception of thinking of graphs as pictures.

Flawed ideas, strategies, and solutions can come from either student's work or from the teacher. If the mistakes or misconceptions come from students in the class, Bray (2013) offers some helpful suggestions for ensuring that the error maker is respected. For example, ask the student for permission to publicly share the mistake or misconception, give the student the choice to explain, acknowledge to the class where there is good reasoning involved in the student's flawed thinking, and express appreciation for the opportunity to analyze the mistake or misconception as a way to improve their classmates' mathematical understanding.

Choosing to publicly treat mistakes and misconceptions in a positive light in your classroom will help students be risk takers and to persevere with challenging tasks. Rather than fearing mistakes and misunderstandings they will appreciate the powerful role these can play in learning.

Representations: Tools for Problem Solving, Reasoning, and Communication

One of the teaching practices identified in *Principles to Actions* (2014) is *use and connect mathematical representations.* The fact that representations made the cut on this short list should give you some sense of its importance in teaching for relational understanding.

Build A Web of Representations

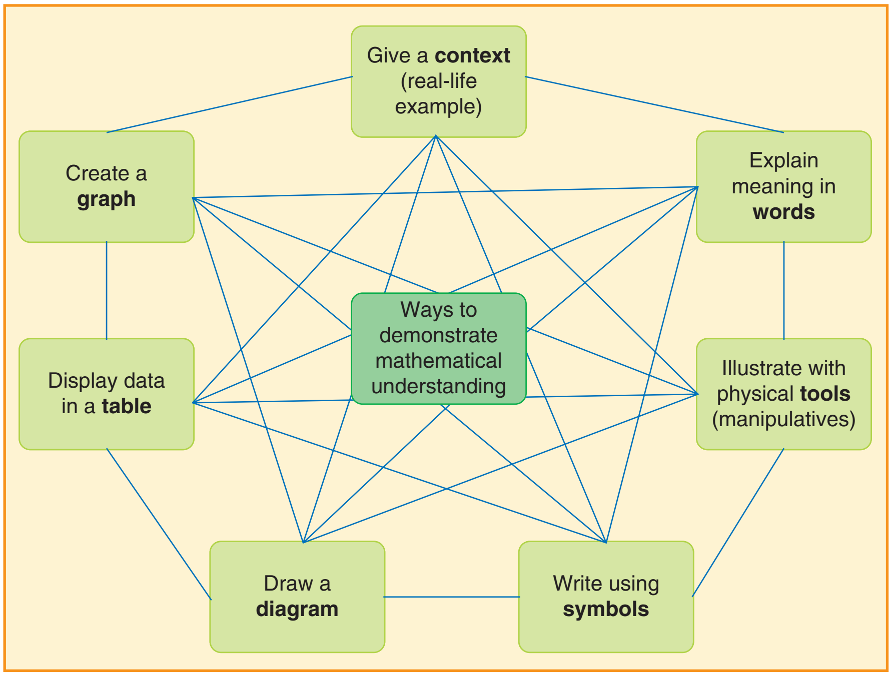
Different representations can illuminate different aspects of a mathematical idea. So to help students build their understanding, they should be encouraged to use, explore, and make connections among multiple representations. Figure 2.1 provides a general Web of Representations that can be applied to any mathematical concept and illustrates the various ways mathematical ideas can be represented. Students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations (Clement, 2004; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Strengthening students' ability to move between and among representations improves their understanding and retention of ideas. For any topic you teach, you can give students the **Translation Task** to complete. (More information is given regarding the use of the Translation Task in Chapter 3, Creating Assessments for Learning.) Fill out one box and ask students to insert the other representations, or you can invite a group to work on all four representations for a given topic (e.g., computation with integers).

Teaching Tip

Pay attention to students' choices of representations, and use those representations as starting points for dialogues with them about their thinking. What they find important may be surprising and informative at the same time.

Figure 2.1

Mathematical understanding can be demonstrated through these different representations of mathematical ideas. Translations between each can help students develop new concepts and demonstrate a richer understanding.



Explore with Tools

A *tool* is any object, picture, or drawing that can be used to explore a concept. CCSS-M includes calculators and manipulatives as tools for doing mathematics (CCSSO, 2010). Manipulatives are physical objects that, regardless of the age of the students, can be used to illustrate and explore mathematical concepts. Choices for manipulatives (including virtual manipulatives) are plentiful—from common objects such as lima beans to commercially produced materials such as Pattern Blocks. A range of manipulatives (e.g., geoboards, base-ten blocks, spinners, number lines) are available in a virtual format, for example, on the National Library of Virtual Manipulatives (NLVM) website and the NCTM Illuminations website. Changes can usually be made to virtual representations more quickly than with physical manipulatives or student-generated drawings, leaving more time for exploration. For example, using interactive graphing software, students can quickly make changes to graphs to help them analyze and interpret how situations change as variables change (see, for example, the applet at Illuminations titled "Constant Cost Per Minute"). As another example, simulation software allows students to perform several trials in just a few seconds, as opposed to having to complete the actual experiment multiple times, again allowing more time for analysis and interpretation of the situation (see, for example, the applet titled "Coin Tossing" at the NLVM website).

Some research suggests benefits to incorporating both physical and virtual manipulatives in instruction (Hunt, Nipper, & Nash, 2011). Physical manipulatives can build the foundation for conceptual understanding while subsequent use of virtual manipulatives can assist learners in bridging to the abstract. For example, an applet at the NLVM website entitled "Grapher" displays a graph of an equation that changes as the equation is modified so that students can see the corresponding results of the changed equation on the graph. The dynamic link between these two representations helps students better understand the effects of changing parameters such as coefficients, signs of terms, and powers of variables on the graph of the function.

Note that a tool does not "illustrate" a concept. While the tool is used to visualize a mathematical concept, an individual has to impose the mathematical relationship on the object (Suh, 2007b; Thompson, 1994). In other words, the manipulative is not the concept but offers a testing ground for emerging ideas. Figure 2.2 shows colored chips commonly used to represent integers. We define one color to represent positive numbers (usually black) and another color to represent negative numbers (usually red, but we have used white here). If a student is able to identify the black chips as "positive" and the white chips as "negative," does this mean the student has constructed the concepts of positive and negative numbers and can operate with them in meaningful ways? No, all you know for sure is that the student has learned the names typically assigned to the manipulatives. There is evidence across a range of grades that students struggle with doing computations with negative numbers (Mukhopadhyay, 1997; Vlassis, 2004). Calculating with integers can become a lesson in memorization when students are rushed to follow rules such as "two like signs become a positive" and "two unlike signs become a negative." Consequently, teachers attempt to support students' work by using manipulatives such as colored chips. However, the concept of "negative" must be created by students in their own minds and imposed on the manipulative used to represent the concept (connecting money to the chips using a profit/loss context can help students construct this meaning). Over time, discussions that explicitly focus on the mathematical concepts can help students make the connections between manipulatives and the related concepts. When you are considering using particular tools in your classroom, take time yourself to try to separate the physical tool from the relationship that you must impose on the tool in order to "see" the concept. This insight will help you support your students as they work with the given tool.

Figure 2.2

Objects and names of objects are not the same as mathematical ideas and relationships between objects.

Names	Models	Relationships
Positive 3	● ● ●	3 chips that are black represent $^{+}3$
Negative 3	○ ○ ○	3 chips that are white represent $^{-}3$
Positive 3	● ● ● ● ○ Equals zero	4 black chips and 1 white chip result in 3 black chips when a black and white chip equal a net gain/loss of 0

Although tools can be used to support the development of relational understanding, used ineffectively, they do not accomplish this goal. The most widespread misuse of manipulatives occurs when teachers tell students, "Do exactly as I do." There is a natural temptation to get out the materials and show students exactly *how to use them*. Students mimic the teacher's directions, and it may even look as if they understand, but they may just be following what they see. A rote procedure with a manipulative is still just that—a rote procedure. The converse is to provide no focus or purpose for using the tools. Neither approach promotes thinking or aids in the development of concepts (Ball, 1992; Stein & Bovalino, 2001).

Drawings are another option for students to represent and illustrate mathematical concepts and are important for a number of reasons. First, when students draw, you learn more about what they do or do not understand. For example, if students are showing the part–part ratio 2:3 with their own drawings, you can observe whether they understand that the whole has five parts, with the first quantity making up $\frac{2}{5}$ of the whole and the second quantity making up $\frac{3}{5}$ of the whole. Second, manipulatives can sometimes restrict how students can model a problem, whereas a drawing allows students to use any strategy they want. Figure 2.3 shows an example of a seventh grader's solution for solving a ratio problem. What does the student's solution tell you about her understanding? Look for opportunities to use students' representations during classroom discussions to help them make sense of the more abstract mathematical symbols and computational procedures. Furthermore, when students create a drawing you tend to get different representations. Have students compare and contrast the various approaches and visuals to facilitate making connections.

Teaching Tip

It is incorrect to say that a manipulative or object "illustrates" or shows a concept. Manipulatives can help students visualize relationships and talk about them, but what they see are the manipulatives, not concepts.

Tips for Using Representations in the Classroom

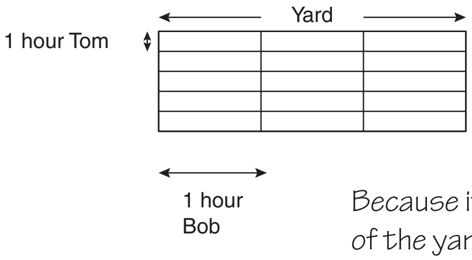
Representations give learners something with which they can explore, reason, and communicate as they engage in problem-based tasks. The goal of using representations is that students are able to manipulate ideas and make connections in a meaningful manner. The following are rules of thumb for using representations in the classroom:

- Introduce new representations or tools by showing how they can represent *the ideas* for which they are intended. But keep in mind that, because the representations are not the concepts, some students may not "see" what you see.
- Encourage students to create their own representations. Look for opportunities to connect these student-created representations to more conventional representations.

Figure 2.3

A seventh grader shows her thinking about a ratio problem.

Bob can mow a yard in 3 hours. Tom can mow the same-size yard in 5 hours.
How long will it take them to mow the same-size lawn if they do it together?



Because it takes Bob an hour to mow $\frac{5}{15}$ of the yard and Tom an hour to mow $\frac{3}{15}$ of the yard, together they can mow $\frac{8}{15}$ of the yard in an hour. Or they can mow 8 parts of the lawn in an hour. After an hour they still have 7 parts to mow. So it will take them $\frac{7}{8}$ of an hour to mow the rest of the yard. Or it takes them $1\frac{7}{8}$ or $\frac{15}{8}$ hours to mow the lawn together.

- Allow students (in most instances) the opportunity to choose their own representations to reason through a problem and to communicate their ideas to others (Mathematical Practice 5: Use appropriate tools strategically). The representations that students choose to use can provide valuable insight into their ways of interpreting and thinking about the mathematical ideas at hand. Note that it is appropriate to encourage a student who is having difficulty to use a particular representation when you believe it would be helpful.
- Ask students to use representations, such as diagrams and manipulatives, when they explain their thinking. This will help you gather information about students' understanding of the idea and also their understanding of the representations that have been used in the classroom. It can also be helpful to other students in the classroom who may be struggling with the idea or the explanation being offered.
- In creating tasks and when facilitating classroom discussions, look for opportunities to make connections among the different representations used (and make sure each is understood). Helping students make these connections is crucial to their learning.

Lessons in the Problem-Based Classroom

Lessons that engage students in problem solving look quite different from traditional lessons that follow the "I-we-you" or "explain, then practice" pattern. Mathematical practices such as modeling mathematics, reasoning quantitatively, and looking for generalizations and structure are not developed in a lecture-style lesson. In contrast, in classrooms where learning is assumed to be a complex process and where inquiry and problem solving are emphasized, worthwhile tasks are posed to challenge students' thinking and students are expected to communicate and justify their ideas. In these kinds of classrooms, preparing a lesson shifts from preparing an agenda of what will happen to creating a "thought experiment" to consider what might happen (Davis, Sumara, & Luce-Kapler, 2008).

The first teaching practice listed in Table 2.1, *establish mathematics goals to focus learning*, is one of the most important. Being clear about the target mathematics that you want

Table 2.5. Teaching mathematics through problem solving lends itself to a three-phase structure for lessons.

Lesson Phase	Teacher Actions in a Teaching Mathematics through Problem-Solving Lesson	
Before	Activate prior knowledge.	Begin with a related but simpler version of the task; connect to students' experiences; brainstorm approaches or solution strategies; estimate or predict whether tasks involve a single computation or are aimed at the development of a computational procedure.
	Be sure the problem is understood.	Have students explain to you what the problem is asking. Go over vocabulary that may be troubling. Caution: This does *not* mean that you are explaining how to *do* the problem—just that students should understand what the problem is about.
	Establish clear expectations.	Tell students whether they will work individually, in pairs, or small groups, or if they will have a choice. Tell them how they will share their solutions and reasoning.
During	Let go!	Although it is tempting to want to step in and "help," hold back and enjoy observing and learning from students.
	Notice students' mathematical thinking.	Base your questions on students' work and their responses to you. Use questions like "Tell me what you are doing"; "I see you have started to [multiply] these numbers. Can you tell me why you are [multiplying]?" [substitute any process/strategy]; "Can you tell me more about . . . ?"; "Why did you . . . ?"; "How does your diagram connect to the problem?"
	Provide appropriate support.	Look for ways to support students' thinking and avoid telling them how to solve the problem: Ensure that students understand the problem (e.g., "What do you know about the problem?"); ask the student what he or she has already tried (e.g., "Where did you get stuck?"); suggest that the student use a different strategy (e.g., "Can you draw a diagram?"; "What if you used cubes to act out this problem?"; "Is this like another problem we have solved?"); create a parallel problem with simpler values (Jacobs & Ambrose, 2008).
	Provide worthwhile extensions.	Challenge early finishers in some manner that is related to the problem just solved. Possible questions to ask are "I see you found one way to do this. Are there any other solutions?; Are any of the solutions different or more interesting than others?" Some good questions for extending thinking are, "What if . . . ?" or "Would that same idea work for . . . ?"
After	Promote a community of learners.	You must teach students about your expectations for this time and how to interact respectfully with their peers. Role-play appropriate (and inappropriate) ways of responding to each other. The "Orchestrating Classroom Discourse" section in this chapter provides strategies and recommendations for how to facilitate discussions that help create a community of learners.
	Listen actively without evaluation.	The goal here is noticing students' mathematical thinking and making that thinking visible to other students. Avoid judging the correctness of an answer so that students are more willing to share their ideas. Support students' thinking without evaluation by simply asking what others think about a student's response.
	Summarize main ideas and identify future problems.	Formalize the main ideas of the lesson, helping to highlight connections among strategies or different mathematical ideas. In addition, this is the time to reinforce appropriate terminology, definitions, and symbols. You may also want to lay the groundwork for future tasks and activities.

students to learn as a result of a particular lesson or lessons helps you to be intentional as you make instructional choices during planning as well as implementation. For example, in planning the lesson, you need to purposefully select a worthwhile task that has maximum potential to illuminate the target learning goals. Without this intentionality, your lesson can lack focus and consequently, not culminate in the desired results. The three-phase lesson format discussed next is intended to support the creation of intentional lessons that support mathematical inquiry.

A Three-Phase Lesson Format

A lesson format that uses the three phases *before*, *during*, *and after* provides a structure for teaching mathematics through inquiry or problem solving. *Before* refers to what happens in the lesson to set up the inquiry, *during* refers to the time during which the students explore the worthwhile problem, and *after* refers to what happens in the lesson after the problem is solved (e.g., discussion, reflection, and making connections). A lesson may take one or more math sessions, but the three-phase structure can also be applied to shorter tasks, resulting in a 10- to 20-minute mini-lesson. Table 2.5 describes appropriate teacher actions and provides some illustrative examples for each phase of the lesson.

Before

The essence of the *before* phase of the lesson is to prepare students to work on the worthwhile task you have purposefully selected. What you do in the *before* phase of the lesson will vary depending on the mathematical goals and the selected task. For example, if your students are familiar with tasks where they are given a "number of the day" and know they are expected to use a variety of representations (words, diagrams, graphs, equations, etc.) and examples to generate as many ideas about the number as they can, all that may be required is to remind students of these expectations. On the other hand, if the task requires students to model the situation with a new manipulative, more time may be needed to familiarize them with the tool. Or if vocabulary needs to be revisited, a related but simpler task could be used in the *before* phase as a way to activate prior knowledge to ensure students understand the terms used in the focus task.

As you plan for the *before* part of the lesson, it is important to analyze the problems you will give to students in order to anticipate their approaches and possible misinterpretations or misconceptions (Wallace, 2007). This process can inform what you do to prepare the students to work on the selected task, without giving away how to solve the task. The more questions raised and addressed prior to the task, the more engaged students will be in the *during* phase.

During

In the *during* phase of the lesson students engage in mathematical activity (alone, with partners, or in small groups) to explore, gather, and record information; make and test conjectures; and solve the mathematical task. In this phase of the lesson you should be engaged in "professional noticing"—that is, in the moment, trying to understand what students know, how they are thinking and approaching the task at hand, and how to respond appropriately to extend students' thinking (Jacobs, Lamb, & Philipp, 2010). In making instructional decisions in the *during* phase you must ask yourself, "Does my action lead to deeper thinking or does it remove the need to think?" These decisions are based on carefully listening to students' ideas and knowing the learning goals of the lesson. This is very different from listening for or leading students toward an answer. Don't be afraid to say that you don't understand a student's strategy. When you are open to learning, you help students become more comfortable with engaging in the learning process.

Students will look to you for approval of their results or ideas. This is not the time to evaluate or to tell students how to solve the problem. When asked whether a result or method is correct, respond by asking, "How can you decide?" or "Why do you think that might be right?" Asking questions such as "How can we tell if that makes sense?" reminds students that the correctness of the answer lies in the justification, not in the teacher's brain or answer key.

Letting go, one of the teacher actions in this phase, includes allowing students to make mistakes. Ask students to explain their process or approach when they make mistakes as well as when they are correct. As they explain they may catch their mistake. Also, in the *after* portion of the lesson, students will have the opportunity to explain, justify, defend, and challenge solutions and strategies. This process of uncovering and working through misconceptions and computational errors emphasizes the important notion that mistakes and misconceptions are opportunities for learning (Boaler, 2016; Dweck, 2006).

Use this time in the *during* phase to identify different representations and strategies students used, interesting solutions, and any misconceptions that arise that you will highlight and address during the *after* phase of the lesson. As you notice the range of ideas, consider how they are related and in what order you might sequence the sharing of solutions in the *after* phase of the lesson (Smith & Stein, 2011).

After

In the *after* phase of the lesson your students will work as a community of learners, discussing, justifying, and challenging various solutions to the problem that they have just worked on. The *after* phase is where much of the learning will occur as students reflect individually and collectively on the ideas they have explored. As in the *during* phase of the lesson, the goal here is noticing students' mathematical thinking, but additionally, in the *after* phase we want to make this thinking visible to other students. By strategically sequencing which diagrams, graphs, equations, notations, ideas, and so forth are shared, you can create spaces for students to take up, try on, connect, and expand on the ideas of others. This is also the time to reinforce precise terminology, definitions, or symbols. After students have shared their solutions, strategies, and reasoning, formalize the main ideas of the lesson, highlighting connections between strategies or different mathematical ideas.

Because this is the place in the lesson where much of the learning will occur, it is critical to plan for and save ample time for this part of the lesson. Twenty minutes is not at all unreasonable for a good class discussion and sharing of ideas. It is not necessary to wait for every student to finish the task before moving into the *after* portion of the lesson. The time they have engaged with the task should prepare them to share and compare ideas.

Variations of the Three-Phase Lesson

The basic lesson structure we have been discussing assumes that the lesson is developed around a task given to the whole class. However, not every lesson is structured in this way. The three-phase format can be applied to other lesson structures, such as mini-lessons and math stations.

Mini-Lessons

A three-phase lesson that capitalizes on the use of routines can be accomplished in as few as 10 minutes. These mini-lessons are not intended to replace the math curriculum or consume most of the instruction time for math, but like in longer lessons, students are expected to use strategies that make sense to them and to explain their thinking. A routine called *Number Talks*, already embraced in many elementary classrooms, can be as equally valuable in middle school classrooms. A number talk engages students in using number relationships and the structures of numbers to do mental computations, followed by sharing their various strategies (Humphreys & Parker, 2015; Parrish, 2014). These brief routines are praised for developing students' number sense as well as their enthusiasm for thinking about and sharing their strategies (Parrish, 2011). Number talks that are appropriate for middle school classrooms could involve computation with decimals, fractions, and percents as well as integers. Also, McCoy, Barnett, and Combs (2013) share seven mathematical routines that can be easily used with a variety of mathematical content and at various grade levels. One routine, *Alike and Different*, requires students to consider how two or more numbers, shapes, properties, graphs, equations, and so on are similar and different. Having students explain and justify their reasons for the similarities and differences they identify is valuable experience in constructing mathematical arguments. McCoy, Barnett, and Combs (2013) also identify ways to increase the cognitive demand of tasks as students become familiar with a given routine.

Once the routine is introduced and understood by students, the *before* part of the mini-lesson involves posing the task to the students and ensuring they understand the vocabulary and any context used. In the *during* portion of the mini-lesson, students spend time developing their own ideas about the task. You can then have the students pair with a classmate and discuss each other's strategies. You can also have small groups discuss or go straight to the whole class to share and compare strategies. Just keep in mind that think-pair-share provides an opportunity to test out ideas and to practice articulating them. For ELLs, students with learning disabilities, and students who are reluctant to participate in larger groups, this offers both a nonthreatening chance to speak and an opportunity to practice what they might later say to the whole class. Like in longer lessons, the *after* part of the mini-lesson involves students sharing, justifying, questioning, and looking for connections between ideas.

Math Stations

Sometimes a mathematical concept or topic can be explored by having students work on different tasks at various classroom locations or *math stations*. Students can work on concepts or topics at math stations as an initial introduction, as a midway exploration, or as a follow-up task that provides practice or allows extension. Because you can decide which students will be assigned to which stations, you can differentiate the content at each station. For example, each station can use different manipulatives, situations, or technology, require students to use a different approach to solve a problem, or vary in terms of the difficulty of the task (e.g., different stations can use different numbers, equations, graphs, and so on that change the level of difficulty). (For more information regarding differentiation see Chapter 4.)

For a given topic, you might prepare four to eight different activities (you can also use the same activity at two different math stations). However, be sure to keep the stations focused on the same topic or concept so that you can help students build connections across the stations. Using stations that focus on a variety of topics will more likely result in a disconnected learning experience for students.

When using math stations, it is still important to think about the *before* phase in which you elicit prior knowledge, ensure the task is understood, and establish clear expectations. For example, to ensure greater student success at the stations, model what happens at each station and review any necessary skills and vocabulary. The *during* phase is still the time where students engage in the task, but with math stations, they stop and rotate to new stations within this phase of the lesson. It is still important to interact with and ask students questions as they engage in tasks from the different stations and even more important to keep track of their strategies, including those you will later highlight. In the *after* phase, you may decide to focus on one particular station, begin with the least challenging station and progress to the most challenging one, or instead of discussing each station, ask students to talk about what they learned about the target topic or concept.

A good task for a math station is one that can be repeated multiple times during one visit. This allows students to remain engaged until you are ready for them to transition to another station or activity. For example, at one station students might play a "game" in which they explore the probability of rolling particular sums. In this game, each student has a game board that shows a row of squares numbered 1 to 12 and, say, seven chips. Each student places his or her chips on numbers of their choice. The students can place all of their chips on one number or place single chips on different numbers. It is their choice. Once players have placed their chips, they take turns rolling two dice. If the student who rolled the dice has a chip on the number that is the sum of the dice, he or she can remove the chip. If the student has multiple chips on that number, he or she can remove only one chip at a time. The goal is to be the first person to remove all the chips from your board. Have students discuss what they notice about which sums seem to occur more often and less often than others. Students can play this game multiple times as they reflect on what is happening across each game and why.

1	2	③	4	⑤	⑥	⑦	8	9	⑩	11	12

Technology-enhanced tasks on the computer or interactive whiteboard that can be repeated can provide the focus of a station, but these tasks must be carefully selected. Among other aspects, you will want to choose technology-based tasks that require students to engage in reflective thought. For example, the applet at Illuminations titled "Visualizing Transformations" offers students opportunities to explore geometric transformations with dynamic software. The dynamic nature of the software allows students to choose a particular transformation and then observe the results of the transformation on a given object. Students can change the orientation of a shape by dragging a vertex, change the distance of a translation and the distance between an object and its reflected image, and change the point as well as the angle of a rotation. As they engage in this interactive environment, they are enhancing their understanding of the results of various transformations. Even if a math station, such as one that uses this applet, is used for independent work, the three-phase model can be implemented by placing a series of reflective questions at the station for students to use as they participate in the tasks.

Life-Long Learning: An Invitation to Learn and Grow

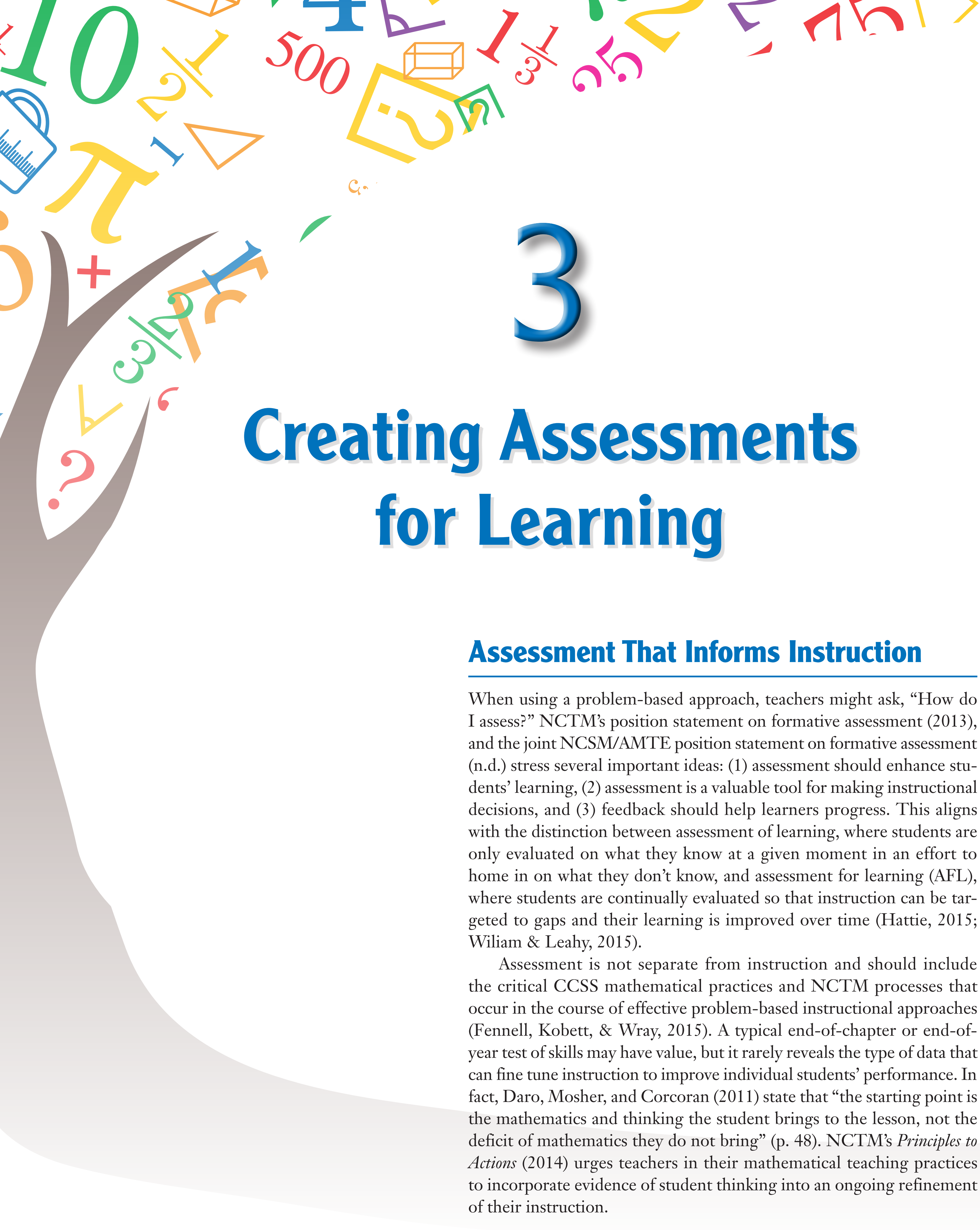
In her book *Building A Better Teacher* (2015), Elizabeth Green attacks the myth of the natural-born teacher—the common notion that good teachers are good because of an innate ability for teaching. Instead she develops the case that teaching is a complex craft that must be taught and developed over time. No matter where you are in your journey as a teacher, there is always more to learn about the content and methodology of teaching mathematics. In fact, the mathematics content and teaching described in this book may not be similar to what you experienced as a student in grades K–8. We know a great deal more about teaching and learning mathematics than we did even five years ago! Just as we would not expect doctors to be using the exact same techniques and medicines that were prevalent when you were a child, teachers' methods should be transformed through the powerful collection of expert knowledge about how to design effective instruction based on such things as how the mind functions and the influence of motivation on learning (Wiggins, 2013).

Planning three-phase inquiry lessons using worthwhile tasks and ensuring that the lesson meets the needs of all students requires intentional and ongoing effort. Questions are likely to surface. You can access responses to seven of the most commonly asked questions about problem-based teaching approaches by **clicking here**. These questions and responses may help as you contemplate how to plan for your students, or consider ways to advocate for teaching through problem solving with other teachers, families, and/or administrators.

The best teachers are always trying to improve their practice through reading the latest article, reading the newest book, attending the most recent conference, or participating in the next series of professional development opportunities. Highly effective teachers never stop learning – they never exhaust the number of new mental connections that they can make. As a result, they never experience teaching as a stale or stagnant profession.

Stop and Reflect

Describe what is (and isn't) meant by "teaching mathematics *through* problem solving." What do you foresee to be opportunities and challenges to implementing problem-based mathematics tasks effectively in your classroom?



3

Creating Assessments for Learning

Assessment That Informs Instruction

When using a problem-based approach, teachers might ask, "How do I assess?" NCTM's position statement on formative assessment (2013), and the joint NCSM/AMTE position statement on formative assessment (n.d.) stress several important ideas: (1) assessment should enhance students' learning, (2) assessment is a valuable tool for making instructional decisions, and (3) feedback should help learners progress. This aligns with the distinction between assessment of learning, where students are only evaluated on what they know at a given moment in an effort to home in on what they don't know, and assessment for learning (AFL), where students are continually evaluated so that instruction can be targeted to gaps and their learning is improved over time (Hattie, 2015; Wiliam & Leahy, 2015).

Assessment is not separate from instruction and should include the critical CCSS mathematical practices and NCTM processes that occur in the course of effective problem-based instructional approaches (Fennell, Kobett, & Wray, 2015). A typical end-of-chapter or end-of-year test of skills may have value, but it rarely reveals the type of data that can fine tune instruction to improve individual students' performance. In fact, Daro, Mosher, and Corcoran (2011) state that "the starting point is the mathematics and thinking the student brings to the lesson, not the deficit of mathematics they do not bring" (p. 48). NCTM's *Principles to Actions* (2014) urges teachers in their mathematical teaching practices to incorporate evidence of student thinking into an ongoing refinement of their instruction.

Assessments usually fall into one of two major categories: summative or formative. *Summative assessments* are cumulative evaluations that take place usually after instruction is completed. They commonly generate a single score, such as an end-of-unit test or a standardized test that is used in your state or school district. Although the scores are important for schools and teachers, used individually they often do not help shape day-to-day teaching decisions.

On the other hand, *formative assessments* are assessments that are used to check students' development during instructional activities, to preassess, or to attempt to identify students' naïve understandings or misconceptions (Hattie, 2009; Popham, 2008; Wiliam & Leahy, 2015). When implemented well, formative assessment is one of the most powerful influences on achievement (Hattie, 2009). It dramatically increases the speed and amount of student learning (Nyquist, 2003; Wiliam, 2007; Wilson & Kenney, 2003) by providing targeted feedback to students and using the results and evidence collected to inform your decision making about next steps in the learning progression.

Wiliam (2010) notes three key processes in formative assessment: (1) Identify where learners are; (2) Identify goals for the learners; and (3) Identify paths to reach the goals. Let's look at an example of this process for sixth graders on a proportional reasoning problem.

Jeff and Pamela are painting fences. Jeff's fence is 8 sections long, and he has 4 sections painted. Pamela's fence is 9 sections long, and she has 5 sections painted. Who is closer to finishing the painting project? Explain how you know.

Students then individually work to complete this task on paper, as a written record of their process and responses. After students have worked, the task is discussed. One student states, "They are the same. They both need to paint four more sections." Another student draws two lines on a paper and says, "Pamela. She is one-half of a section more than halfway done. Jeff is only halfway done."

The information gathered from observing these students reveals very different paths for the next steps. This teacher is at the first step in Wiliam's three key processes, noting where students are in their learning. Moving into the second step, the teacher notes that the first student is still thinking additively, and will need to develop proportional reasoning through more targeted instruction, while the latter student is thinking multiplicatively, and is therefore ready to move to more challenging proportional tasks.

If summative assessment can be described as a digital snapshot, formative assessment is like streaming video. One is a picture of what a student knows that is captured in a single moment of time; the other is a moving picture that demonstrates active student thinking and reasoning. In the following pages and throughout Part 2 of this book in Formative Assessment Notes, we focus on four basic methods for using of formative assessments to evaluate students' understanding: observations, questions, interviews, and tasks. Here we discuss each method in depth.

Observations

All teachers learn useful bits of information about their students every day. When the three-phase lesson format suggested in Chapter 2 is used, the flow of evidence about student performance increases dramatically, especially in the *During* and *After* portions of lessons. If you have a systematic plan for gathering this information while observing and listening to students during regular classroom instruction, at least two very valuable results occur. First, information that may have gone unnoticed is suddenly visible and important. Second, observation data gathered systematically can be combined with other data and used in planning lessons, providing feedback to students, conducting parent conferences, and determining grades.

Depending on the information you are trying to gather, several days to two weeks may be required to complete a single observation of how a whole class of students is progressing on a standard. Shorter periods of observation will focus on a particular cluster of concepts or skills or on particular students. Over longer periods, you can note growth in mathematical processes or practices, such as developing problem solving, representation, or reasoning. To use observation effectively, take seriously the following maxim: Only try to collect data on a reasonable number of students in a single class period.

Anecdotal Notes

The act of *professional noticing* is a process where you observe learners through a focus on three phases: (1) attending; (2) interpreting; and (3) deciding (Jacobs, Lamb, & Philipp, 2010). You want to collect anecdotal notes on these phases to help understand learners' strategies and thinking as you plan potential next steps. That means you *attend* to everything such as if the student is nodding his head or if she is using her fingers to count, or if she is creating appropriate models, or if he is using strategies that he can clearly describe and defend. Then *interpret* those student gestures, comments, drawings, and actions by making notes of possible strengths and the level of sophistication of their conceptual understanding. Then you can move to noting *decisions* for subsequent instructional actions.

One system for recording your professional noticing is to write these anecdotal notes on an electronic tablet and store them in a spreadsheet or in a multicolumn table that documents such things as students' use of mathematical practices. In any case, focus your observations on approximately five students a day. The students selected may be members of one or two cooperative groups or a group previously identified as needing additional support.

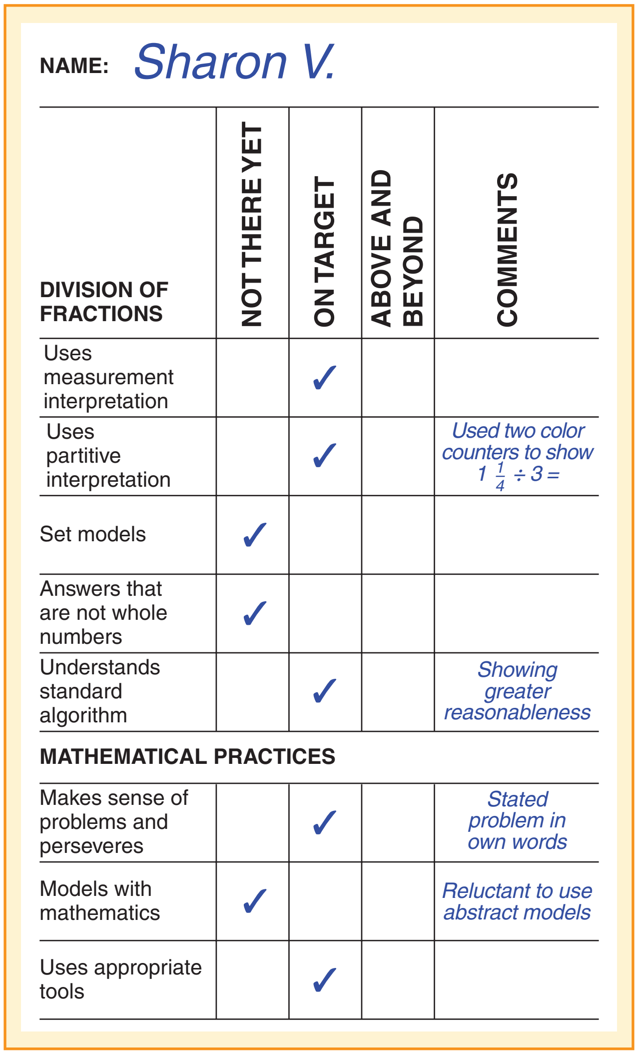
Checklists

To help focus your attention, a checklist with several specific processes, mathematical practices, or content objectives can be devised (see Figure 3.1). As you can see there is a place for comments that should concentrate on big ideas and conceptual understanding. For example, you will probably find a note such as "is beginning to estimate answers to multiplication of decimals to locate the proper placement of the decimal" more useful than "cannot accurately multiply decimals."

Another Observation Checklist involves listing all students in a class on one to three pages (see Figure 3.2). Across the top of the page are specific abilities or common misconceptions to look for possibly based on learning progressions. Pluses and minuses, checks, or codes can be entered in the grid. A full-class checklist is more likely to be used for long-term objectives such as problem-solving processes, strategic use of representations or tools, and such skill areas as basic fact fluency or computational estimation. Dating entries or noting specifics about observed performance is also helpful.

Figure 3.1

A focused checklist and rubric that can be printed for each student.



NAME: Sharon V.

DIVISION OF FRACTIONS	**NOT THERE YET**	**ON TARGET**	**ABOVE AND BEYOND**	**COMMENTS**
Uses measurement interpretation		✓		
Uses partitive interpretation		✓		*Used two color counters to show* $1\frac{1}{4} \div 3 =$
Set models	✓			
Answers that are not whole numbers	✓			
Understands standard algorithm		✓		*Showing greater reasonableness*
MATHEMATICAL PRACTICES				
Makes sense of problems and perseveres		✓		*Stated problem in own words*
Models with mathematics	✓			*Reluctant to use abstract models*
Uses appropriate tools		✓		

Figure 3.2

A class observation checklist.

Transformations *Mental computation adding 2-digit numbers* Names	Not There Yet *Can't do mentally*	On Target *Has at least one strategy*	Above and Beyond *Uses different methods with different numbers*	Comments
Lalie		✓ *3-18-2017* *3-21-2017*		
Pete	✓ *3-20-2017*	✓ *3-24-2017*		*Difficulty with any problems using rotation*
Sid			✓+ *3-20-2017*	*Can do dilations on grid*
Lakeshia		✓		*Struggles with reflections on a coordinate grid*
George		✓		
Pam	✓			*Beginning to use slides and reflections*
Maria		✓ *3-24-2017*		

Questions

Probing student thinking through questioning can provide better data and more insights to inform instructional next steps. As you circulate around the classroom to observe and evaluate students' understanding, your use of questions is one of the most important ways to formatively assess in each lesson phase. Have these **Question Probes** on a tablet or in print as you move about the classroom to prompt and probe students' thinking.

To make sure you are asking critical thinking questions you may want to consider videotaping yourself and having a friend score how many high level or how many recall questions you are asking. Use a matrix such as the Cognitive Rigor Matrix (Simpson, Mokalled, Ellenburg, & Che, 2014/2015) which is a blend of Bloom's Taxonomy and Webb's Depth of Knowledge (2002) for additional information.

Getting students used to responding to these questions (as well as accustomed to asking questions about their thinking and the thinking of others) helps prepare them for the more intensive questioning used in interviews.

Interviews

"An assessment system designed to help steer the instruction system must give good information about direction as well as distance to travel. A system that keeps telling us we are not there yet is like a kid in the back seat whining 'are a we there yet?' " (Daro et al., 2011, p. 51). Interviews, particularly diagnostic interviews, are a means of getting in-depth information about a student's knowledge of concepts and strategy use to provide needed navigation. The diagnostic interview is usually a one-on-one investigation of a student's thinking about a particular concept, process, or mathematical practice that lasts from three to ten minutes. The challenge of diagnostic interviews is that they are assessment opportunities, not teaching opportunities, making it hard to watch students make errors and not respond immediately. The interviews are used to listen to students' descriptions of their strategies and probe their understanding with the purpose of discovering both strengths and gaps.

To start, select a problem that matches an essential understanding for the topic students are studying and have paper, pencils, and a variety of materials available (particularly manipulatives used during previous instruction). Also, be ready to jot down notes about emerging understandings, common methods you expect students to use, or common misunderstandings that you anticipate. Then ask the student to solve the problem and make sure the student verbalizes his or her thinking at several points. Encourage multiple representations by asking the student to demonstrate his or her thinking using materials or drawings. Fennell, Kobett, and Wray call this the "show me" approach (2015, p. 56).

Teaching Tip

These interviews can be time intensive but they have the potential to provide information that you simply cannot get in any other way. So how can you accomplish this? Think of these interviews as tools to be used for only a few students at a time, not for every student in the class. Briefly interview a single student while others work on a task or are in learning centers. Some teachers work with one student at an interactive whiteboard and record the whole conversation, any written work, or use of virtual manipulatives. Other times a paraprofessional or student teacher can interview.

Sometimes students self-correct a mistake but, more frequently, you can unearth a student's misunderstanding or reveal what strategies are mastered. When you focus on exploring common errors and pitfalls, you can then build greater sophistication in students' conceptual understanding (Bray & Santagata, 2014). Examples of diagnostic interviews might include tasks such as

- What is the unit rate (cost per cap) if we paid $102.00 for 17 caps?
- Solve $6\frac{1}{4} \times 2\frac{3}{4} =$ ________. Draw a picture to illustrate the solution.
- [The student is given an isosceles right triangle drawn on cm grid paper, where the sides are 8 cm]. Find the length of the hypotenuse of this triangle.

Also, see a **Sample Interview for Middle Grades**, and/or **Student Observation and Interview: Learning through Problems**.

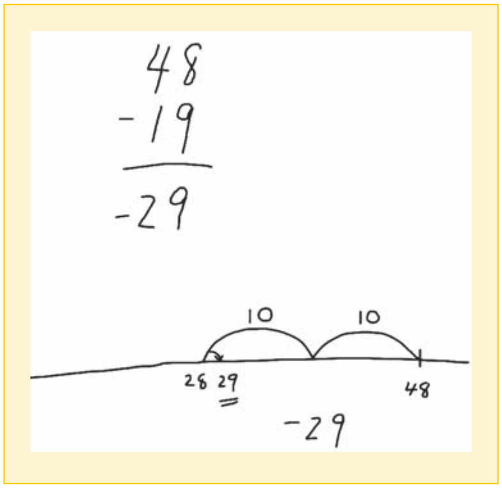
After examining hundreds of research studies, Hattie (2009) found that feedback that teachers received from students on what they knew and did not know was critical in improving students' performance. That is precisely what diagnostic interviews are designed to do! For example, are you sure that your students have a good understanding of operations with integers, or are they just doing exercises according to poorly understood procedures? Remediation will be more successful if you can pinpoint why a student is having difficulty before you try to fix the problem.

Let's look at an actual example of a diagnostic interview.

Mr. Jergens noticed Jeremy was demonstrating difficulty with subtracting integers. Jeremy gave unreasonable estimates and was not able to predict whether the result would be negative or positive. To better understand Jeremy's thinking in terms of what he understood and where some gaps might be, Mr. Jergens planned a diagnostic interview. Using an adaptation of a problem he read in a journal (Tillema, 2012), he asked Jeremy to write the answer to $(+48) - (-19) =$ __. Jeremy wrote the problem vertically, subtracted, and quickly answered (-29). Mr. Jergens resisted the temptation to immediately correct him, and instead probed further asking Jeremy to talk about his thinking. Jeremy said that he needed to take away the 19 from the 48 and, because there was a negative sign, needed to show the "opposite" as the answer, so it was "negative 29." Second, Mr. Jergens asked Jeremy to show the computation on a number line, trying to stimulate some of the ideas discussed in a previous math period. Showing fluency with the "jump" movements on an open-number line, Jeremy used a "take-away" approach and reached the same answer (see Figure 3.3). Third, Mr. Jergens asked Jeremy to tell a story to go with the problem. Jeremy responded that it was 48 degrees one day and 19 degrees below zero the next. What was the difference in temperatures? Mr. Jergens asked Jeremy to compare his story to the number line jumps he just illustrated, asking where those temperatures would be located on the number line and what the difference was between them. The student said, "67? I am not sure." Mr. Jergens asked why he thought he had

Figure 3.3

Student's work on a diagnostic interview task.



gotten two different answers and which answer he thought was correct. The student quietly pondered and then pointed to the original equation and said, "This one is right."

Although this interview revealed that the student had a good grasp of the use of a number line for whole-number subtraction, it did reveal gaps in his full understanding of subtraction of integers. In the first two questions, Jeremy focused on a "take-away" model of subtraction, but in his story used a "difference" model (what is the difference or distance between these two numbers?). Notice that Mr. Jergens used one problem for the interview, but asked for the student to consider three different approaches to that problem. Importantly, he asked Jeremy to relate his thinking from one representation (story) to another (number line). The cognitive dissonance caused by getting two different answers provided more insights into Jeremy's thinking, in particular showing that Jeremy has more confidence that an algorithm will get the "right" answer than other representations. Planning could then begin for future instruction based on the actual evidence from the student.

There is no one right way to plan or structure a diagnostic interview. In fact, flexibility is a key ingredient. You should, however, have an overall plan that includes an easier task and a more challenging task in case you have misjudged your starting point. Also, did you notice that Ms. Jergens had appropriate instructional materials ready for the student to use? Be sure you have materials available that have potential to provide insight into students' understanding. Also be prepared to probe students' thinking with question like these:

- Can you explain what you just did?
- How would you explain this to a sixth grader?
- What does this [point to something on the paper] stand for?
- Why did you solve it that way?
- Can you show me what you are thinking with the materials?
- Why do you think you got two different answers? Which one do you think is correct?
- If you tried to do this problem again, which approach would you try first?

In each case, it is important to explore whether students (1) understand what he or she did, and (2) can describe connections between approaches and/or representations.

Consider the following suggestions as you implement your diagnostic interview:

- *Avoid revealing whether the student's answer is right or wrong.* Often facial expressions, tone of voice, or body language can give a student clues. Instead, use a response such as "Can you tell me more?" or "I think I know what you are thinking."
- *Wait silently for the student to give an answer.* Give ample time to allow a student to think and respond. Only then should you move to rephrasing questions or probing for a better understanding of a student's thoughts. After the student gives a response (whether it is accurate or not), wait again! This second wait time is even more important because it encourages the student to elaborate on initial thoughts and provide more information.
- *Avoid interjecting clues or teaching.* The temptation to interrupt is sometimes overwhelming. Watch and listen. Your goal is to use the interview not to teach but to find out where the student is in terms of conceptual understanding and procedural fluency.

- *Give opportunities for students to share their thinking without interruption.* Encourage students to use their own words and ways of writing things down. Correcting language or spelling words can sidetrack the flow of students' explanations.

The benefits of diagnostic interviews become evident as you plan instruction that capitalizes on students' strengths while recognizing possible weaknesses and confusion. Also, unlike large-scale testing, you can always ask another question to find out more when the student is taking an incorrect or unexpected path. You may also discuss results of interviews with colleagues to gain shared insights (Stephan, McManus, & Dehlinger, 2014). These insights are invaluable in moving students to mathematical proficiency, as there is perhaps no better method for developing instruction that supports student understanding than having students explain their thinking and a team of teachers sharing a conversation about evidence.

Tasks

Tasks refer to products that include performance-based tasks, writing, and student self-assessments. Good assessment tasks for either instructional or formative assessment purposes should permit every student, regardless of mathematical ability, to demonstrate his or her knowledge, skill, or understanding.

Problem-Based Tasks

Problem-based tasks are tasks that are connected to actual problem-solving activities used in instruction. When problem-based tasks are used for assessment and evaluation, the intent is to find what students do know (e.g., students can solve linear equations when given in a context), rather than just identify what they do not know (e.g., students can't analyze and solve linear equations). The result is a broad description of the ideas and skills that students possess—for example, "Adam can write a linear equation given a graph of several different functions but has difficulty calculating the slope." High-quality tasks permit every student to demonstrate their abilities (Rigelman & Petrick, 2014; Smith & Stein, 2011) and include real-world or authentic contexts that interest students or relate to recent classroom events. Of course, be mindful that English language learners may need support with context as challenges with language should not overshadow the attention to their mathematical ability.

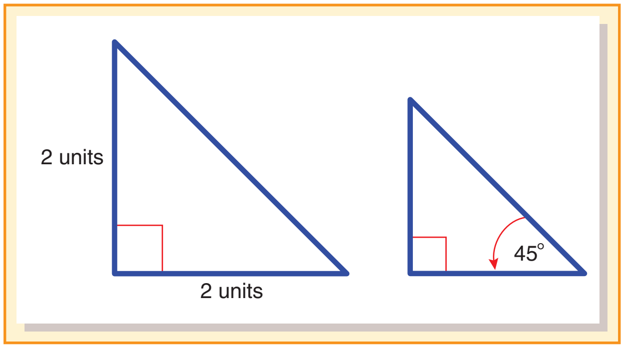
Problem-based tasks have several critical components that make them effective. They:

- Focus on an important mathematics concept or skill aligned to valued learning targets.
- Stimulate the connection of students' previous knowledge to new content.
- Allow multiple solution methods or approaches that incorporate a variety of tools.
- Offer opportunities for students to correct themselves along the way.
- Provide occasions for students to confront common misconceptions.
- Encourage students to use reasoning and explain their thinking.
- Create opportunities for observing students' use of mathematical processes and practices.
- Generate data for instructional decision making as you listen to your students' thinking.

Notice that the following examples of problem-based tasks are not elaborate, yet when followed by a discussion, each can engage students for most of a class session (also see **Problem-Based Tasks**). What mathematical ideas and practices are required to successfully respond to each of these tasks?

TWO TRIANGLES (GRADES 7-8):

Learning targets: (1) Classify two-dimensional shapes into categories based on their properties. (2) Attend to precision by clearly applying definitions to define categories.



Tell everything you can about these two triangles. Given what you wrote about the two triangles, determine which of the following statements are true: The large triangle is an isosceles triangle; the small triangle is an isosceles triangle; the big triangle has an area of 2 square units; the small triangle has an area of 1 square unit; the large triangle has at least one angle that measures 45 degrees; the small triangle has at least one angle that measures 30 degrees; the two triangles are similar. Explain your thinking.

ALGEBRA GRAPHING FUNCTIONS (GRADE 8):

Learning targets: (1) Compare and analyze quadratic functions. (2) Build a logical argument for a conjecture using reasoning.

Does the graph of $y = x^2$ ever intersect the graph of $y = x^2 + 2$? What are some ways that you could test your conjecture? Would your conjecture hold true for other equations in the form of $y = x^2 + b$? Within all quadratic functions of the form $y = ax^2 + b$, when would your conjecture hold true?

Much can be learned about students' understanding in a discussion that follows students solving the task individually. Watch this video called **My Favorite No** (**www.youtube.com/watch?v=Rulmok_9HVs**) to see how a teacher uses students' incorrect answers to develop a conversation about mathematics. Students must develop the habit of sharing, writing, and listening to justifications. In particular, it is important for students to compare and make connections between strategies and debate ideas in order to assist them in organizing their thoughts, thinking about their position, and analyzing the positions of others.

Translation Tasks

One important assessment option is what we refer to as a *translation task*. Using four possible representations for concepts (see Figure 2.3), students are asked to demonstrate understanding using words, models, and numbers for a single problem. As students move between these representations, there is a better chance that a concept will be formed correctly and integrated into a rich web of ideas.

So what is a good way of structuring a translation task? With use of a template based on a format for assessing concept mastery from Frayer, Fredrick, and Klausmeier (1969) (see Figure 3.4) and the **Translation Task Template**, you can give students a computational equation and ask them to:

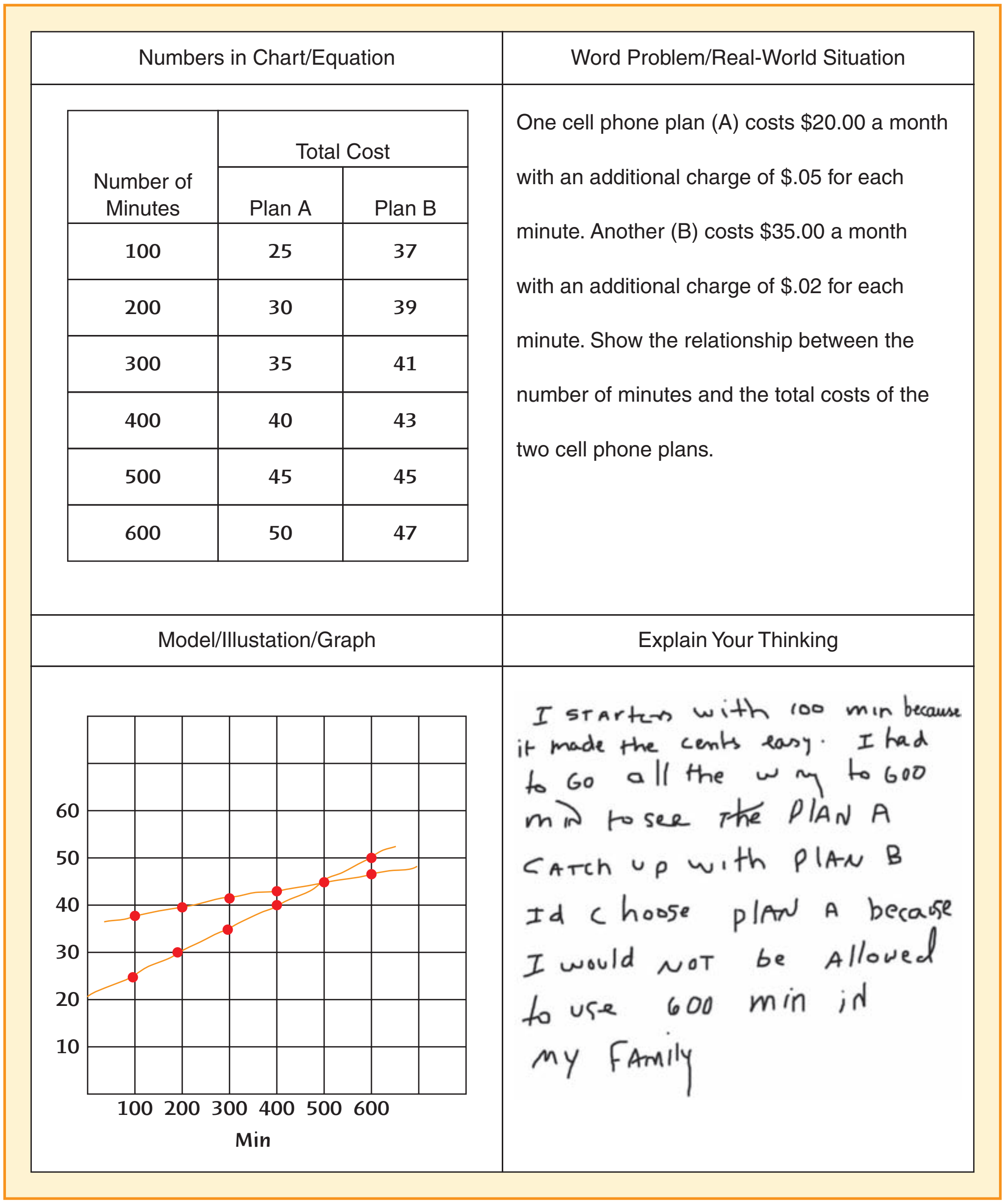
- Write a word problem that matches the equation
- Illustrate the equation with materials or drawings
- Explain their thinking about the process of arriving at an answer

In particular, students' ability to communicate how they solved a problem is critical for open-response questions on many summative assessments (Parker & Breyfogle, 2011).

Translation tasks can be used for whole-class lessons or for individual or small-group diagnosis. For example, seventh-grade students might be given a situation about the cost of cell phone plans in the section titled "Word Problem/Real-World Situation" in Figure 3.4. Their task could be to create a table of values for different numbers of minutes in the "Numbers in Chart/Equations" area, use the values to make a graph in "Model/Illustration/

Figure 3.4

Translation task template with example task.



Numbers in Chart/Equation

Number of Minutes	Total Cost	
	Plan A	Plan B
100	25	37
200	30	39
300	35	41
400	40	43
500	45	45
600	50	47

Word Problem/Real-World Situation

One cell phone plan (A) costs $20.00 a month with an additional charge of $.05 for each minute. Another (B) costs $35.00 a month with an additional charge of $.02 for each minute. Show the relationship between the number of minutes and the total costs of the two cell phone plans.

Model/Illustation/Graph

Explain Your Thinking

I STArted with 100 min because it made the cents easy. I had to Go all the way to 600 min to see the PlAN A CATch up with PlAN B Id choose plAN A because I would NOT be Allowed to use 600 min in my FAmily

Graph" and explain to another person in writing what plan would be the best under what circumstances (for which number of minutes a month is each plan best?) in "Explain Your Thinking." Students could also be asked to write the function that goes with each of the plans (translating to written symbols) in "Numbers in Chart/Equation."

Think about translation tasks when you want to find out more about students' thinking. If students represent ideas in various forms and can explain why these representations are similar or different, you can use this valuable information to recognize misconceptions they may have and then identify the type of activity you can provide to advance learning. Here are two other options for tasks that can be used with the translation. Remember that a translation task may start in any quadrant of the template, and then the student proceeds to fill in the other three sections.

Consider the following two "starters":

- In the "Equation" box, write this:

  $3x + 2 = 11$. Solve for x.

  Students then make a corresponding model, create a word problem, and explain how they solved the problem in the three other areas.
- In the section for "Real-World Situation" write this:

  Seven soccer teams are playing in a tournament. Each team will play all the others only once. How many games will be played?

  Students should write a corresponding equation, make an illustration or model to demonstrate the solution to this problem, and in the last section explain to a friend how they approached the problem.

Writing

As an assessment tool, writing in journals, exit slips, or other formats provide a unique window to students' perceptions and the way they are thinking about an idea. **Watch this video** (**https://www.youtube.com/watch?v=dIk0LEmtHl4**) on using exit slips. Students can make sense of problems, express early ideas about concepts, unearth confusion, connect representations, justify a claim with evidence, or even clarify strategy use (Casa, 2015). When students explain their thinking about their solutions to a task in writing prior to class discussions, the written record serves as a rehearsal for the class conversation. Students who otherwise have difficulty thinking on their feet now have a script to support their contributions. Call on more reluctant talkers first so that their ideas are heard and valued. They can also summarize a learning situation through such prompts as:

Concepts and Processes

- Write an explanation for a new or younger student of when does the ratio of 3 : 6 mean the same thing as 4 : 8 and when do they mean something different?
- Explain to a student who was absent today what you learned about how slope describes the relationship between one variable and another.
- If you were stuck today in solving a problem, where in the problem did you have trouble?
- After you got the answer to today's problem, what did you do so that you were convinced your answer was reasonable? How sure are you that you got the correct answer?
- Write a word problem that goes with this (equation, graph, diagram, model).

In Mathematical Practice 3, mathematically proficient students "build a logical progression of statements to explore the truth of their conjectures" and are able to "justify their conclusions, communicate them to others, and respond to the arguments of others" (NGA, 2010, pp. 6–7). Helping students pull evidence to show how they answered a problem often requires showing them the work of other students. By showing exemplars and counterexamples from real or "created" peers, students begin to identify elements of a sound argument and cohesive communication (Lepak, 2014).

Additionally, student writing is an excellent form of communication with parents during conferences. Writing shows evidence of students' thinking, telling parents much more than any grade or test score.

Students' Self-Assessment and Reflection

Wiliam (2015) stresses that a key strategy in effectively using formative assessment is the activation of students as "owners of their own learning" (p. 169). Stiggins (2009) suggests that students should be informed partners in understanding their progress in learning and how to enhance their growth in understanding concepts. They should use their own assessment results to move forward as learners as they see that "success is always within reach" (p. 420). Student self-assessment should not be your only measure of students' learning or dispositions, but rather a record of how *students perceive* their strengths and weaknesses as they begin to take responsibility for their learning.

You can gather student self-assessment data in several ways, including preassessments that catch areas of confusion or misconceptions prior to formally assessing students on particular content or by regularly using exit slips (paper slips or a web application with a quick question or two) when students are concluding the instructional period (Wieser, 2008). **Watch this video** (**https://www.youtube.com/watch?v=1ejQcTTwEtA**) that is applicable to any grade to see a "stoplight" approach that shows how a teacher uses students' responses to see how they evaluate where they are.

As you plan for student self-assessment, consider what you need to know to help you find better instructional strategies and revised learning targets. Convey to your students why you are having them do this activity—they need to grasp that they must play a role in their mastery of mathematics rather than just focus on completing a task. Encourage them to be honest and candid. Use open-ended prompts such as:

- How well do you think you understand the work we have been doing on integers during the last few days? What is still causing you difficulty?
- Write two important things you learned in class today (or this week).
- Which problem(s) on the activity sheet/quiz did you find most challenging? Which were easiest?

Discussions of how students can improve can start when they analyze their own mistakes or have discussions with other students about which answers they think are correct. When students get back a test—make sure they use the feedback and revisit any errors and confirm that they understand what they need to learn next or how to revise. This attention to using feedback and mistakes to improve one's understanding moves students from a performance orientation to a mastery orientation (Pintrich, 2003).

Although in general, it takes additional time to infuse students' self-assessments and formative assessment into the daily schedule, allowing students to take part in the assessment process is motivating and encourages them to monitor and adapt their approaches to learning. Remember, start the process of incorporating these assessment ideas in this chapter over time building strategy by strategy (Petit & Bouck, 2015) and growing your ability to effectively assess students.

Rubrics and Their Uses

Problem-based tasks tell us a great deal about what students know, but how do we analyze and use this information? These assessments yield an enormous amount of information that must be evaluated by examining more than a simple count of correct answers. A *rubric* is a scale based on predetermined criteria with two important functions: (1) It permits students to see what is central to excellent performance, and (2) it provides you with scoring guidelines that support equitable analysis of students' work.

In a teaching-through-problem-solving approach, you will often want to include criteria and performance indicators on your rubrics such as the following:

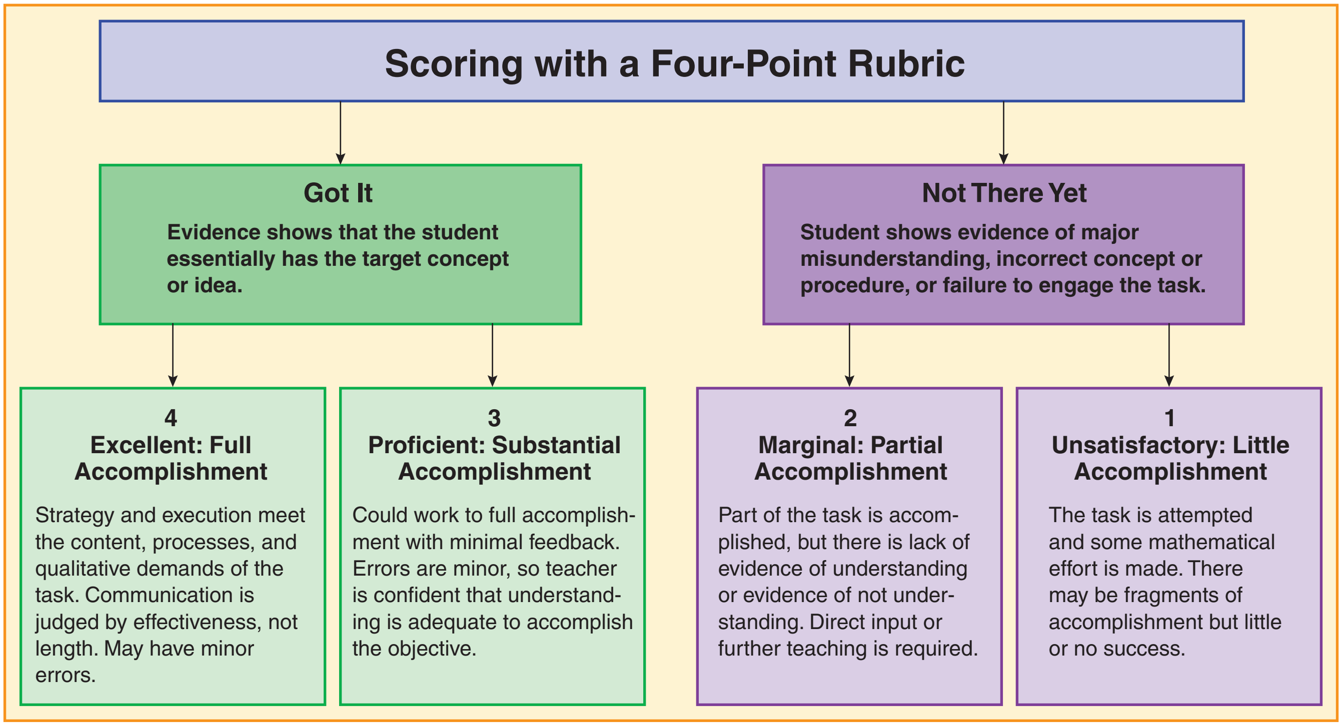
- Solved the problem(s) accurately and effectively.
- Persevered and demonstrated resilience when facing a challenging problem.
- Justified and explained strategy use or arguments.
- Used logical reasoning.
- Expressed a grasp of numerical relationships and/or mathematical structure.
- Incorporated multiple representations and/or multiple strategies.
- Demonstrated an ability to appropriately select and use tools and manipulatives.
- Communicated with precise language and accurate units.
- Identified general patterns of ideas that repeat, making connections from one big idea to another.

Generic Rubrics

Generic rubrics identify categories of performance instead of specific criteria for a particular task and therefore can be used for multiple assignments. The generic rubric allows a teacher to score performances by first sorting into two broad categories, as illustrated in the **four-point rubric** shown in Figure 3.5. Then you to separate each category into two additional levels as shown. A rating of 0 is given for no response, no effort, or for responses that are completely off task. The advantage of this scale is the relatively easy initial sort into "Got It" or "Not There Yet."

Figure 3.5

A four-point generic rubric.



Scoring with a Four-Point Rubric

Got It

Evidence shows that the student essentially has the target concept or idea.

Not There Yet

Student shows evidence of major misunderstanding, incorrect concept or procedure, or failure to engage the task.

4
Excellent: Full Accomplishment

Strategy and execution meet the content, processes, and qualitative demands of the task. Communication is judged by effectiveness, not length. May have minor errors.

3
Proficient: Substantial Accomplishment

Could work to full accomplishment with minimal feedback. Errors are minor, so teacher is confident that understanding is adequate to accomplish the objective.

2
Marginal: Partial Accomplishment

Part of the task is accomplished, but there is lack of evidence of understanding or evidence of not understanding. Direct input or further teaching is required.

1
Unsatisfactory: Little Accomplishment

The task is attempted and some mathematical effort is made. There may be fragments of accomplishment but little or no success.

Another possibility is to use your three- or four-point generic rubric on a reusable form (see **Four-Point Rubric**), as in Figure 3.5. This method is especially useful for planning purposes. But there are times when generic rubrics do not give enough definition of specific criteria for a particular task. For those instances, try a task-specific rubric.

Task-Specific Rubrics

Task-specific rubrics include specific statements, also known as *performance indicators* that describe what students' work should look like at each rubric level and, in so doing, establish criteria for acceptable performance on that particular task (see Figure 3.6 and **Anecdotal Note Rubric**). Initially, it may be difficult to predict what student performance at different levels will or should look like, but your criteria depend on your knowledge and experience with students at that grade level and your insights about the task or mathematical concept. One important part of setting performance levels is predicting students' common misconceptions or their expected approaches to similar problems.

To facilitate developing performance levels, write out indicators of "proficient" or "on target" performances before using the task. This excellent self-check ensures that the task is likely to accomplish your purpose. If you find yourself writing performance indicators in terms of the number of correct responses, you are most likely looking at drill or practice exercises, not the problem-based tasks for which a rubric is appropriate. Like athletes who continually strive for better performances rather than "good enough," students should always recognize opportunities to excel. When you take into account the total performance (processes, strategies, answers, justifications, extensions, and so on), it is always possible to "go above and beyond."

Early in the year, discuss your rubric (such as Figure 3.6) with the class and post it prominently. Make it a habit to discuss students' performance on tasks in terms of the rubric. For example, if you are using the anecdotal note rubric, rubric language can be used informally: "George, the rubric states to get an Above and Beyond you need to solve the problem with two different representations and explain your thinking. Is that what you did?" This approach lets students know how well they are doing and encourages them to persevere by giving specific areas for improvement. You might also have students use the rubric to self-assess their work, having them explain reasons for their ratings. Then target follow-up instruction in response to their gaps and misunderstandings building on their identified strengths.

Figure 3.6

Record names in a rubric used during an activity or for a single topic over a period of several days.

Observation Rubric

Partition Regions into Equal Shares 3/17

Above and Beyond Clear understanding. Communicates concept in multiple representations. Shows evidence of using idea without prompting. Can partition rectangles and circles into two, four, and eight equal shares. Explains that partitioning the same wholes into more shares makes smaller shares.	Sally Latania Greg	Zal
On Target Understands or is developing well. Uses designated models. Can partition regions into equal shares and describes as "halves" and "fourths." May need prompt to compare halves and fourths.	Lavant Julie George Maria	Tanisha Lee J.B. John H.
Not There Yet Some confusion or misunderstanding. Only models idea with help. Needs help to do activity. No confidence.	John S.	Mary

Stop and Reflect

Consider the task "Two Triangles" on page 46. Assume you are creating a task-specific four-point rubric to share with your seventh graders. What task-specific indicators would you use for level 3 and level 4 performances? Start with a level 3 performance, then think about level 4. Try this before reading further.

Determining performance indicators is always a subjective process based on professional judgment. Here is one possible set of indicators for “Two Triangles” task:

Level 3: Accurately describes the properties of the two triangles. Gives correct results and reasoning for the first questions about the basic properties but an incorrect result for the question about similarity.

Level 4: Determines the true statements by using words, pictures, and numbers to explain and justify the results and describe how they were obtained. Demonstrates knowledge of the properties of two-dimensional shapes and the relationship to similarity.

Teaching Tip

When you return papers, review the indicators with students, including examples of correct answers and successful responses. This will help students understand how they could have done better. Often it is useful to show anonymous student work. Let students decide on the score for the anonymous student. Importantly, students need to see models of what a level 4 performance looks like.

What about level 1 and level 2 performances? Here are suggestions for the same task:

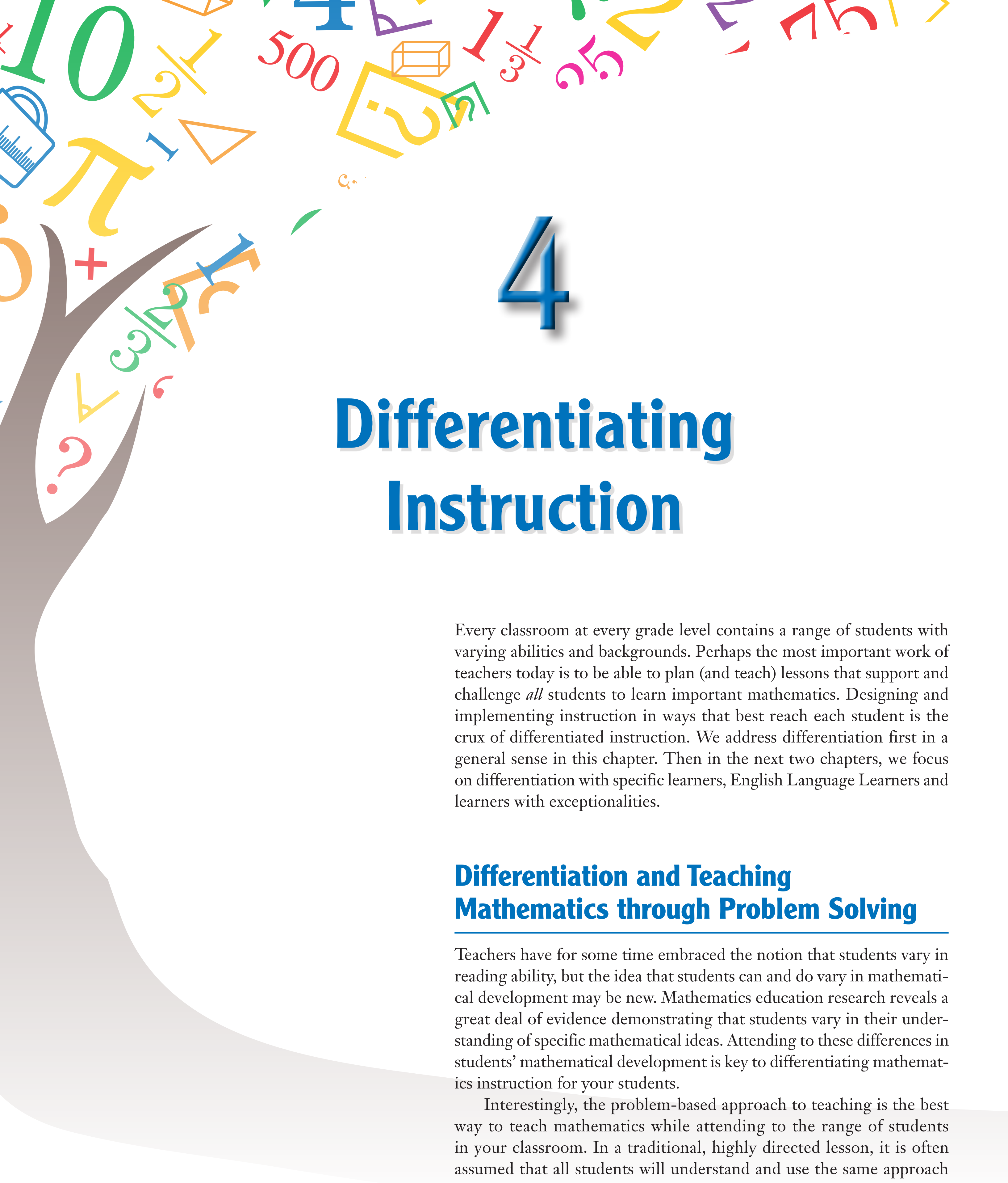
Level 2: Uses some reasoning about the properties of triangles but fails to identify all true statements and an understanding of similarity.

Level 1: Shows some effort to describe the triangle but demonstrates little or no understanding of all of the properties of triangles and the connections to similarity.

Unexpected methods and solutions happen. Don’t limit students to demonstrating their understanding only as you thought they would when there is evidence that they are accomplishing your objectives in different ways. Such occurrences can help you revise or refine your rubric for future use.

Stop and Reflect

How can having students assess peers’ work (both strong and weak responses) support their ability to generate more in-depth responses?



4

Differentiating Instruction

Every classroom at every grade level contains a range of students with varying abilities and backgrounds. Perhaps the most important work of teachers today is to be able to plan (and teach) lessons that support and challenge *all* students to learn important mathematics. Designing and implementing instruction in ways that best reach each student is the crux of differentiated instruction. We address differentiation first in a general sense in this chapter. Then in the next two chapters, we focus on differentiation with specific learners, English Language Learners and learners with exceptionalities.

Differentiation and Teaching Mathematics through Problem Solving

Teachers have for some time embraced the notion that students vary in reading ability, but the idea that students can and do vary in mathematical development may be new. Mathematics education research reveals a great deal of evidence demonstrating that students vary in their understanding of specific mathematical ideas. Attending to these differences in students' mathematical development is key to differentiating mathematics instruction for your students.

Interestingly, the problem-based approach to teaching is the best way to teach mathematics while attending to the range of students in your classroom. In a traditional, highly directed lesson, it is often assumed that all students will understand and use the same approach and the same ideas as determined by the teacher. Students not ready to understand the ideas presented by the teacher must focus their attention on following rules or directions without developing a conceptual or relational understanding (Skemp, 1978). This, of course, leads to endless difficulties and can leave students with misunderstandings or in need of significant remediation. In contrast, in a problem-based classroom, students are expected to approach problems in a variety of ways that make sense to *them*, bringing to each problem the skills and ideas that they own. So, with a problem-based approach to teaching mathematics, differentiation is already built in to some degree.

To illustrate, let's consider a sixth-grade classroom in which the teacher posed the following task to students:

Suppose you and a friend are making a drink from a powder mix. You use 3 tablespoons of the mix with every 2 cups of water, while your friend uses 4 tablespoons of mix with every 3 cups of water. Which mixture will be stronger, or will both mixtures be the same strength?

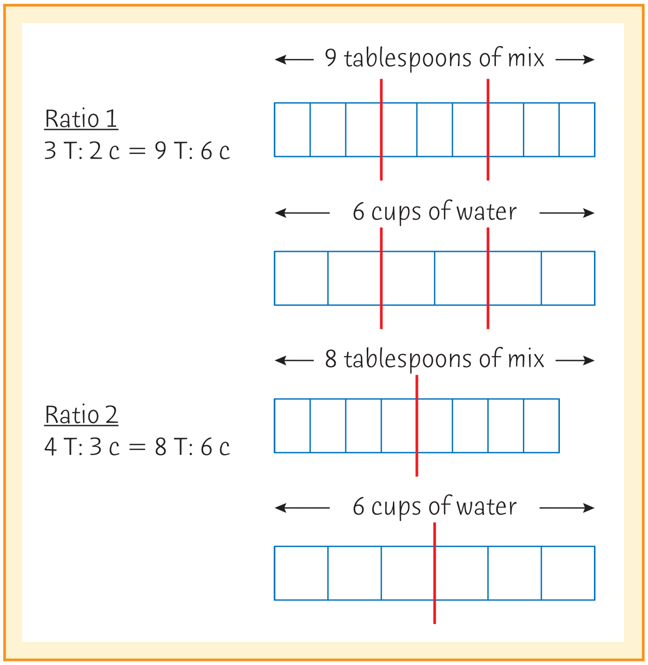
The teacher asked the students to be ready to explain how they knew. Below are some of the students' explanations:

Sam: I made it so each mixture had 6 cups of water. I knew I could do that because the first ratio used 2 cups and the second ratio used 3 cups, and 6 is a multiple of both 2 and 3. So I multiplied each part of the first ratio by 3 to get 9 tablespoons to 6 cups, and I multiplied each part of the second ratio by 2 to get 8 tablespoons to 6 cups. They now have the same amount of water, but the first ratio has more powder (9 tablespoons), so it has to be stronger.

Carmen: I did something similar, but I changed the ratios so that each has 1 cup of water. The first ratio was 3 tablespoons to 2 cups. Dividing both amounts by 2, the ratio became $\frac{3}{2}$ tablespoons to 1 cup. The second ratio was 4 tablespoons to 3 cups, and after I divided the amounts by 3, it became $\frac{4}{3}$ tablespoons to 1 cup. So now, I can compare how much mix there is because the amount of water is the same. Because $\frac{3}{2} > \frac{4}{3}$, there is more mix in the first ratio, so the first ratio will be a stronger drink.

Figure 4.1

Nora's solution showing that one mixture is stronger than another.



Nora: Mine is like Sam's, but I used pictures because just thinking about the numbers confused me (Figure 4.1). But I used the same ideas Sam did, so that each ratio had 6 cups of water. Then I knew by looking at my picture that the first mixture had more powder, so it must be stronger.

Edwin: I changed the ratios so that they had the same amount of powder. So I divided the parts of the first ratio by 3 to get 1 tablespoon to $\frac{2}{3}$ cup, and I divided the parts of the second ratio by 4 to get 1 tablespoon to $\frac{3}{4}$ cup. Now they have the same amount of powder, but the second ratio has more water because $\frac{3}{4} > \frac{2}{3}$, so the second mixture dilutes the powder more than the first mixture. So the first mixture has to be stronger.

It makes sense to some students to rewrite the ratios so that the amounts of water are equivalent, while to other students, it makes sense to make the amounts of powder equivalent. One student had to use a diagram to help her understand the relationships in the problem. If the teacher had expected all students to use diagrams, then many of the students might have used a less-efficient method than they would have used independently. If the teacher had expected all the students to recognize which ratio was larger by changing the amounts to decimals and comparing the decimals (a procedure sometimes taught to solve problems like this one), then some students might have been confused because they still needed to explicitly think about unit ratios to make the comparison. Also, the cognitive demand of the task would have been lowered! Instead, the teacher allowed the students to use their own ideas to determine which ratio was larger. This expectation and the recognition that different students will approach and solve the same problem in various ways honors students' varying mathematical development and sets the stage for differentiated mathematics instruction. In addition, by listening to how different students approached the task, the teacher has acquired important information that can be used to plan subsequent instruction that meets a variety of students' needs.

The Nuts and Bolts of Differentiating Instruction

Differentiation is an instructional approach that requires a shift from focusing on the "middle-of-the-road" student to attending to all students. As overwhelming as this may sound, differentiation does not require a teacher to create individualized lessons for each and every student in the classroom. Rather, it requires emphasizing three basic ideas (Sousa & Tomlinson, 2011):

- Planning lessons around meaningful content, grounded in authenticity.
- Recognizing each student's readiness, interest, and approach to learning.
- Connecting content and learners by modifying content, process, product, and the learning environment.

Planning Meaningful Content, Grounded in Authenticity

Before you begin to think about differentiation, you first need to know where you want students to "be" at the end of the learning experience. You must be explicitly aware of the content that students should know, understand, and be able to do after engaging in a given lesson or sequence of lessons. This awareness enables you to effectively guide students' learning by varying or differentiating instruction. If you do not have a clear idea about the specific learning outcomes, identifying how and when to differentiate can be difficult. In fact, Tomlinson (1999) claims that "If the 'stuff' [content] is ill conceived, the 'how' [differentiation] is doomed" (p. 16).

Note that the content must be authentic and grounded in important mathematics that emphasizes the big ideas in ways that require students to develop relational understanding. Authentic content engages students with the heart of mathematics by requiring them to be problem solvers and creators of knowledge. Through this kind of engagement, students also develop a productive disposition toward mathematics and see it as sensible, useful, and worthwhile.

Recognizing Students as Individuals

Knowing each student in the context of learning requires finding out who he or she is as an individual on traits such as readiness, interests, and learning profile. *Readiness* refers to a student's proficiency with the knowledge, understanding, and skills embedded in specific learning goals. *Interest* means a student's attraction to particular topics, ideas, and events.

Using contexts that are interesting and familiar to students enhances their attention and motivation to engage and achieve (Sousa & Tomlinson, 2011). A *learning profile* identifies how a student approaches learning—how each student prefers to learn (e.g., in groups, alone); prefers to process and reason about information (e.g., by listening, observing, participating, or through talking; by thinking about details first and then the big picture or vice versa; by doing one task at a time or multitasking); and prefers to use or demonstrate what has been learned (e.g., writing, verbalizing, drawing). When deciding how to structure the environment, tasks, and assessments, consider students' preferences for learning and you will greatly facilitate the learning process. This is not to say that you must narrow learning experiences to only students' preferences all the time. That is simply impossible to do. What is possible is to look for opportunities to provide students some learning choices. Furthermore, knowing students' preferences alerts you to when they may need additional supports or guidance when those preferences are not possible to incorporate.

Information about your students' traits can inform how you might modify different elements of the classroom (e.g., Sousa & Tomlinson, 2011; Tomlinson, 2003). You can gather information pertaining to students' readiness by using preassessments several days before a given unit so that you have time to analyze the evidence and assess each student's readiness for the unit. You can also use surveys, typically at the beginning and midpoint of the year, to gather information about students' interests and learning profiles. Interest surveys give students opportunities to share personal interests (e.g., what they like to do after school, on the weekends, and during the summer; what school subjects they find most interesting and why) and information about pets, siblings, and extracurricular activities. Use your students' interests to provide contexts for the mathematics they are learning to increase their motivation and engagement. Learning profile surveys or questionnaires also help students think about what helps them learn and what does not, such as preferring to work in pairs versus alone, being able to work with background noise, and needing to process ideas verbally (Figure 4.2). Teacher observation can also provide valuable insights into student learning profiles. By recording students' information on index cards, you can quickly refresh your memory by looking through the cards as you plan lessons. You can also sort the cards to help you create groups based on interests or learning profiles.

Connecting Content and Learners

A critical component of differentiated lesson planning is determining how to modify four classroom elements to help the learner better connect with the content (Tomlinson, 2003). These four classroom elements are content, process, product, and the learning environment.

Figure 4.2

Learning profile inventory.

When working on a task, I like to . . .	I like to work . . .	When working, I like the room to be . . .	When working, I like . . .	When learning about a new idea, I like to . . .	When sharing information, I like to . . .
☐ sit at my desk ☐ sit somewhere other than at my desk ☐ stand ☐ lie on the floor ☐ other	☐ with a partner ☐ in a small group ☐ alone ☐ other	☐ warm ☐ cool ☐ dim, with the lights off ☐ bright ☐ other	☐ quiet ☐ noise ☐ music ☐ other	☐ hear about it ☐ read about it ☐ see visuals about it ☐ use materials to explore it ☐ talk about it ☐ other	☐ talk ☐ show ☐ write ☐ other

Content: What You Want Each Student to Learn

Generally, what is learned (the big ideas) should be relatively the same for all students. However, content can still be differentiated in terms of depth (level of complexity) and breadth (connecting across different topics) (Murray & Jorgensen, 2007; Small, 2012). Students' readiness typically informs the level of complexity or depth at which the content is initially presented for different groups of students. Interest and learning profiles tend to inform differentiation geared toward breadth.

An example of a depth adaptation for developing understanding and skill with organizing, representing, and interpreting data is a mini-lesson in which all students organize and represent data and answer questions based on the data. However, some students may have a smaller set of data to deal with, or they may be asked to answer given questions about the data, while others, who are ready for more sophisticated content, are asked to generate their own questions about the data. An example of a breadth adaptation for the same objective is to allow students a choice in terms of the kind of data with which to work. For example, based on their interests, students might choose to work with data pertaining to sports, books, science, gaming, or pets. By working with data from various contexts, students not only learn something about those contexts, but also can begin to see the broader applications of organizing, representing, and interpreting data.

Process: How Students Engage in Thinking about Content

Although the big ideas of a learning experience remain relatively stable when differentiating, how students engage with and make sense of the content—the process—changes. Tomlinson (1999) described the process as students "taking different roads to the same destination" (p. 12). You can use different strategies or encourage students to take different "roads" to increase access to the essential information, ideas, and skills embedded in a lesson (Cassone, 2009; Tomlinson, 2003). For example, the use of manipulatives, games, and relevant and interesting contextual problems provides different ways for students to process their ideas while engaging with content.

Teaching Tip

Be sure that the differentiated tasks you ask students to do are closely aligned with the learning objectives of the lesson.

The mathematical process standards in the *Principles and Standards for School Mathematics* (NCTM, 2000), which served as a basis for the Standards for Mathematical Practice in the *Common Core State Standards* (CCSSO, 2010), lend themselves well to differentiating how students engage with and make sense of content. In particular, the process standard of representation emphasizes the need to think about and use different ways to represent mathematical ideas, which can help students make connections between concepts and skills. With the process standard of communication, students can use verbal or written communication as they share their reasoning, depending on their strengths. In addition, the process standard of problem solving allows for differentiation because of the myriad of strategies that students can use—from drawing a diagram or using manipulatives to solving a simpler problem and looking for patterns.

Due to students' different levels of readiness, it is imperative that they be allowed to use a variety of strategies and representations that are grounded in their own ideas to solve problems. You can facilitate students' engagement in thinking about the content through a variety of methods. For example, teachers may

- Use visuals or graphic organizers to help students connect ideas and build a structure for the information in the lesson.
- Provide manipulatives to support students' development of a concept.
- Provide manipulatives other than those previously used for the same content to expand students' understanding.

- Use an appropriate context that helps students build meaning for the concept and that employs purposeful constraints that can highlight the significant mathematical ideas.
- Share examples and nonexamples to help students develop a better understanding of a concept.
- Gather a small group of students to develop foundational knowledge for a new concept.
- Provide text or supplementary material in a student's native language to aid understanding of materials written or delivered in English.
- Set up learning centers or a tiered lesson (a lesson that offer learners different pathways to reach a specific learning goal).

Product: How Students Demonstrate What They Know, Understand, and Are Able to Do after the Lesson Is Over

The term *product* can refer to what a student produces as a result of completing a single task or to a major assessment after an extended learning experience. The products for a single task are similar to the ways students share their ideas in the *After* portion of a lesson (described in Chapter 2), which could include explaining their ideas with manipulatives, through a drawing, in writing, or verbally. The products related to an extended experience can take the form of a project, portfolio, test, write-up of solutions to several problem-based inquiries, and so on. An important feature of any product is that it allows a variety of ways for students to demonstrate their understanding of essential content.

Learning Environment: The Logistics, Physical Configuration, and Tone of the Classroom

Consider how the physical learning environment might be adapted to meet students' needs. Do you have a student who prefers to work alone? Who prefers to work in a group? Who can or cannot work with background noise? Who prefers to work in a setting with brighter or dimmer lighting? Attending to these students' needs can affect the seating arrangement, specific grouping strategies, access to materials, and other aspects of the classroom environment. In addition to the physical learning space, establishing a classroom culture in which students' ideas and solutions are respected as they explain and justify them is an important aspect of a differentiated classroom. Refer to the recommendations provided in Chapter 2 pertaining to facilitating effective classroom discussions and establishing a supportive and respectful learning environment.

Differentiated Tasks for Whole-Class Instruction

One challenge of differentiation is planning a task focused on a target mathematical concept or skill that can be used for whole-class instruction while a variety of students' needs are met. Let's consider two different kinds of tasks that can meet this challenge: parallel tasks and open questions (Murray & Jorgensen, 2007; Small, 2012).

Parallel Tasks

Parallel tasks are two or three tasks that focus on the same big idea but offer different levels of difficulty. The tasks should be created so that all students can meaningfully participate in a follow-up discussion with the whole class. You can assign tasks to students based on their readiness, or students can choose which task to work on. If they choose a task that is too difficult, they can always move to another task. Consider how the following parallel tasks emphasize the big ideas of volume and surface area (and the relationship between them), but at different levels of difficulty.

TASK 1:

A cube has a volume of 729 cubic units. What are the dimensions and the surface area?

TASK 2:

A prism has a volume of 90 cubic units and a surface area of 146 square units. What could the dimensions be?

Stop and Reflect

Which of the two tasks do you think would be more difficult, and why?

Both tasks provide opportunities for students to work with volume and surface area, but the use of a prism instead of a cube in the second task increases the level of difficulty for a number of reasons. First, because the dimensions of a cube are all the same, students are looking for one number that when cubed equals 729, whereas in the latter task because the dimensions can vary, students must consider different factor trios for 90. Second, task 2 has an added constraint in determining the dimensions because the surface area is given.

You can facilitate a whole-class discussion by asking questions that are relevant to both tasks. For example, with respect to the previous two tasks, you could ask the whole class the following questions:

- In general, what does your object look like?
- What is the relationship between the object's dimensions and the object's volume?
- What is the relationship between the object's dimensions and the object's surface area?
- How do you know the dimensions you have identified are correct?
- How are surface area and volume different?
- When would you want to know the surface area of a box? When would you want to know its volume?

Although the students work on different tasks, because the tasks are focused on the same big idea, these questions allow them to extend their thinking as they hear others' strategies and ideas.

For many problems involving computation, you can simply insert multiple sets of numbers to vary the difficulty. In the following problem, students are permitted to select the first, second, or third number in each set of brackets. Giving a choice increases motivation and helps students become more self-directed learners (Bray, 2009; Gilbert & Musu, 2008).

LEARNING OBJECTIVE: (CCSS-M: 6.NS.B.3)

Compute fluently with multi-digit numbers

Task: It costs [$1.50, $1.65, $2.25] to play a game at the county fair. Mark played the game [9, 16, 43] times while the fair was in town. How much money did Mark spend on playing the game?

The following parallel tasks for seventh graders focus on the big idea of measures of center in statistics.

TASK 1:

A data set has a mean of 50 and a median of 20. If there are 10 data values, what could those values be?

TASK 2:

A data set has a mean of 30 and a median of 20. If there are 5 data values, what could those values be?

With the first task, the teacher provides a task for students who are ready to work with a larger set of data. Requiring them to use an even number of data values also increases the challenge when finding the median. The parallel task still offers an opportunity for students to work with mean and median but lowers the difficulty by using fewer data values, an odd number of data points, and a mean that is closer to the median.

In thinking about how to create parallel tasks, once you have identified the big idea you wish to focus on, consider how students might differ in their reasoning about that idea. The size of the numbers involved, the operations students can use, and the degree of structure inherent in the task are just a few things to consider as you create parallel tasks. Start with a task from your textbook and then modify it to make it suitable for a different developmental level. The original task and the modified task will serve as the parallel tasks offered simultaneously to your students. If you number the parallel tasks and allow students to choose the task they will work on, be sure there are instances in which the more difficult task is the first one. This randomness will ensure that the students consider both options before they choose their task.

Open Questions

Many questions found in textbooks are closed, meaning there is one answer and often only one way to get there. These kinds of questions cannot meet the needs of the range of learners in your classroom. Alternatively, open questions are broad-based questions that can be solved in a variety of ways or that can have different answers. Because these kinds of questions invite meaningful responses from students who are at varying developmental levels, they more readily meet the needs of a range of learners (Small, 2012). Following are two examples of open questions. Both questions can have different answers and can also be solved in a variety of ways.

- The probability of an event is $\frac{1}{5}$. What could the event be?
- The product of three numbers is -120. What could the three numbers be?

Stop and Reflect

How would you solve each of these tasks? Can you think of at least two different strategies and at least two different answers for each task?

Open questions have a high level of cognitive demand, as described in Chapter 2, because students must use more than recall or do more than merely follow steps in a procedure. There are ample opportunities for them to approach the problems at their own level, which means open questions automatically accommodate student readiness. Consequently, when given an open question, most students can find something appropriate to contribute, which helps to increase their confidence in doing mathematics and can provide you insight into their level of understanding.

You can use a variety of strategies, such as the following, to create open questions (Small, 2012; Sullivan & Lilburn, 2002):

- Give the answer and ask for the problem.
- Replace a number in a given problem with a blank or a question mark.
- Offer two situations or examples and ask for similarities and differences.
- Create a question in which students have to make choices.

The two previous examples of open questions illustrate the first strategy of giving an answer and asking for the problem. Here are examples of using the other three strategies to convert standard questions to open questions.

STRATEGY:

Replace a number in a given problem with a blank or a question mark.

Standard Question	Open Question
$\begin{array}{r} 2.3 \\ \times\ 0.68 \\ \hline \end{array}$	Fill in the ? with values so that the result is greater than 1 and less than 2. $\begin{array}{r} ?.3 \\ \times\ 0.6? \\ \hline \end{array}$

STRATEGY:

Offer two situations or examples, and ask for similarities and differences.

Standard Question	Open Question
Graph $4y = 2x + 8$.	Describe similarities and differences between these linear equations: $4y = 2x + 8$ and $3y = -6x + 6$.

STRATEGY:

Create a question that can generate a range of possible answers so that students have to make choices.

Standard Question	Open Question
Use a tree diagram to show the outcomes of rolling a six-sided die and flipping a coin.	Use a tree diagram to show the possible outcomes of a two-step experiment. Your solution should describe the experiment that your tree diagram is illustrating.

Facilitating follow-up discussions is also important when you use open questions. While students work on an open question, walk around and observe the strategies they are using and the answers they are finding. During this time, plan which students will share their ideas during the follow-up discussion, ensuring that a variety of strategies and answers are examined. During the discussion, look for opportunities to help students make connections between different ideas that are shared. For instance, suppose for the second task above, one student explains that he graphed both equations by creating a table of values and noticed that both lines intersected the y-axis at (0, 2). Another student explains that she divided each equation by the coefficient of the variable y before she created a table of values so that she could use smaller numbers in her computations. You could ask the class to look at the second student's modified equations in light of the first student's comment that both lines intersected the y-axis at (0, 2) and see whether the class could make a conjecture about the equation of a line and its y-intercept. Asking questions that help students build connections can support those who need additional help to track on significant ideas and can also challenge students to extend their understanding. You could also do a gallery walk during which students posted their solutions for others to view. With the open question related to creating tree diagrams, students could go on a gallery walk and use sticky notes to write questions or comments to give feedback to their peers about their examples and the related illustration or explanations.

Teaching Tip

Make sure students understand the vocabulary used in tasks before they begin working independently. For example, before students start working on the task where they are to use a tree diagram, have a discussion about the meaning of the terms *tree diagram, outcomes,* and *experiment*.

Tiered Lessons

In a tiered lesson, you set the same learning goals for all students, but different pathways are provided to reach those learning goals, thereby creating the various tiers. First, you need to decide which category you wish to tier: content, process, or product. If you are new to preparing tiered lessons, tier only one category until you become more comfortable with the process. Once you decide which category to tier, determine the challenge of each of the defined tiers based on student readiness levels, interests, and learning profiles (Kingore, 2006; Murray & Jorgensen, 2007; Tomlinson, 1999). Murray and Jorgensen (2007) suggest starting by creating three tiers to make the process more manageable: a regular tier or lesson, an extension tier that provides extra challenge, and a scaffolding tier that provides more background or support. Once you have this framework, you can design as many tiers as needed to meet your students' needs. All tiered experiences should have the following characteristics (Sousa & Tomlinson, 2011):

- Address the same learning goals.
- Require students to use reasoning.
- Be equally interesting to students.

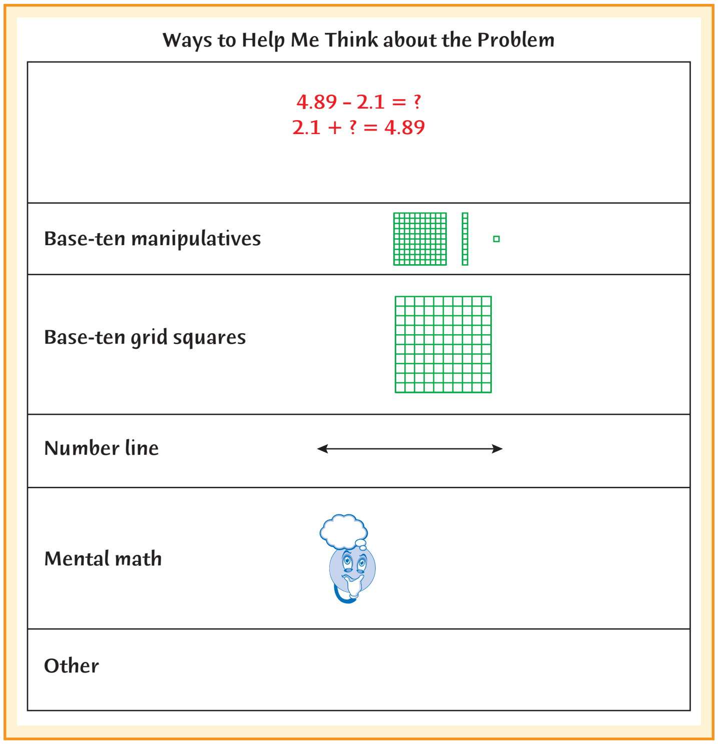
We have already considered some ways to tier the content by using parallel tasks and open questions. However, varying the degree of challenge is not just about the *content.* You can also use any of the following four aspects to tier lessons (Kingore, 2006):

- *Degree of assistance:* If some students need additional support, you can partner students, provide examples, help them brainstorm ideas, or provide a cue sheet (Figure 4.3).
- *Structure of the task:* Some students, such as students with disabilities, benefit from highly structured tasks. However, gifted students often benefit from a more open-ended structure.
- *Complexity of the task(s):* Make tasks more concrete or more abstract and/or include more difficult problems or applications.
- *Complexity of process:* As you think about your learners, ask yourself these questions: How quickly should I pace this lesson? How many instructions should I give at one time? How many higher-level thinking questions are included as part of the task(s)?

Consider in the following examples how the original task is modified to change the level of the challenge. Assume that the students do not know the standard algorithm for dividing fractions.

Figure 4.3

Problem-solving cue sheet.



Ways to Help Me Think about the Problem

4.89 - 2.1 = ?
2.1 + ? = 4.89

Base-ten manipulatives

Base-ten grid squares

Number line

Mental math

Other

ORIGINAL TASK (GRADE 6):

Elliot is making cookies. Each recipe calls for $\frac{1}{2}$ cup of sugar. If he has $\frac{3}{4}$ cup of sugar, how many recipes can he make? Explain how you know. (Assume he can make part of a recipe.)

The teacher has distributed fraction bars to the students to model the problem, and paper and pencil to illustrate and record how they solved the problem. The teacher asks the students to model the problem and be ready to explain their solution.

ADAPTED TASK (GRADE 6):

Elliott is making cookies. Each recipe calls for a certain amount of sugar. Elliot has some sugar. How many recipes can he make? Explain how you know. (Assume he can make part of a recipe.)

The teacher asks the students what is happening in this problem, how they might solve the problem, and what tools might help them solve the problem. Then the teacher distributes task cards that tell how much sugar is needed for each recipe and how much sugar Elliot has. The teacher has varied the difficulty of the tasks by considering how easy it is to relate the fractions to each other.

Card 1 (easier)

Each recipe uses $\frac{1}{3}$ cup of sugar.

Elliot has $\frac{4}{3}$ cups of sugar.

Card 2 (middle)

Each recipe uses $\frac{1}{2}$ cup of sugar.

Elliot has $\frac{4}{3}$ cups of sugar.

Card 3 (advanced)

Each recipe uses $\frac{5}{6}$ cup of sugar.

Elliot has $\frac{3}{4}$ cups of sugar.

In each case, the students must use words, pictures, models, or numbers to show how they figured out the solution. Various tools are provided (Cuisenaire rods, fraction bars, and grid paper) for the students' use.

Stop and Reflect

Which of the four aspects that change the challenge of tiered lessons was addressed in the adapted task?

You would preassess your students to determine the best ways to use these task cards. One option is to give students only one card, based on their current academic readiness (e.g., easy cards to those who struggle with division of fractions). A second option is to give out cards 1 and 2 based on readiness, then use card 2 as an extension for those who successfully complete card 1, and card 3 as an extension for those who successfully complete card 2. In each of these cases, you will need to record at the end of the lesson which students were able to model and explain the various levels of the problems so that the next lesson can be planned appropriately. Notice that this tiered lesson addresses both the complexity of the task (difficulty of different cards) and the process (instructions are broken down by starting with the no-numbers scenario).

The following example illustrates how to tier a lesson based on *structure*. Notice that the different tasks vary in how open-ended the work is, yet all tasks focus on the same learning goal of applying the Pythagorean theorem in a real-world problem.

LEARNING OBJECTIVE: (CCSS-M: 8.G.B.7)

Apply the Pythagorean Theorem

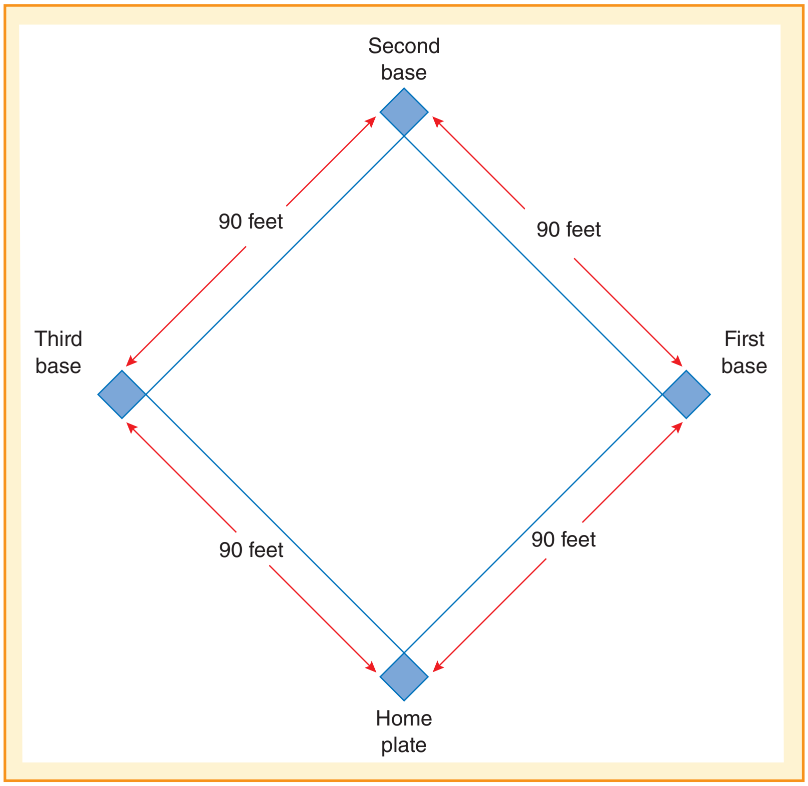
Suppose students have been reading and discussing the World Series and so are familiar with the game of baseball. Give them a diagram of a baseball diamond with regulation dimensions (Figure 4.4). Students will be given different tasks to solve, based on their learning needs and prior success in applying the Pythagorean theorem.

- *Group A:* Identify and solve at least two situations that would require you to use the Pythagorean theorem to find distances on the baseball diamond. (open-ended)
- *Group B:* Suppose the second baseman is standing on second base with the ball and a runner is running toward home plate. How far will the second baseman need to throw the ball to throw the runner out at home plate? (slightly structured)
- *Group C:* Suppose the second baseman is standing on second base with the ball and a runner is running toward home plate. How far will the second baseman need to throw the ball to throw the runner out at home plate? Use the following suggestions and questions to work through the problem:

1. On your diagram of the baseball diamond, draw a line between second base and third base, another line between third base and home plate, and another line between second base and home plate.
2. What shape have you drawn? What do you know about the distance between bases? Which distance are you trying to find?
3. Use the Pythagorean theorem ($a^2 + b^2 = c^2$) to help you find the unknown distance. (most structured)

Figure 4.4

Application of the Pythagorean theorem.



The three tiers in this lesson reflect different degrees of difficulty in terms of task structure. However, all students are working on the same learning objective, and they all must engage in reasoning to use the Pythagorean theorem to find unknown distances.

In Chapter 6 you will read about response to intervention (RtI), a multitier student-support system that offers struggling students increasing levels of intervention. We want to distinguish between the tiers in RtI and tiered lessons used in differentiation. In RtI the tiers refer to the different degrees of intervention offered to students as needed—from the first tier that occurs in a general education setting and involves the core instruction for all students based on high-quality mathematics curriculum and instructional practices, to the upper tier that could involve one-on-one instruction with a special education teacher. Tiered lessons used in differentiation would be an avenue to offer high-quality core instruction for all students in the first tier or level of RtI.

Flexible Grouping

Allowing students to collaborate on tasks supports and challenges their thinking and increases their opportunities to communicate about mathematics and build understanding. In addition, many students feel that working in groups improves their confidence, engagement, and understanding (Nebesniak & Heaton, 2010). Even students who prefer to work alone need to learn the life skill of collaboration and should be provided opportunities to work with others.

Determining how to place students in groups is an important decision. Avoid grouping by ability. This kind of grouping, although well-intentioned, perpetuates low levels of learning and actually increases the gap between more and less dependent students. **Watch this video** (**https://www.youtube.com/watch?v=R4iAwShVIBE**) that shares some of the negative effects of ability grouping. Instead, consider using *flexible grouping*, in which the size and makeup of small groups vary in a purposeful and strategic manner (Murray & Jorgensen, 2007). When coupled with differentiation strategies, flexible grouping gives all students the chance to work successfully in groups.

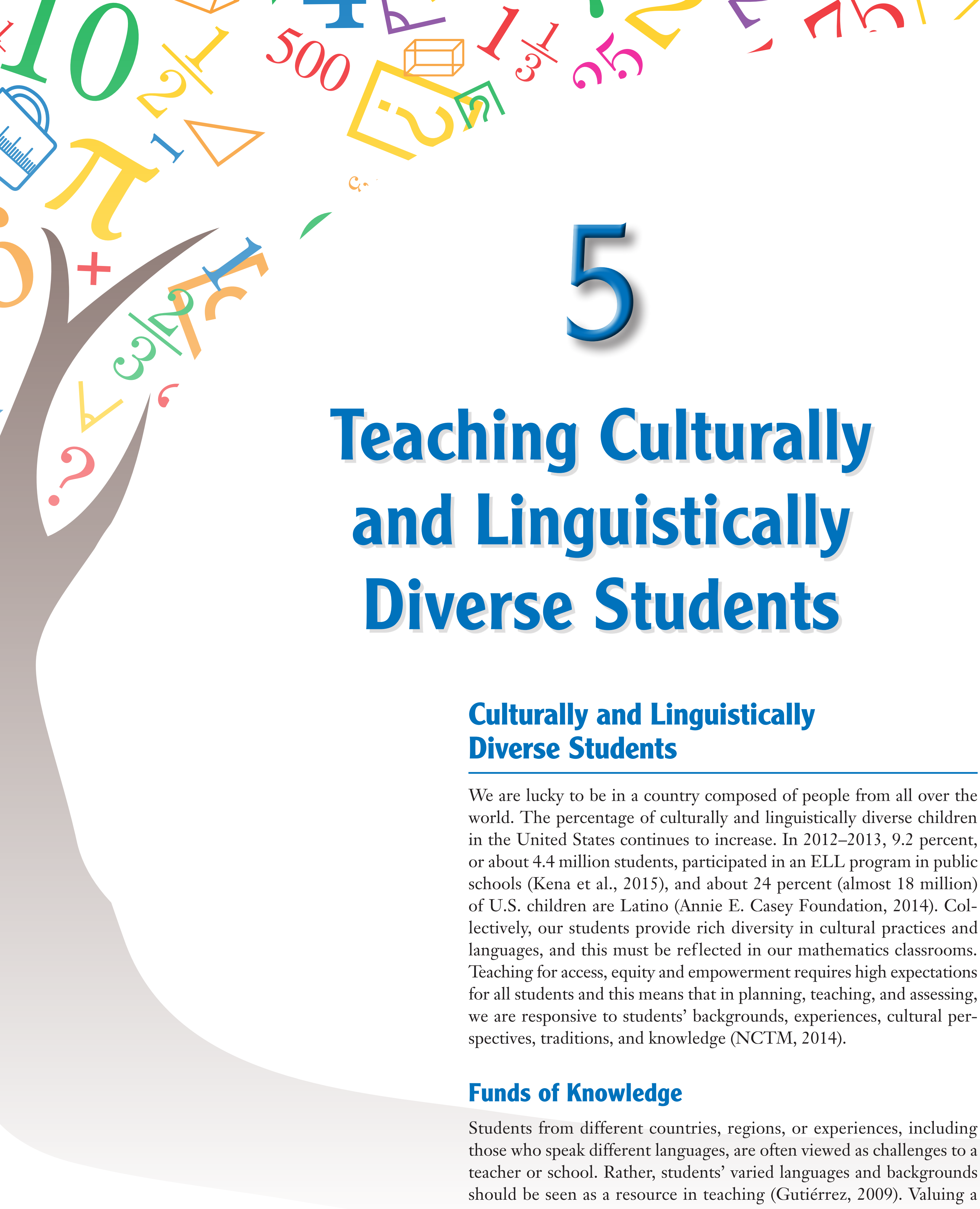
Flexible groups can vary based on students' readiness, interests, language proficiency, and learning profiles, as well as the nature of the tasks. For example, sometimes students can work with a partner because the nature of the task best suits two people working together. At other times, flexible groups might be created with four students because their assigned task has enough components or roles to warrant a larger team. Note that although it can be tempting to occasionally place struggling learners with more capable students, consistently pairing struggling learners with more capable students is not helpful for either group. The idea behind flexible grouping is that groups can and do easily change in response to all students' readiness, interests, and learning profiles and the nature of the task they will be doing.

Regardless of how you group your students, the first key to successful grouping is individual accountability. While the group is working together on a product, individuals must be able to explain the content, the process, and the product. Second, and equally important, is building a sense of shared responsibility within a group. At the start of the year, it is important to engage students in **Team-Building Activities** and to set expectations that all group members will participate in the assigned group task(s) and that all group members will be responsible for ensuring that the entire group understands the concept.

Reinforcing individual accountability and shared responsibility may create a shift in your role as the teacher. When a member of a small group asks you a question, pose the question to the whole group to find out what the other members think. Students will soon learn that they must use teammates as their first resource and seek teacher help only when the whole group needs help. Also, when you are observing groups, rather than asking Amanda what she is doing, ask Kyle to explain what Amanda is doing. Having all students participate in the oral report to the whole class also builds individual accountability. Letting students know that you may call on any member to explain what the group did is a good way to ensure that all group members understand what they did. In addition, having students individually write and record their strategies and solutions is important. Using these techniques will increase the effectiveness of grouping, which in turn will help students learn mathematical concepts more successfully.

Stop and Reflect

Why is teaching mathematics through problem solving (i.e., a problem-based approach) a good way to differentiate instruction and reach all students in a classroom?



5

Teaching Culturally and Linguistically Diverse Students

Culturally and Linguistically Diverse Students

We are lucky to be in a country composed of people from all over the world. The percentage of culturally and linguistically diverse children in the United States continues to increase. In 2012–2013, 9.2 percent, or about 4.4 million students, participated in an ELL program in public schools (Kena et al., 2015), and about 24 percent (almost 18 million) of U.S. children are Latino (Annie E. Casey Foundation, 2014). Collectively, our students provide rich diversity in cultural practices and languages, and this must be reflected in our mathematics classrooms. Teaching for access, equity and empowerment requires high expectations for all students and this means that in planning, teaching, and assessing, we are responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge (NCTM, 2014).

Funds of Knowledge

Students from different countries, regions, or experiences, including those who speak different languages, are often viewed as challenges to a teacher or school. Rather, students' varied languages and backgrounds should be seen as a resource in teaching (Gutiérrez, 2009). Valuing a person's cultural background is more than a belief statement; it is a set of intentional actions that communicate to the student, "I want to know about you, I want you to see mathematics as part of your life, and I expect that you can do high-level mathematics." In getting to know students, we access their funds of knowledge—the essential knowledge or information that they use to survive and thrive (Moll et al.,1992; Chao, Murray, & Gutiérrez, 2014). Unfortunately, too many teachers view non-Asian and non-European ELLs as behind academically and socially, and lacking the skills needed to succeed (Chval & Pinnow, 2010; Vollmer, 2000).

Instead of teaching students from a deficit model (i.e., focusing on their lack of knowledge and experience), we can connect students' experiences at home and with family to those of the mathematics classroom. Family and community activities, such as playing games, weaving, cooking, and story-telling, can serve as cultural and linguistic resources in learning mathematics.

Mathematics as a Language

Mathematics is commonly referred to as a "universal language," but this is not the case. Conceptual knowledge (e.g., what division is) is universal. Procedures (e.g., how you divide or factor) and symbols are culturally determined and are not universal. For example, in the United States, the common algorithm for the division of fractions is to invert the second fraction and multiply, but in Mexico, the common algorithm is to cross-multiply (Perkins & Flores, 2002):

$$\frac{7}{8} \div \frac{1}{4} = \frac{28}{8}$$

Figure 5.1 illustrates the process for factoring that is used in other countries, including Mexico.

Commas and periods are sometimes used in reverse; for example, 1, 4 means one and four tenths, and 1.400.000 means 1.4 million (Secada, 1983). In Mexico, textbooks may refer to angle B as $\hat{B}$ rather than as $\angle B$, or to angle $\widehat{ABC}$ as ABC rather than as $\angle ABC$, as is done in the United States. An ELL might not recognize the angle symbol and could confuse it with the "less than" symbol. What is called "billions" in the United States is called "thousand millions" in Mexico (Perkins & Flores, 2002). Treating mathematics as a universal topic can lead to inequities in the classroom as students from other cultures may not recognize the symbols and processes being used in their class, even if they know the content, and therefore may not be able to participate and learn.

What we value and how we engage in "doing mathematics" is part of cultural practice. For example, mental mathematics is highly valued in other countries, whereas in the United States, recording every step is valued. Could you follow the factoring and common-multiple approach illustrated in Figure 5.1? It looks different from the factor trees or factor lists commonly taught in the United States. The critical equity question, though, is not just whether you can follow an alternative approach, but how will you respond when you encounter a student using such an approach?

- Will you require the student to use trees or lists (disregarding the way he or she learned it)?
- Will you ask the student to elaborate on how he or she did it?
- Will you have the student show other students his or her way of thinking?

The latter two responses communicate to the student that you are interested in his or her way of knowing mathematics, and that there are many ways in which different people and different cultures approach mathematics. Supporting

Teaching Tip

Instead of requiring students to write all their steps, ask them to think aloud as they solve a problem, or ask how they did it in their head.

Figure 5.1

Steps and thought processes for finding factors and multiples in Mexico.

Step in the Algorithm			Explanation or Think Aloud for the Step
5.1a. Finding Prime Factors			
150			Start with the first prime number and see how many times it can be used as a factor. Record the remaining factor under the original number.
150	2		2 won't work a second time, so try the next prime number, 3.
75			
150	2		3 won't work again, so try the next prime number, 5.
75	3		
25			
150	2		5 will work a second time, leaving 1 as the remaining factor, so the factoring is complete. The prime factors are the numbers in the right-hand column.
75	3		
25	5		
5	5		
1			
5.1b. Finding the Least Common Multiple			
24	18	2	As in 5.1a, the process begins with trying the first prime number, 2. 2 is a common factor.
12	9		
24	18	2	2 will work again for 24, so factors for 24 are continued. The next prime to try is 3.
12		2	
6		2	
3	9	3	
24	18	2	3 is a common factor. 3 works a second time for 18. This leaves factors of 1 for each column. The least common multiple is found by multiplying the values in the right-hand column: $2 \times 2 \times 2 \times 3 \times 3 = 72$.
12		2	
6		2	
3	9	3	
	3	3	
1	1	1	

Source: Adapted from Perkins & Flores (2002).

a range of strategies for algorithms is an important way to show that you value students as individuals and is a good way to gain insights into possible culturally influenced strategies (Gutiérrez, 2015).

Culturally Responsive Mathematics Instruction

Culturally responsive mathematics instruction includes attention to mathematical thinking, language, and culture. And, it applies to all students, including students from different ethnic groups, different socioeconomic levels, and so on. It includes consideration for content, relationships, cultural knowledge, flexibility in approaches, use of accessible learning contexts (i.e., contexts familiar or interesting to students), a responsive learning community, and working in cross-cultural partnerships (Aguirre & del Rosario Zavala, 2013; Averill, Anderson, Easton, Te Maro, Smith, & Hynds, 2009). Culturally responsive mathematics instruction can improve the performance of all students, as well as narrow the academic performance gap (Boaler, 2008; Kisker, Lipka, Adams, Rickard, Andrew-Ihrke, Yanez, &

Millard, 2012). Table 5.1 lists four **Aspects of culturally responsive mathematics instruction**, along with questions to guide planning, teaching, and assessing.

Communicate High Expectations

Too often, our first attempt to help students, in particular ELLs, is to simplify the mathematics and/or remove the language from the lesson. Simplifying or removing language can take away opportunities to learn. Culturally responsive instruction stays focused on the big ideas of mathematics (i.e., is based on standards such as the *Common Core State Standards*) and helps students engage in and stay focused on those big ideas. In addition to focusing on the big ideas, using tasks worthy of groupwork, emphasizing multiple representations, incorporating student justifications and presentations are features of classrooms that support equitable opportunities to learn mathematics (Cabana, Shreve, & Woodbury, 2014; Dunleavy, 2015). For example, a critical area in the sixth grade is division of fractions, including being able to "understand and explain why the procedures for dividing fractions make sense" (CCSSO, 2010, p. 44). Stories involving the division of fractions can be carefully selected for the use of contexts that are familiar to ELLs and for the use of visuals (e.g., rectangular plots of land or trays of brownies). Rather than every story having a new context, the stories can focus on the same theme (and connect to the English that students are learning in their English-as-a-second-language instruction, if possible). The teacher can incorporate opportunities for students to share their approaches to dividing fractions, for example, and illustrate (with

Table 5.1. Aspects of culturally responsive mathematics instruction.

Aspect of Culturally Responsive Instruction	Teacher Reflection Questions
Communicate high expectations.	Does teaching focus on understanding big ideas in mathematics? Are students expected to engage in problem solving and generate their own approaches to problems? Are connections made among mathematical representations? Are students justifying their strategies and answers, and are they presenting their work?
Make content relevant.	In what ways is the content related to familiar aspects of students' lives? In what ways is prior knowledge elicited/reviewed so that all students can participate in the lesson? To what extent are students asked to make connections between school mathematics and mathematics in their own lives? How are student interests (events, issues, literature, or pop culture) used to build interest and mathematical meaning?
Attend to students' mathematical identities.	In what ways are students invited to include their own experiences within a lesson? Are story problems generated from students and teachers? Do stories reflect the real experiences of students? Are individual student approaches presented and showcased so that each student sees his or her ideas as important to the teacher and peers? Are alternative algorithms shared as a point of excitement and pride (as appropriate)? Are the multiple modes used to demonstrate knowledge (e.g., visuals, explanations, models) valued?
Ensure shared power.	Are students (rather than just the teacher) justifying the correctness of solutions? Are students invited (expected) to engage in whole-class discussions in which they share ideas and respond to one another's ideas? In what ways are roles assigned so that every student feels that he or she is contributing to and learning from other members of the class? Are students given a choice in how they solve a problem? In how they demonstrate knowledge of the concept?

visuals) how they thought about it. In this way, ELLs are able to learn the important content (connecting story situation to equation to meaning for division) and engage in classroom discourse.

Make Content Relevant

There are really two components to making content relevant. One is to think about the *mathematics*: Is the mathematics itself presented meaningfully, and is it connected to other content? The second is to select *contexts*: Is the mathematics presented so that it connects to relevant/authentic situations in students' lives?

Mathematical Connections

Helping students see that mathematical ideas are interrelated will fill in or deepen their understanding of and connections to previously taught content. For example, consider the eighth-grade problem in Figure 5.2.

You may recognize that this task connects several representations, including graphs, tables, and situations. Just having students plot ratios on a graph to illustrate the relationship takes away opportunities to emphasize the meaning among the representations and to connect what they already know about fractions to proportional reasoning and algebra. Instead, provide language support that help students make sense of the task and, in so doing, build strong mathematical connections and a strong understanding of mathematics.

Context Connections

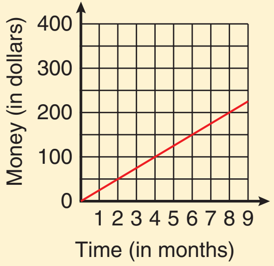
Making content relevant is also about contexts. Questions can emphasize the context-content connection. For the task in Figure 5.2, you might ask, "What is the monthly rate for each girl?" At these rates how much will each save in a year? Using problems that connect students to developmentally appropriate social or peer connections is one way to contextualize learning. Another is to make connections to historical or cultural contexts. Seeing mathematics from various cultures provides opportunities for students to "put faces" on mathematical contributions. For example, students could analyze freedom quilts, which tell stories about the Underground Railroad (Neumann, 2005), or flags of various countries; this allows them to explore *Common Core State Standards* content by looking at the ratio of colors in an area (grade 6)

Figure 5.2

Possible eighth grade task for graphing proportional relationships, interpreting the unit rate as the slope of the graph and comparing two different proportional relationships represented in different ways (Standard 8.EE.5 in CCSS-M).

Who is saving more money?

Marta's Savings



Catarina's Savings

Months	Money Saved
1	$30
2	$60
3	$90
4	$120

or creating scale drawings of the design (grade 7). They could also look for or design their own quilts that include translations, rotations, reflections, and dilations (grade 8). Analyze existing textbook lessons to see if the contexts used actually help students make sense of the mathematics—if they do not, then adapt the contexts so that they do (Drake et al., 2015). Additionally, sometimes data is provided in textbook tasks that can be replaced with data from your community, making it much more relevant and interesting to students, as well as potentially teaching students something important about their community (Simic-Muller, 2015). Using everyday situations can increase student participation, use of different problem strategies, and consequently help students develop a productive disposition (Tomaz & David, 2015).

Attend to Students' Mathematical Identities

A focus on student's mathematical identities overlaps with the previous category, but it merits its own discussion. A student's *mathematical identity* is their disposition toward mathematics and sense of competence as learner and a contributor in mathematics classrooms (Cobb, Gresalfi, & Hodge, 2009). Whether it is intentional or not, all teaching is identity work, as students are constantly adapting and redefining themselves based on their experiences in mathematics classrooms (Gutiérrez, 2015). Our goal is to develop productive dispositions in all students (i.e., the tendency to believe that steady effort in math learning pays off and to see oneself as an effective learner and doer of mathematics [NRC, 2001]). There are a number of ways that teachers can shape students' mathematical identities. One way to do this is "assigning competence" (Boaler & Staples, 2014, p. 27). As the teacher listens during small group work, they hear different contributions from students. During later discussion, the teacher can attribute ideas to individuals, saying something like, "That relates to the strategy Nicolas used." This strategy recognizes Nicolas as capable in mathematics, influencing how he perceives himself, as well as how other classmates might perceive him.

Additionally, telling stories about their own lives, or asking students to tell stories, makes mathematics relevant to students and can raise student achievement (Turner, Celedón-Pattichis, Marshall, & Tennison, 2009). Table 5.2 provides ideas for making mathematics relevant to a student's home and community.

Read the following teacher's story to see how she incorporated family history and culture into her class by reading *The Hundred Penny Box* (Mathis, 1986). In Mathis's story, a 100-year-old woman remembers an important event in every year of her life as she turns over each of her 100 pennies. Each penny is more than a coin; it is a "memory trigger" for her life.

Table 5.2. Where to find mathematics in students' homes and community.

Where to Look	What You Might Ask Students to Record and Share (and Mathematics That Can Be Explored)
Grocery store or market advertisements	• Cost of an item of which they bought more than one (multiplication) • Cost of an item that came with a quantity (e.g., dozen eggs) (division) • Better buy of two items of different sizes (division) • Shapes of different containers (geometry) • Different types/brands of different foods they select, such as what kind of bread (data)
Photographs	• A person they admire (data) • A favorite scene (geometry, measurement) • 2-D and 3-D shapes in their home or neighborhood • Flowers (multiplication with number of petals, algebraic thinking with growing patterns)
Artifact (game or measuring device) from their culture or that is a favorite	• The game may naturally involves mathematics • Measuring devices can be used to explore fractions and decimals

> Taking a cue from the book, I asked all the students to collect one penny from each year they were alive, starting from the year of their birth and not missing a year. Students were encouraged to bring in additional pennies their classmates might need. Then, the students consulted with family members to create a penny time line of important events in their lives. Using information gathered at home, they started with the year they were born, listing their birthday and then recording first steps, accidents, vacations, pets, births of siblings, and so on.

Students in grades 6–8 can prepare a "life line" of their key events, too, determining where between zero (i.e., the day they were born) and 1 (their current age) a memorable event happened (or between 0 and 12 years). The number line is an important model to use in teaching rational numbers, and in this context, students can build the meaning of rationale numbers on the number line. Importantly, number line investigations should focus on the distance between two quantities (the intervals), and in this context, they can focus on the amount of time between events. For example, you can ask students what fraction of their life they have spent in school. Considering events that happened before their birth introduces negative numbers. They can ask parents, grandparents, and other relatives about key events that happened before they were born and add these to their life lines. Or, they can find coins dated before they were born and determine where they would go on their life lines.

Ensure Shared Power

When we think about creating a positive classroom environment, one in which all students feel as if they can participate and learn, we are addressing considerations related to power. The teacher plays a major role in establishing and distributing power, whether intentional or not. In many classrooms, the teacher has the power—telling students whether answers are right or wrong (rather than having students determine correctness through reasoning), dictating processes for how to solve problems (rather than giving choices for how students will engage in the problems), and determining who will solve which problems (rather than allowing flexibility and choice for students). Effective teachers encourage students to make mathematics contributions and validate reasoning, reaching a higher level of rigor (Gresalfi & Cobb, 2006). The way that you assign groups, seat students, and call on students sends clear messages about who has power in the classroom. The "assigning competence" strategy just described in student mathematical identities is a teacher move to distribute power (who knows the math in our class). When teachers position ELLs as contributing in meaningful ways to the group, other students begin to see ELLs in similar ways, eliciting and valuing their ideas (Yoon, 2008). And, when teachers delegate authority to marginalized students, they learn more (Dunleavy, 2015).

Teaching Tip

In reflecting on teaching, focus on student participation. Which students struggled most? Were there multiple entry points? How might certain students have been encouraged to participate more?

Teaching Strategies that Support Culturally and Linguistically Diverse Students

Culture and language are interwoven and interrelated. Therefore, teaching strategies that support diverse learners often attend to a students' background, as well as their language. Creating effective learning opportunities for ELLs involves integrating the principles of bilingual education with those of standards-based mathematics instruction. When learning about mathematics, students may be learning content in English for which they do not know the words in their native language. In middle school, these words are many. Examples include *transformation*, *proportional*, *transversal*, *polygon*, *system of equations*, *cylinder*, and *sphere*. In addition, there are many words in middle-school mathematics that are familiar to students but take on specialized meaning in mathematics, such as *mean*, *multiply*, *expression*, *reflection*, *translation*, *factor*, *acute*, and *similar*.

Stop and Reflect

Consider recent lessons you have taught, or look at four or five pages in a textbook. See how many words you can identify that are unique or take on a specialized meaning in mathematics. Can you name ten others?

Story problems may be difficult for ELLs not just because of the language but also because the sentences in story problems are often structured differently from sentences in conversational English (Janzen, 2008). ELLs need to use both English and their native language to read, write, listen, and speak as they learn appropriate content—a position similarly addressed in NCTM standards documents and position statements. The strategies discussed in this section are the ones that appear most frequently in the literature as critical to increasing the academic achievement of ELLs in mathematics classrooms (e.g., Celedón-Pattichis & Ramirez, 2012; Echevarria, Vogt, & Short, 2008). Table 5.3 **Reflective questions for planning and teaching mathematics lessons for ELLs** offers an "at-a-glance" format of some reflective questions related to instructional planning and teaching to support ELLs.

Teaching Tip

Any one of the categories in Table 5.3 could be the focus of a lesson study, discussions with colleagues, or the basis for individual reflection. The importance lies not in the specific suggestions, but in the concept of having an eye on language development *and* mathematics content.

Focus on Academic Vocabulary

ELLs enter the mathematics classroom from homes in which English is not the primary language of communication. Although a person may develop conversational English language skills in a few years, it takes as many as seven years to learn *academic language*, which

Table 5.3. Reflective questions for planning and teaching mathematics lessons for ELLs.

Process	Mathematics Content Considerations	Language Considerations
Reflective Questions for Planning		
1. Determine the mathematics.	• What mathematical concepts (aligned to grade-level standards) am I teaching? • What student-friendly learning objectives will I post? • How does this mathematics concept connect to other concepts students have learned?	• What language objectives might I add (e.g., reading, writing, speaking, and listening)? • What visuals or words will I use to communicate the content and language objectives?
2. Consider student needs.	• How can I connect the content to be taught to content that students have learned? Or, how will I fill in gaps if students don't have prerequisite content needed for the lesson?	• What context or models might I select that are a good match to students' social and cultural backgrounds and previously learned vocabulary?

Table 5.3. Reflective questions for planning and teaching mathematics lessons for ELLs. (*continued*)

Process	Mathematics Content Considerations	Language Considerations
3. Select, design, or adapt a task.	• What task can I use that addresses the content identified in No. 1 and the needs of my students identified in No. 2? • How might I adapt a task so that it has multiple entry and exit points (i.e., is challenging and accessible to a range of students)?	• What context might I use that is meaningful to the students' cultures and backgrounds? • What language pitfalls does the task have? Which of these will I eliminate, and which of these need explicit attention? • Which words or phrases, even if familiar to students, take on new meaning in a mathematics context (e.g., homonyms, homophones, and words such as *mean, similar, find*)?
Reflective Questions for Teaching		
1. Introduce the task (the *Before* phase).	• How will I introduce the task in a way that elicits prior mathematics knowledge needed for the task? • Is a similar task needed to build background related to the content (or would such a preview take away from the purpose or challenge of the task)?	• How can I connect the task to students' experiences and to familiar contexts? • What key vocabulary do I want to introduce so that the words will be used throughout the lesson? (Post key vocabulary in a prominent location.) • What visuals and real objects can I use that bring meaning to the selected task? • How can I present the task in visual, written, and oral formats? • How will I be sure that students understand what they are to do in the *During* phase?
2. Work on the task (the *During* phase).	• What hints or assists might I give as students work that help them focus without taking away their thinking? • What extensions or challenges will I offer for students who successfully solve the task? • What questions will I pose to push the mathematics identified in the learning goals?	• Have I grouped students for both academic and language support? • Have I encouraged students to draw pictures, make diagrams, and/or use manipulatives? • Have I used strategies to reduce the linguistic demands (e.g., graphic organizers, sentence starters such as, "I solved the problem by . . .," recording tables, and concept maps) without hindering the problem solving?
3. Debrief and discuss the task and the mathematics (the *After* phase).	• How will students report their findings? • How will I format the discussion of the task? • What questions will I pose to push the mathematics identified in the learning goals?	• In what ways can I maximize language use in nonthreatening ways (e.g., think–pair–share)? • How can I encourage and reinforce different formats (multiple exit points) for demonstrating understanding of the lesson content? • How might I provide advance notice, language support, or rehearsal to English language learners so that they will be comfortable speaking to their peers? • Am I using appropriate "wait time"?
Formative Assessment		
Throughout lesson and unit	• What questions will I ask during the lesson, or what will I look for in the students' work as evidence of learning the objectives (*During* and *After* phases)? • What follow-up might I provide to students who are not demonstrating understanding of the mathematics?	• What words will I use in my questions to be sure the questions are understood? How might I use a translator to assist in assessing? • If a student is not succeeding, how might I diagnose whether the problem is with language, content, or both? • What accommodations can I provide to be sure I am accessing what the students know?

is the language specific to a content area such as mathematics (Cummins, 1994). Academic language is harder to learn because it is not used in a student's everyday world. In addition, there are unique features of the language of mathematics that make it difficult for many students, in particular those who are learning English. Teaching the academic language of mathematics evolves over time and requires thoughtful and reflective instructional planning.

Honor Use of Native Language

Valuing a student's native language is one of the ways you value his or her cultural heritage. In a mathematics classroom, students can communicate in their native languages while continuing their English language development (Haas & Gort, 2009; Moschkovich, 2009; Setati, 2005). For example, a good strategy for students working individually or in small groups is having them think about and discuss the problem in their preferred language. If a student knows enough English, then the presentation in the *After* phase of the lesson can be shared in English. If the student knows little or no English and does not have access to a peer who shares his or her native language, then a translator, the use of a Web-based dictionary, or a self-made mathematics-focused dictionary can be a strong support. Students within the small group can also be coached to use visual aids and pictures to communicate. Bilingual students will often code-switch, moving between two languages. Research indicates that the practice of code-switching supports mathematical reasoning because students select the language in which they can best express their ideas (Moschkovich, 2009).

Certain native languages can support learning mathematical words. Because English, Spanish, French, Portuguese, and Italian all have their roots in Latin, many math words are similar across languages (Celedón-Pattichis, 2009; Gómez, 2010). For example, *aequus* (Latin), *equal* (English), and *igual* (Spanish) are cognates. See if you can figure out the English mathematical terms for the following Spanish words: *número*, *hexágano*, *ángulo*, *triángulo*, *álgebra*, *circunferencia*, *ángulo*, *triángulo*, *quadra*, and *cubo*. Students may not make this connection if you do not point it out, so it is important to explicitly teach students to look for cognates.

Use Content and Language Objectives

If students know the purpose of a lesson, they are better able to make sense of the details when they are challenged by some of the oral or written explanations. When language expectations are explicitly included, students will know that they will be responsible for reaching certain language goals alongside mathematical goals and will be more likely to attempt to learn those skills or words. Here are two examples of dual objectives:

1. Students will write and evaluate expressions with variables from story situations (mathematics).
2. Students will describe in writing and speaking how the expression connects to the situation (language and mathematics).

Explicitly Teach Vocabulary

Intentional vocabulary instruction must be part of mathematics instruction for all students. There is strong evidence that teaching a set of academic vocabulary words intensively across several days using a variety of instructional activities supports ELLs (Baker et al., 2014). Vocabulary support can happen throughout a lesson, as well as reinforced before or after a lesson. These additional opportunities can reinforce understanding as they help students learn the terminology. Examples include these:

- Self-made math dictionaries that link concepts and terms with drawings or clip art pictures (Kersaint, Thompson, & Petkova, 2009)
- Foldables of key words for a topic (for example, see *Dinah Zike's Teaching Mathematics with Foldables* (Zike, n.d.), a free download)

- Games focused on vocabulary development (e.g., "Pictionary" or "$10,000 Pyramid")
- Interactive word walls, including visuals and translations
- Graphic organizers that look at multiple ways to help define a term (for example, **Vocabulary Reference Card Template**)

All students benefit from an increased focus on language; however, too much emphasis on vocabulary can diminish the focus on mathematics. Importantly, the language support should be *connected* to the mathematics and the selected task or activity (Livers & Bay-Williams, 2014).

As you analyze a lesson, you must identify terms related to the mathematics and to the context that may need explicit attention. Consider the following eighth-grade constructed response item (low level of difficulty), released from the 2007 National Assessment of Educational Progress (NCES, 2011).

Teaching Tip

Not all vocabulary should be "previewed" because the term (and its concept) can sometimes be better understood after some exploration has occurred.

Three tennis balls are to be stacked one on top of another in a cylindrical can. The radius of each tennis ball is 3 cm. To the nearest whole centimeter, what should be the minimum height of the can? Explain why you chose the height that you did. Your explanation should include a diagram.

In order for students to engage in this task, they need visuals or illustrations for the context words, such as *tennis balls* and *cylindrical can*. They also need support for the mathematical terminology they must learn and retain. The words *cylinder*, *radius*, and *height* are listed in the problem, but *sphere* and *diameter* are words that are also important to this task. Students need to know what is meant by *diagram* so that they can create one.

Foster Student Participation during Instruction

Student participation is important to learning (Tomaz & David, 2015; Wager, 2014). Facilitating discourse that provides access to ELLs is critical. This includes (1) efforts to ensure that ELLs understand and have the background for engaging in the focus task(s), and (2) the need to put structures in place for student participation throughout the lesson.

Build Background Knowledge

Similar to building on prior knowledge, building background also takes into consideration native language and culture as well as content (Echevarria, Vogt, & Short, 2012). If possible, use a context and appropriate visuals to help students understand the task you want them to solve. This is a nonthreatening and engaging way to help students make connections between what they have learned and what they need to learn. For example, if you are teaching operations with positive and negative numbers (integers), you can build background by using actual thermometers, which provide a visual and a context. You might display pictures of places covered in snow and position them near pictures of thermometers with low temperatures to build the meaning of the negative values (e.g., -10 is colder and therefore less than -3).

Some aspects of English and mathematics are particularly challenging to ELLs (Whiteford, 2009/2010). For example, teen numbers sound a lot like their decade number—if you say *sixteen* and *sixty* out loud, you can hear how similar they are. And, decimal fractions (e.g., 0.15) sound like whole numbers (e.g., 1500). Emphasizing the *n* in *teen* or the *ths* in decimal fractions helps ELLs hear the difference. Remember, too, that the U.S. measurement system may be unfamiliar to ELLs. When encountering content that may be unfamiliar or difficult for ELLs, devote additional time to building background so that students can engage in the mathematical tasks without also having to navigate language and background knowledge.

Use Comprehensible Input

Comprehensible input means that the message you are communicating is understandable to students. Modifications include simplifying sentence structures and limiting the use of nonessential or confusing vocabulary (Echevarria, Vogt, & Short, 2012). Note that these modifications do not lower expectations for the lesson. Sometimes, teachers put many unnecessary words and phrases into questions, making them less clear to nonnative speakers. Compare the following two sets of teachers' instructions:

Not Modified:

You have a worksheet in front of you that I just gave out. For every situation, I want you to determine the total surface area for the shapes. You will be working with your partner, but each of you needs to record your answers on your own paper and explain how you got your answer. If you get stuck on a problem, raise your hand.

Modified:

Please look at your paper. (Holds paper and points to the first picture.) You will find the surface area. What does surface area mean? (Allows wait time.) How can you calculate surface area for the shapes? ("Calculate" is more like the Spanish word *calcular*, so it is more accessible to Spanish speakers.) Talk to your partner. (Points to mouth and then to a pair of students as she says this.) Write your answers. (Makes a writing motion over paper.) If you get stuck on a problem (shrugs shoulders and looks confused), raise your hand (holds hand up).

Notice that three things have been done: sentences have been shortened, confusing words have been removed, and related gestures and motions have been added to the oral directives. Also notice the "wait time" the teacher gives. It is very important to provide extra time after posing a question or giving instructions to allow ELLs time to translate, make sense of the request, and then participate.

Another way to provide comprehensible input is to use a variety of tools to help students visualize and understand what is verbalized. In the preceding example, the teacher models the instructions. Effective tools include manipulatives, real objects, pictures, visuals, multimedia, demonstrations, and literature. When introducing a lesson, include pictures, real objects, and diagrams. For example, with surface area of rectangular solids, show different boxes that have been cut open into their nets (flat shapes that can be folded to form three-dimensional solids). Ask students, "How many tiles will cover each of the faces of the box?" And, as you ask, physically move some tiles on top of the box to illustrate. Review terms for the box (net, side, length, width, height) and label a box for reference.

Engage Students in Discourse That Reflects Language Needs

Discourse, or the use of classroom discussion, is essential for *all* learning (Cirillo, Steele, Otten, Herbel-Eisenmann, McAneny, & Riser, 2014), but is particularly important for ELLs, who need to engage in productive language (writing and speaking), not just receptive language (listening and reading) (Baker et al., 2014). As noted in *Application of Common Core State Standards for English Learners*:

> ELLs are capable of participating in mathematical discussions as they learn English. Mathematics instruction for ELL students should draw on multiple resources and modes available in classrooms—such as objects, drawings, inscriptions, and

> gestures—as well as home languages and mathematical experiences outside of school. Mathematical instruction should address mathematical discourse and academic language (CCSSO, 2011, p. 2).

There are strategies you can use in classroom discourse that help ELLs understand and participate in discussions. Practicing an explanation first with a partner can increase participation. Revoicing, using gestures and visuals, inviting sharing and justification, and asking other students to respond to ELLs' ideas are all ways to support participation (Shein, 2012; Turner, Dominguez, Maldonado, & Empson, 2013). *Revoicing* is a research-based strategy that helps ELLs hear an idea more than once and hear it restated with the appropriate language applied to concepts. Students from other countries often solve or record problems differently, so inviting ELLs to share how they solved a problem can enhance the richness of discussion about a task. Importantly, ELLs cannot always explain their ideas fully, but rather than call on someone else; press for details. This pressing, or expansion move (Choppin, 2014), helps the teacher decide whether the idea makes sense *and* it helps other students make sense of the ideas (Maldonado, Turner, Dominguez, & Empson, 2009). Note that these teacher moves also attribute ideas to students, thereby enhancing their mathematical authority (power), as discussed earlier in this chapter.

Teaching Tip

Making the strategies of ELLs public and connecting their strategies to others supports the learning of all students while building the confidence of the ELLs.

Plan Cooperative/Interdependent Groups to Support Language Development

The use of cooperative groups is a valuable way to support ELLs (Baker et al., 2014). For ELLs, groups provide the opportunity to use language, but only if the groups are carefully formed in a way that considers students' language skills. Placing an ELL with two English-speaking students may result in the ELL being left out. On the other hand, grouping all Spanish speakers together prevents these students from having the opportunity to hear and participate in mathematics in English. Consider placing a bilingual student in a group with a student who has limited English, or place students who have the same first language together with native speakers so that they can help one another understand and participate (Garrison, 1997; Khisty, 1997).

CLASSROOM VIGNETTE

The strategies just described are subtle moves in teaching. As you read the following vignette, look for strategies that the teacher applies to provide support for ELLs while keeping expectations high.

Ms. Evers is teaching a seventh-grade lesson that involves determining whether a situation is proportional, and if so, what is the constant of proportionality. The lesson asks students to analyze a situation in which servings of trail mix are being made. Ms. Evers has four ELLs in her class, including a child from Korea who knows very little English and three children from Mexico who speak English at varying levels. All are recent arrivals in the United States. These students are not familiar with trail mix, or with the academic language of rates and proportions. Ms. Evers knows she needs to build background to ensure that they can participate in the lesson.

The lesson begins with Ms. Evers placing cereal mix in two different bowls, one with twice the amount of mix as the other (neither bowl has enough for the whole class). She holds up the bowl with the smaller portion and asks what is in her mix. Students say Cheerios and chocolate chips (Ms. Evers emphasizes the word *mix* by saying she *mixed* Cheerios and chocolate chips in her *recipe*). As students say the terms, she shows the bags of each. She then shows the bowl with more in it and asks, "This has more, but is it the same *mix*?" She asks students to think (points at her head) yes or no, then share (points at her mouth) with a partner. After think–pair time, she gets some ideas from student pairs. Then, Ms. Evers says that the big idea today is to figure out if the mixes in the two bowls are in the same *ratio* of Cheerios to chocolate chips.

Ms. Evers then uses the context she started with—a mix of Cheerios and chocolate chips—and presents a table of how to make the mix. (The first two columns in the table represent the actual bowls.)

Ms. Evers asks, "Is this the same mix for all the different-sized recipes?" She gives students materials (grid paper, counters, cubes) and says, "Use these tools to show and tell me how you know if these are the same mix or not. Are the food items in the same *ratio*?"

Mix	For 1 Person	For 2 People	For 3 People	For 4 People
Cups of Cheerios	2	4	6	8
Cups of chocolate chips	1	2	3	4

This lesson is split into two parts. After students share their illustrations and explanations indicating that the ratios are the same, Ms. Evers asks students, "If we know how many chocolate chips, can we figure out how many Cheerios?" She asks students to work with their partners to answer the question and to see whether they can write an equation that fits all mixes. When the students share these responses, Ms. Evers teaches the new words—*proportional* and *constant of proportionality*, having students say the words, describe the words, and add the words to their personal math dictionaries.

Stop and Reflect

What specific strategies to support ELLs can you identify?

There are a number of strategies that provided support for ELLs: recognizing the potential language support for *mix* and *ratio* initially, and later for *proportional* and *constant of proportionality*. Ms. Evers anticipated that the context (trail mix) might be unfamiliar, so she prepared two of the mixes to build background (and interest!). She employed a think–pair–share technique, concrete models (the cereal and chips), and scaffolded the lesson by separating it into part 1 (on ratios) and part 2 (on determining the constant). Most importantly, Ms. Evers did not diminish the challenge of the task with these strategies. If she had altered the task—for example, by having students find missing values in the table and telling them that they were related in the same way—she would have taken the reasoning out of the proportional reasoning lesson. Conversely, if she had simply asked students to decide whether the mixes were proportional and to find the constant of proportionality, she might have kept her expectations high but failed to provide the support that would enable all students to succeed.

Assessment Considerations for ELLs

If a teacher wants to understand what a student knows about mathematics, then the student should be able to communicate that understanding in a way that is best for the student, even if the teacher may need a translation. Research shows that ELLs perform better when a test is given in their native language (Robinson, 2010). Several strategies can assist teachers in using formative assessments with ELLs, including tasks with multiple entry and exit points, diagnostic interviews, tasks that limit the linguistic load, accommodations, and self-assessment.

Select Tasks with Multiple Entry and Exit Points

An aspect of teaching mathematics through problem solving that is important, particularly for ELLs, is to select tasks carefully. If a problem can be solved in multiple ways, an ELL is more likely to be able to design a strategy that makes sense and then illustrate that strategy. Inviting students to show and/or explain their strategy provides options for ELLs to use words and pictures to communicate their thinking.

Use Diagnostic Interviews

When ELLs do not get a correct answer or cannot explain a response, it is easy to assume it is a lack of mathematical understanding rather than a language issue. Diagnostic interviews provide a chance to observe what content or language the student does or does not understand. Consider the following task focused on percent increase (grade 7).

Marissa is a cupcake designer. In January she sold 480 cupcakes. Because of Valentine's day in February, she thinks she can do better. If she wants to sell 30% more in February, how many cupcakes will she be selling in February?

Stop and Reflect

If a student missed this problem, what do you think might be the reason?

Students might struggle with this problem because of a lack of understanding of percentage increase. Or, the struggle may be due to context-related vocabulary, such as *sold*, *sell*, *selling*, *cupcakes* and *designer*. Also, it could also be due to the sentence structure, in this case the "If . . . then" style common in math story problems. Diagnostic interviews have revealed that the word *if* can prevent students from comprehending what the sentence is asking (a challenge for native English speakers as well) (Fernandez, Anhalt, & Civil, 2009). The fact that there are many possible reasons why a student might not be able to solve a task, some related to language and some to mathematics, is a strong argument for using diagnostic interviews. If we misdiagnose the reason for a student's struggles, our interventions will be misguided.

Diagnostic interviews also can be used before instruction in order to assess the mathematical and language needs of students. Hearing an ELL's interpretation of a problem and seeing how he or she approaches the problem provide valuable insights. For example, a student might say "three-hundredths *from* seven-tenths" rather than "seven-tenths *minus* three-hundredths," which indicates the way the student talks about subtraction at home. Using and connecting both ways of talking about subtraction strengthen everyone's understanding.

Limit Linguistic Load

If you are trying to assess student understanding, look for language that can interfere with students' understanding the situation (e.g., unneeded elaboration in a story, difficult or unfamiliar vocabulary). Removing pronouns such as *they*, *this*, *that*, *his*, and *her* and using actual names can assist ELLs in understanding some problems. For example, the television problem above could be rewritten as follows:

The sales team sold 48 televisions in April. The sales team wants to sell 30 percent more televisions in May so that they will earn a bonus. How many televisions must the sales team sell in May to receive the bonus? Justify your solution.

Of course, this particular problem could be adapted further by using illustrations or manipulatives.

Teaching Tip

To reduce the linguistic load for students, pick one scenario/context, and stay with it for an entire lesson or series of lessons.

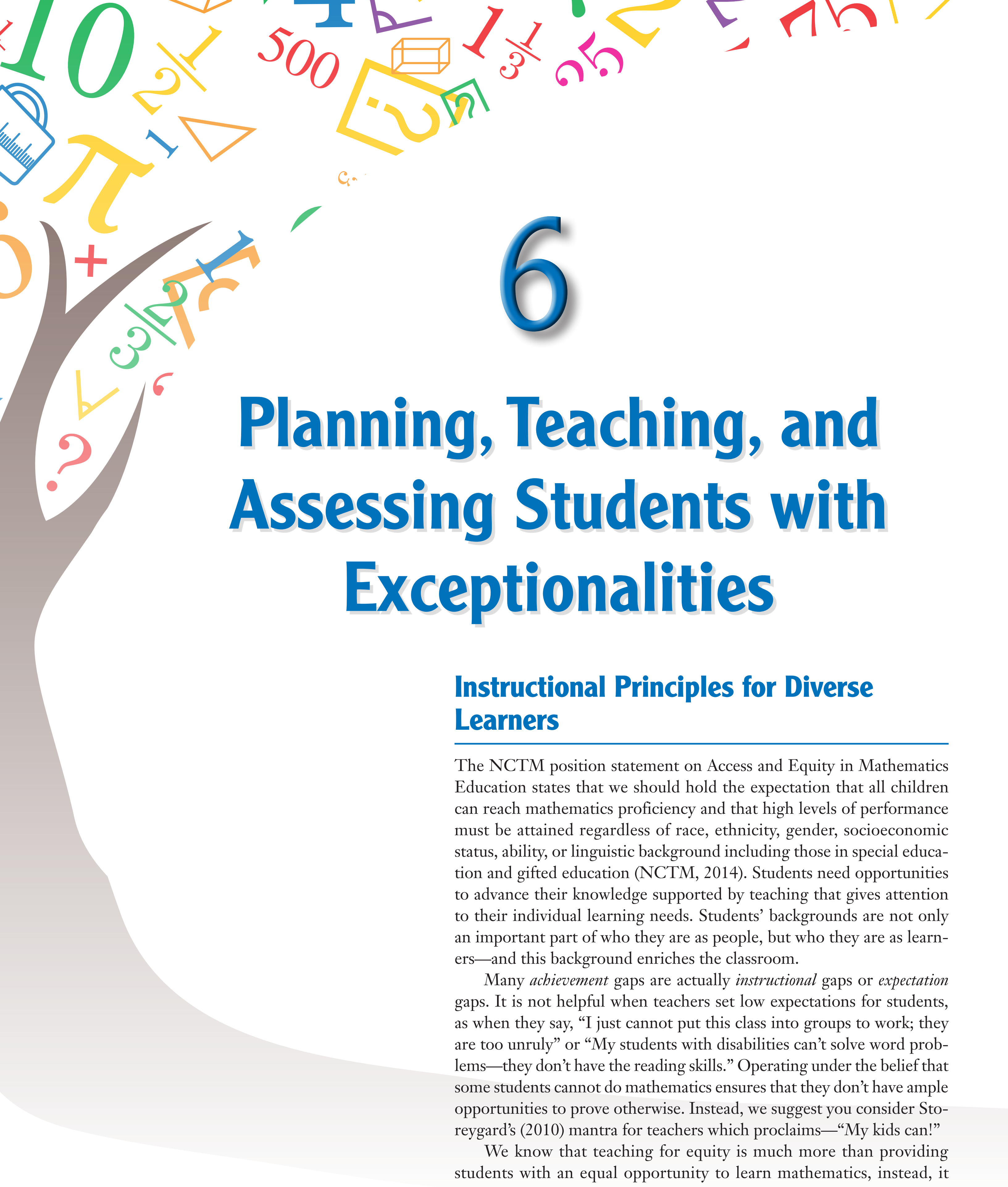
If there are more problems in a lesson or assessment, staying with the same scenario/context allows students to focus their thinking on the mathematics without getting bogged down in different contexts. For example, instead of using a different percentage increase situation, stay with television sales but change the problem details (what is known, unknown). This reduces the linguistic load but keeps the mathematical challenge high.

Provide Accommodations

For assessing, *providing accommodations* refers to strategies for making sure that the assessment itself is accessible to children. This might mean allowing students to hear the question (students often can understand spoken English better than written English), shortening the assessment, or extending the time (Kersaint, Thompson, & Petkova, 2009). In addition, you can refer to word walls and provide sentence starters so that the ELL knows what type of response you want. For example, “My equation fits the story because . . .” In general, the goal with assessment accommodations, like teaching accommodations, is to put structures in place so that ELLs can understand what you want them to learn and you can understand what they have learned.

Stop and Reflect

The goal of equity is to offer every student access to important mathematics. What might you have on a list of things to do (and things not to do) that support equity, access and empowerment?



6

Planning, Teaching, and Assessing Students with Exceptionalities

Instructional Principles for Diverse Learners

The NCTM position statement on Access and Equity in Mathematics Education states that we should hold the expectation that all children can reach mathematics proficiency and that high levels of performance must be attained regardless of race, ethnicity, gender, socioeconomic status, ability, or linguistic background including those in special education and gifted education (NCTM, 2014). Students need opportunities to advance their knowledge supported by teaching that gives attention to their individual learning needs. Students' backgrounds are not only an important part of who they are as people, but who they are as learners—and this background enriches the classroom.

Many *achievement* gaps are actually *instructional* gaps or *expectation* gaps. It is not helpful when teachers set low expectations for students, as when they say, "I just cannot put this class into groups to work; they are too unruly" or "My students with disabilities can't solve word problems—they don't have the reading skills." Operating under the belief that some students cannot do mathematics ensures that they don't have ample opportunities to prove otherwise. Instead, we suggest you consider Storeygard's (2010) mantra for teachers which proclaims—"My kids can!"

We know that teaching for equity is much more than providing students with an equal opportunity to learn mathematics, instead, it attempts to attain equal outcomes for all students by being sensitive to individual differences. How you will maintain equal outcomes and high expectations while providing for individual differences with strong support can be challenging. Equipping yourself with an ever-growing collection of instructional strategies for a variety of students is critical. A strategy that works for one student may be completely ineffective with another, even for a student with the same exceptionality. Addressing the needs of *all* means providing access and opportunity for:

- Students who are identified as struggling or having a disability.
- Students who are mathematically gifted.
- Students who are unmotivated or need to build resilience.

You may think, "I do not need to read the section in this chapter on mathematically gifted students because they will be pulled out for math enrichment." Students who are mathematically talented need to be challenged in the daily core instruction, not just when they are participating in a gifted program.

One of the basic tenets of education is individualizing the content taught and methods used for students who struggle, particularly those with special needs. Mathematics learning disabilities are best thought of as cognitive differences, not cognitive deficits (Lewis, 2014). Students with disabilities often have mandated individualized education programs (IEPs) that guarantee access to grade-level mathematics content—preferably in a general education classroom. This legislation also implies that educators consider individual learning needs not only in terms of *what* mathematics is taught but also *how* it is taught.

Essential in making decisions about how you can adapt instruction to meet individual learner's needs is the use of accommodations and modifications. An accommodation is a response to the needs of the environment or the learner; it does not alter the task. For example, you might write down directions for a student instead of just saying them orally. A modification changes the task, making it more accessible to the student. For example, if students are asked to find the surface area of a three-dimensional shape, you might break the shape up into all of its faces (as a net) and ask the student to find the area of each face and then combine those amounts. Then the next shape they will attempt without the modification. When modifications result in an easier or less demanding task, expectations are lowered. Modifications should be made in a way that leads back to the original task, providing scaffolding or support for learners who may need it. Complete an **Accommodation or Modification Needs** table to reflect on how you will plan for students in your classroom who have special needs. Record the evidence that you are adapting the learning situation.

In this chapter, we share research-based strategies that reflect these equity principles while providing appropriate accommodations and modifications for the wide range of students in your classroom.

Prevention Models

In many school systems, a systematic process for achieving higher levels performance for all students often includes a multitiered system of support (MTSS) frequently called response to intervention (RtI). This approach commonly emphasizes ways for struggling students to get immediate assistance and support rather than waiting for students to fail before they receive help or also for identifying students who are far exceeding standards and need additional challenges. Multitiered models are centered on the three interwoven elements: high quality curriculum, instructional support (interventions), and formative assessments that capture students' strengths and weaknesses. Often this model is used to determine whether low achievement is due to a lack of high-quality mathematics (i.e., "teacher-disabled students") (Baroody, 2011; Ysseldyke, 2002) or due to an actual learning disability. This model can also help determine more intensive instructional options for students who may need to have additional advanced mathematical challenges beyond what other students study.

Response to Intervention

RtI (https://www.youtube.com/watch?v=nkK1bT8ls0M) is a multitiered student support system that is often represented in a three-tier triangle format. As you might guess there are a variety of RtI models developed by school systems as they structure their unique approaches to meeting student needs.

As you move up the tiers the number of students involved decreases, the teacher–student ratio decreases, and the level of intervention increases. Each tier in the triangle represents a level of intervention with corresponding monitoring of results and outcomes, as shown in Figure 6.1. The foundational and largest portion of the triangle (Tier 1) represents the core instruction for **all** students based on high quality mathematics curriculum, highly-engaging instructional practices (i.e., manipulatives, conceptual emphasis, etc.) and progress monitoring assessments. For example, if using a graphic organizer in Tier 1 core instruction the following high-quality practices would be expected in the three phases of the lesson—Before, During, and After:

Before: States purpose, introduces new vocabulary, clarifies concepts from needed prior knowledge in a visual organizer, and defines tasks of group members (if groups are being used).

During: Displays directions in a chart, poster, or list; provides a set of guiding questions in a chart with blank spaces for responses.

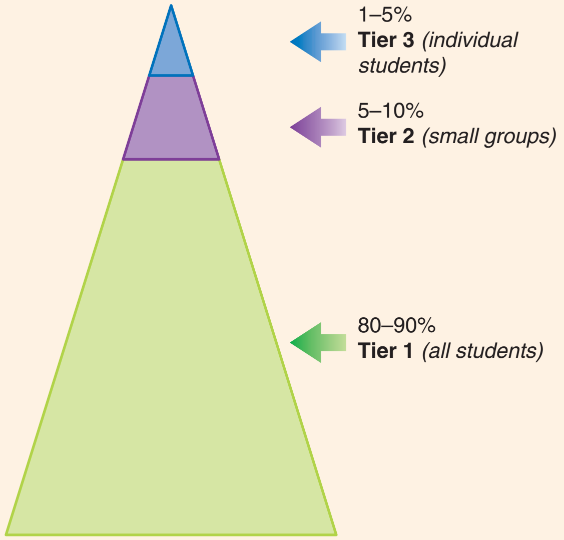
After: Facilitates a discussion to highlight or make more explicit the significant concepts or skills and then presents a summary and list of important concepts as they relate to one anothe**r.**

Tier 2 represents students who did not reach the level of achievement expected during Tier 1 instruction. Students who move to Tier 2 should receive supplemental targeted instruction (interventions) using more explicit strategies with systematic teaching of critical skills and concepts, more intensive and frequent instructional opportunities, and more supportive and precise prompts (Torgesen, 2002). The National Council of Teachers of Mathematics' position statement on Interventions (2011) endorses the use of interventions that increase in intensity as students demonstrate continuing struggle with learning mathematics. Interventions may require "heroic action to preclude serious complications" (Fuchs & Fuchs, 2001, p. 86).

If further assessment reveals students have made favorable progress, the extra interventions are faded and discontinued. But, if difficulties and struggles still remain, the interventions can be adjusted in intensity, and in rare cases, students are referred to the next tier of support. Tier 3 is for students who need more intensive assistance, which may include comprehensive mathematics instruction or a referral for special education evaluation or special education services. Instructional strategies for the three tiers are outlined in Table 6.1.

Figure 6.1

Response to intervention—using effective prevention stratrgies for all children.



Common Features across Tiers

- **Research-Based Practices:** Prevention begins with practices based on students' best chances for success.
- **Data-Driven:** All decisions are based on clear objectives and formative data collection.
- **Instructional:** Prevention and intervention involve effective instruction, prompts, cues, practice, and environmental arrangements.
- **Context-Specific:** All strategies and measures are selected to fit individual schools, classrooms, or students.

Source: Based on Scott, Terence, and Lane , Holly. (2001). *Multi-tiered interventions in Academic and Social Contexts.* Unpublished manuscript, University of Florida, Gainesville.

Table 6.1. Interventions for teaching mathematics in a multitiered model.

Tiers	Interventions
Tier 1	**Highly qualified regular classroom teacher:** • Incorporates high-quality, engaging, and rigorous curriculum • Expects all students will be challenged • Builds in CCSSO Standards for Mathematical Practice and NCTM process standards • Commits to teaching the curriculum as defined • Uses multiple representations such as manipulatives, visual models, and symbols • Monitors progress to identify struggling students and students who excel at high levels for possible interventions • Uses flexible student grouping • Fosters active student involvement • Communicates high expectations
Tier 2	**Highly qualified regular classroom teacher, with collaboration from other highly qualified educators (i.e., special education teacher):** • Works with students (often in small groups) in engaging, high-quality, and rigorous supplemental sessions outside of the core instruction • Conducts individual diagnostic interviews to target a student's strengths and weaknesses to facilitate next instructional steps • Slices back (Fuchs & Fuchs, 2001) to material from a previous grade to ramp back up to grade-level curriculum • Collaborates with special education, gifted, and/or ELL specialists • Creates lessons that emphasize the big ideas (focal points) or themes • Incorporates CSA (concrete, semi-concrete, abstract) approach • Shares thinking in a think-aloud to demonstrate how to make problem-solving decisions • Incorporates explicit systematic strategy instruction (summarizes key points and reviews key vocabulary or concepts prior to the lesson) • Models specific behaviors and strategies, such as how to handle measuring materials or geoboards • Uses mnemonics or steps written on cards or posters to help students follow, for example, the stages of problem-solving • Uses peer-assisted learning, where another student can provide help to a student in need • Supplies families with additional instructional support materials to use at home • Encourages student use of self-regulation and self-instructional strategies such as revising notes, writing summaries, and identifying main ideas • Teaches test-taking strategies and allows the students to use a highlighter on the test to emphasize important information
Tier 3	**Highly qualified special education teacher:** • Works one-on-one with students • Uses tailored instruction based on specific areas of strengths and weaknesses • Modifies instructional methods, motivates students, and adapts curricula • Uses explicit contextualization of skills-based instruction

Progress Monitoring

A key to guiding students' movement within the multitiered prevention model is the monitoring of students' progress. One way that you can collect evidence of student knowledge of concepts through the use of diagnostic interviews (examples are described in Chapter 3 and throughout the book in a feature called Formative Assessment Notes) (Hodges, Rose, &

Hicks, 2012). Another approach is to assess students' growth toward fluency in basic facts, an area that is well-documented as a barrier for students with learning disabilities (Mazzocco, Devlin, & McKenney, 2008). Combining instruction with short daily assessments proved to be a way to help students not only be better at remembering the facts but better at generalizing to other facts (Woodward, 2006). The collection of information gathered from these assessments will reveal whether students are making the progress expected or if more intensive instructional approaches need to be put into place.

Implementing Interventions

NCTM has shared a set of effective, research-based strategies (NCTM, 2007b) for teaching the subset of students for whom the initial core instruction was not effective (the students needing Tier 2 or Tier 3 interventions). These strategies include systematic and explicit instruction, think-alouds, concrete and visual representations of problems, peer-assisted learning activities, and formative assessment data provided to students and teachers. These interventions, proven to be effective for students with disabilities, may represent principles different from those used at Tier 1.

Explicit Strategy Instruction

Explicit instruction is characterized by highly structured, teacher-led instruction on a specific strategy. When engaging in this explicit instruction you do not merely model the strategy and have students practice it, instead you try to illuminate the decision making along the way—a process that may be troublesome for these particular learners without support. In this instructional strategy, after you assess the students so that you know what to target, you use a tightly organized sequence from modeling the strategy to prompting students through the model, to practice. Your instruction uses these teacher-led explanations of concepts and strategies, including the critical connection building and meaning making that help learners relate new knowledge with concepts they know. Let's look at a classroom teacher using explicit instruction:

> *As you enter Mr. Logan's classroom, you see a small group of students who are struggling with understanding volume and being able to calculate volume using the measurements of length, width, and height. They are seated at a table listening to the teacher's detailed explanation and watching his demonstration. The students are using manipulatives, as suggested by Mr. Logan, and moving through carefully selected tasks. He tells the students to take out the open box (rectangular prism) and asks them how they could find a way to fill the box to see how much it holds. Mr. Logan asks, "Is* ***volume*** *a word you know?" He suggests that they first think about filling just the bottom of the box and how that might help them. The students use the available cubes to cover a layer in the bottom of the box. Mr. Logan writes their responses on the adjacent board as* $5 \times 12 = 60$. *Then, he asks them to talk about their reasoning process by asking the question, "What are some things you need to keep in mind as you place the cubes?" He then shows a box with cubes covering the bottom layer, but with gaps between the cubes. He says, "I have only 50 cubes covering the bottom. Can that be correct, too? "The students take turns answering these questions aloud. Then he asks them to estimate how many layers of cubes they might need to fill the box and records their responses. During the lesson Mr. Logan frequently stops the group, interjects points of clarification, and directly highlights critical components of the task. For example, he asks, "Do you need to complete each layer with cubes to find the volume of the box? Is there another way to calculate the number of cubes that will be needed to fill the box?" Mr. Logan then turns the box so that a different side forms the base and asks, "What will the volume be now?" He then has*

students talk about what the base and height would be, helping students see that any two of the three dimensions can be used to form the base. Vocabulary words, such as ***volume, base, height****, and* ***area****, are written on the "math word wall" nearby and the definitions of these terms are reviewed and reinforced throughout the lesson. At the completion of the lesson, students are given several similar pictures of rectangular solids and several examples with just the dimensions given as independent practice.*

A number of aspects of explicit instruction can be seen in Mr. Logan's approach to teaching volume concepts. He employs a teacher-directed teaching format, carefully describes the use of manipulatives, and incorporates a model-prompt-practice sequence. This sequence starts with verbal instructions and demonstrations with concrete models, followed by prompting, questioning, and then independent practice. Note that he also explicitly presents a non example when he places the cubes in the bottom of the box in a haphazard way leaving gaps. The students are deriving mathematical knowledge from Mr. Logan's oral, written, and visual clues.

As students with disabilities solve problems, explicit strategy instruction can help guide them in carrying out tasks. First ask them to read and restate the problem, draw a picture, or make a model with materials, develop a plan by linking this problem to previous problems, write the problem in a mathematical sentence, break the problem into smaller pieces, carry out operations, and check answers using a calculator or other appropriate tools. These self-instructive prompts, or self-questions, structure the entire learning process from beginning to end. Unlike more inquiry-based instruction, the teacher models these steps and explains components using terminology that is easily understood by students who struggle—students who did not discover them independently through initial Tier 1 or 2 activities. Yet, consistent with what we know about how all students learn, students are still doing problem solving (not just skill development).

Concrete models can support explicit strategy instruction. For example, consider a lesson on multiplication of fractions using the example $\frac{2}{3} \times \frac{1}{4}$. Create a square from paper, fold into fourths vertically and shade one long vertical strip. Then have students fold the paper horizontally to show thirds. You might ask, "How can I use the folds for the thirds to color two-thirds of the shaded fourth?" In contrast, a teacher with a more inquiry-oriented approach might say, "Using grid paper can you show me a representation for two-thirds times one-fourth?" Although initially more structured, the use of concrete models in this fashion will provide students with disabilities with greater access to abstract concepts and will also lead to generalizing a procedure for finding the product.

There are a number of possible advantages to using explicit strategy instruction for students with disabilities. This approach helps you make more explicit for these students the covert thinking strategies that others use in mathematical problem solving. Although students with disabilities hear other students' thinking strategies in the *After* phase of each lesson, they frequently cannot keep up with the rapid pace of the sharing. Without extra time to reprocess the conversation, students with disabilities may not have access to these strategies. More explicit approaches are also less dependent on the student's ability to draw ideas from past experience or to operate in a self-directed manner.

There are some aspects of explicit strategy instruction that have distinct disadvantages for students with special needs, particularly the times students must rely on memory—often one of their weakest skills. There is also the concern that highly teacher-controlled approaches promote prolonged student dependency on teacher assistance. This is of particular concern for students with disabilities because many are described as passive learners.

Students learn what they have the opportunity to practice. Students who are never given opportunities to engage in self-directed learning (based on the assumption that this is not an area of strength) will be deprived of the opportunity to develop skills in this area. In fact, the best explicit instruction is scaffolded, meaning it moves from a highly structured, single-strategy approach to multiple models, including examples, and nonexamples. It also includes immediate error correction followed by the fading of prompts to help students move to independence. To be effective, explicit instruction must include making mathematical relationships explicit (so that students, rather than only learning how to do that day's mathematics, make connections to other mathematical ideas). Because making connections is a major component in how students learn, it must be central to learning strategies for students with disabilities.

Concrete, Semi-Concrete, Abstract (CSA)

The CSA (concrete, semi-concrete, abstract) intervention has been used in mathematics education for a variety of topics for years (Dobbins, Gagnon, & Ulrich, 2014; Griffin, Jossi, & van Garderen, 2012; Heddens, 1964; Hunter, Bush, & Karp, 2014). Based on Bruner and Kennedy's stages of representation (1965), this model reflects concrete representations such as manipulative materials that encourage learning through movement or action (enactive stage) to semi-concrete representations of drawings or pictures (iconic stage) and learning through abstract symbols (symbolic stage). Built into this approach is the return to visual models and concrete representations as students need or as students begin to explore new concepts or extensions of concepts learned previously. As students share reasoning that shows they are beginning to understand the mathematical concept, there can be a shift to semi-concrete representations. This is not to say that this is a rigid approach that only moves to abstraction after the other phases. Instead, it is essential that there is parallel modeling of number symbols throughout this approach to explicitly relate the concrete models and visual representations to the corresponding numerals and equations. CSA also includes modeling the mental conversations that go on in your mind as you help students articulate their own thinking. Used particularly in a combination with explicit strategy instruction, this approach has met with high levels of success for students with disabilities (Flores, Hinton, & Strozier, 2014; Mancl, Miller, & Kennedy, 2012, Miller & Kaffar, 2011).

Peer-Assisted Learning

Students with special needs also benefit from other students' modeling and support (McMaster & Fuchs, 2016). The basic notion is that students learn best when they are placed in the role of an apprentice working with a more skilled peer or "expert." Although the peer-assisted learning approach shares some of the characteristics of the explicit strategy instruction model, it is distinct because knowledge is presented on an "as-needed" basis as opposed to a predetermined sequence. The students can be paired with older students or peers who have more sophisticated understandings of a concept. At other times, tutors and tutees can reverse roles during the tasks. Having students with disabilities "teach" others is an important part of the learning process, so giving students with special needs a chance to explain their thinking to a peer or younger student is a valuable learning tool.

Think-Alouds

When you use a "think-aloud" as an instructional strategy you demonstrate the steps to accomplish a task while verbalizing the thinking process and reasoning that accompany the actions. Remember, don't start with where your thinking is; assess and start where the student's thinking is. Let's look at an example. Consider a problem in which sixth-grade students are given the task of determining how much one hamburger costs given that 13 hamburgers cost $78.00. The think-aloud strategy would involve talking through the steps and identifying the reasons for each step as you progress through the task. As you write down the important information, you might state, "I need to find how much one hamburger would cost, the *rate*. How can I use the information we were given to find that amount? I know I have to use the total cost, but what other information do I need?" All of this dialogue occurs before the choice of an operation is finalized. Often teachers share alternatives about how else they could have carried out the task. When using this metacognitive strategy, try to talk about and model possible approaches (and the reasons behind these approaches) in an effort to make your invisible thinking processes visible to students.

Although you will choose any of these strategies as needed for interventions, your goal is always to work toward high student responsibility for learning. Movement to higher levels of understanding of content can be likened to moving up a hill. For some, formal stair steps with support along the way is necessary (explicit strategy instruction); for others ramps with encouragement at the top of the hill will work (peer-assisted learning). Other students can find a path up the hill on their own with some guidance from visual representations (CSA approach). All people can relate to the need to have different support during different times of their lives or under different circumstances, and it is no different for students with special needs (see Table 6.2). Yet students with special needs must eventually learn to create a path

Table 6.2. Common stumbling blocks for students with disabilities.

Stumbling Blocks for Students	What Will I Notice?	What Should I Do?
Trouble forming mental representations of mathematical concepts	• Can't interpret a number line for fractions or integers • Has difficulty going from a story about a garden plot (finding area) to a graph or dot paper representation	• Explicitly teach the representation—for example, exactly how to draw a diagram (e.g., partition the number line) • Use larger versions of the representation (e.g., number line or grid paper) so that students can move on (e.g. add or subtract integers) or interact with the model
Difficulty accessing numerical meanings from symbols (issues with number sense)	• Has difficulty with the use of variables • Does not understand the meaning of the equal sign • Can't interpret if an answer is reasonable	• Explicitly teach multiple ways of representing an unknown, showing the variations at exactly the same time (e.g., closed containers as models that hold an unknown quantity, algebra tiles) • Use a number balance to support understanding of the equal sign • Use multiple representations for a single problem to show how it would be carried out in a variety of ways rather than assigning multiple problems that are similar..
Difficulty keeping numbers and information in working memory	• Gets confused when multiple strategies are shared by other students during the *After* portion of the lesson • Forgets how to start the problem-solving process • Has difficulty keeping track of the meaning of the values in proportional situations (what varies with what)	• Record in writing the ideas of other students during discussions • Incorporate a chart that lists the main steps in problem solving as an independent guide or make bookmarks with questions the students can ask themselves as self-prompts • Use word labels with each numeric value
Lacks organizational skills and the ability to self regulate	• Loses track of steps in a process • Writes computations in a way that is random and hard to follow	• Use routines as often as possible or provide self-monitoring checklists to prompt steps along the way • Use graph paper to record problems or numbers • Create math word walls as a reference
Misapplies rules or overgeneralizes	• Applies rules such as "Always add a zero on the end of a number when you multiply it by ten" too literally, resulting in errors such as $2.5 \times 10 = 2.50$ • Mechanically applies algorithms—for example, adds $\frac{7}{8}$ and $\frac{12}{13}$ and generates the answer $\frac{19}{21}$.	• Always give examples as well as counter examples to show how and when "rules" should be used and when they should not. • Tie all rules into conceptual understanding, don't emphasize memorizing rote procedures or practices

to new learning on their own, as that is what will be required in the real world after formal education ends. Leaving students with only knowing how to climb steps with support and then later in life having them face hills without constant assistance and encouragement from others will not help them attain their life goals.

Teaching and Assessing Students with Learning Disabilities

Students with learning disabilities often have very specific difficulties with perceptual or cognitive processing that may affect memory; general strategy use; attention; the ability to speak or express ideas in writing; the ability to perceive auditory, visual, or written information; or the ability to integrate abstract ideas (Berch & Mazzocco, 2007). Although specific learning needs and strategies that work for one student may not work for another, there are some general ideas that can help as you plan instruction for students with special needs. The following questions should guide your planning:

1. What organizational, behavioral, and cognitive skills are necessary for the students with special needs to derive meaning from this activity?
2. Which students have significant weaknesses in any of these skills or concepts?
3. What are the children's strengths?
4. How can I provide additional support in these areas of weakness so that students with special needs can focus on the conceptual task in the activity? (Karp & Howell, 2004, p. 119)

Each phase of the lesson evokes specific planning considerations for students with disabilities. Some strategies apply throughout a lesson. The following discussion is based on Karp and Howell (2004) and although not exhaustive provides some specific suggestions for offering support throughout the lesson while maintaining the challenge.

1. *Structure the Environment*
    - *Centralize attention*. Move the student close to the board or teacher. Face students when you speak to them and use gestures. Remove competing stimuli.
    - *Avoid confusion*. Word directions carefully and ask the student to repeat them. Give one direction at a time. Use the same language for consistency. For example, when teaching decimals, talk about base-ten materials as ones, tenths, and hundredths rather than interchanging those names with "flats," "rods," and "cubes," which emphasize shape rather than value.
    - *Create smooth transitions*. Ensure that transitions between activities have clear directions and there are limited chances to get off task.
2. *Identify and Remove Potential Barriers*
    - *Help students remember*. Recognize that memory is often not a strong suit for students with disabilities and therefore develop mnemonics (memory aids) for familiar steps or write directions that can be referred to throughout the lesson. For example, **STAR** is a mnemonic for problem solving: **S**earch the word problem for important information; **T**ranslate the words into models, pictures, or symbols; **A**nswer the problem; **R**eview your solution for reasonableness (Gagnon & Maccini, 2001).
    - *Provide vocabulary and concept support*. Explicit attention to vocabulary and symbols is critical throughout the lesson. Preview essential terms and related prior knowledge/concepts, create a "math wall" of words and symbols to provide visual cues, and connect symbols to their precise meanings.

Teaching Tip

Note that searching the word problem for important information is different from identifying "key words" as the use of a "key word approach" is not effective.

- *Use "friendly" numbers*. Instead of using \$6.13 use \$6.00 to emphasize conceptual understanding rather than mixing computation and conceptual goals. Incorporate this technique when computation and operation skills are *not* the lesson objective.
- *Vary the task size*. Assign students with special needs fewer problems to solve. Some students can become frustrated by the enormity of the task.
- *Adjust the visual display*. Design assessments and tasks so that there is not too much on a single page. The density of words, illustrations, and numbers on a page can overload students. Find ways to put one problem on a page, increase font size, or reduce the visual display.

3. *Provide Clarity*
    - *Repeat the timeframe*. Give students additional reminders about the time left for exploring materials, completing tasks, or finishing assessments. This helps students with time management.
    - *Ask students to share their thinking*. Use the think-aloud method or think-pair-share strategy.
    - *Emphasize connections*. Provide concrete representations, pictorial representations, and numerical representations. Have students connect them through carefully phrased questions. Also, connect visuals, meanings, symbols, and words. For example, when teaching part-part and part-whole ratios, you can bring a group of eight students to the front of the class, placing part of the group (e.g., those wearing red) to the left and the other part (e.g., those not wearing red) to the right. Point out the part-part (red-not red) relationship and the part-whole (red-total) relationship with gestures as you ask students to explain how the following symbols connect to the situation 5:3, 3 to 8, $\frac{3}{5}$, 5:8 and $\frac{5}{8}$.
    - *Adapt delivery modes*. Incorporate a variety of materials, images, examples, and models for visual learners. Some students may need to have the problem or assessment read to them or generated with voice creation software. Provide written instructions in addition to oral instructions.
    - *Emphasize the relevant points*. Some students with disabilities may inappropriately focus on the color of a cube instead of the quantity of cubes when filling a prism to measure volume.
    - *Support the organization of written work*. Provide tools and templates so students can focus on the mathematics rather than the creation of a table or chart. Also use graphic organizers, picture-based models, and paper with columns or grids
    - *Provide examples and nonexamples*. Give examples of scalings of triangles that are dilations and scalings of triangles that are not dilations, or in another case relationships that are functions and relationships that are not functions. Help students focus on the characteristics that differentiate the examples from those that are not examples.
4. *Consider Alternative Assessments*
    - *Propose alternative products*. Provide options for how to demonstrate understanding (e.g., a verbal response that is written by someone else, voice recorded, or modeled with a manipulative). Use voice recognition software or word prediction software that can generate a whole menu of word choices when students type a few letters.
    - *Encourage self-monitoring and self-assessment*. Students with learning disabilities often need support in self-reflection. Asking them to review an assignment or assessment to

explain what was difficult and what they think they got right, can help them be more independent and take greater responsible for their learning.

- *Consider feedback charts.* Help students monitor their growth by charting progress over time.

5. *Emphasize Practice and Summary*
    - *Consolidate ideas.* Create study guides that summarize the key mathematics concepts and allow for review. Have students develop their own study guides.
    - *Provide extra practice.* Use carefully selected problems (not a large number) and allow the use of familiar physical models.

Not all of these strategies will apply to every lesson or every student with special needs, but as you are thinking about a particular lesson and certain individuals in your class, you will find that many of these will allow your students to engage in the task and accomplish the learning goals of the lesson. Explore **Strategies for Making Math Accessible** for a handy collection of cards that you can use to think about particular students as you plan. The Center for Applied Special Technology (CAST) website contains resources and tools to support the learning of all students, especially those with disabilities, through universal design for learning (UDL).

Adapting for Students with Moderate/Severe Disabilities

Students with moderate/severe disabilities (MSD) often need extensive modifications and individualized supports to learn mathematics. This population of students may include those with severe autism, sensory disorders, limitations affecting movement, cerebral palsy, processing disorders such as intellectual disabilities and combinations of multiple disabilities.

Originally, the curriculum for students with severe disabilities was called “functional,” in that it often focused on life-related skills such as managing money, telling time, using a calculator, measuring, and matching numbers to complete such tasks as entering a telephone number or identifying a house number. Now directives and assessments have broadened the curriculum to address the grade-level expectations in the *Common Core State Standards* (CCSSO, 2010) or other curriculum policy documents.

At a basic level, students work on developing number sense, use measuring tools, compare graphs, explore place-value concepts (often linked to money use), use the number line, and compare quantities. When possible, the content should be connected to life skills and when appropriate features of jobs. Shopping skills or activities in which food is prepared are both options for mathematical problem solving. At other times, link mathematical learning objectives to everyday events in a practical way. For example, when percent is considered, students can calculate their test grades or figure the price of an item given an online coupon. Students can also undertake a small project such as exploring two different handicapped access ramps at the school and finding out how the ramps are alike (e.g., slope) and how they are different (e.g., length).

Do not believe that all basic facts must be mastered before students with moderate or severe disabilities can move forward in the curriculum; students can learn geometric or measuring concepts without having mastered all basic facts. Geometry for students with moderate and severe disabilities is more than merely identifying shapes, but is in fact critical for orienting in the real world. Concepts such as parallel and perpendicular lines and curves and straight sides become helpful for interpreting maps of the local area. Students who learn to count bus stops and judge time can be helped to successfully navigate their world.

The handout **Math Activities for Students with Moderate or Severe Disabilities** offers ideas across the curriculum appropriate for teaching students with moderate to severe disabilities. Also, look at the **Additional Strategies for Supporting Students with Moderate/Severe Disabilities** handout for more ideas on how you can modify grade-level instruction.

Planning for Students Who Are Mathematically Gifted

Students who are mathematically gifted include those who have high ability or high interest. Some may be gifted with an intuitive knowledge of mathematical concepts, whereas others have a passion for the subject even though they may work hard to learn it. Many students' giftedness becomes apparent to parents and teachers when they grasp and articulate mathematics concepts at an age earlier than expected. They are often found to easily make connections between topics of study and frequently are unable to explain how they quickly got an answer (Rotigel & Fello, 2005).

Many teachers have a keen ability to spot talent when they note students who have strong number sense or visual/spatial sense (Gavin & Sheffield, 2010). Note that these teachers are not pointing to students who are fast and speedy with their basic facts, but those who have the ability to reason and make sense of mathematics.

Do not wait for students to demonstrate their mathematical talent; instead develop it through a challenging set of tasks and inquiry-based instruction (VanTassel-Baska & Brown 2007). Generally, as described in the RtI model, high-quality core instruction is able to respond to the varying needs of diverse learners, including the talented and gifted. Yet for some of your gifted students, the core instruction may prove not to be enough of a challenge.

Pre-assessing students by curriculum-based tests and also other measures such as concept maps prior to instruction allows the evaluation of what the student already knows and in some cases identifies how many grade levels ahead they might be (Rakow, 2012). Without this information the possibility of targeting the next steps and adaptations becomes guess work.

There are four basic categories for adapting mathematics content for gifted mathematics students: *acceleration and pacing*, *depth*, *complexity*, and *creativity* (Johnsen, Ryser & Assouline, 2014). In each category, your students should apply, rather than just acquire, information. The emphasis on applying, implementing, and extending ideas must overshadow the mental collection of facts and concepts.

Acceleration and Pacing

Acceleration recognizes that your students may already understand the mathematics content that you plan to teach. Some teachers use "curriculum compacting" (Reis & Renzulli, 2005) to give a short overview of the content and assess students' ability to respond to mathematics tasks that would demonstrate their proficiency. Allowing students to increase the pace of their own learning can give them access to curriculum different from their grade level while demanding more independent study. But, moving students to higher mathematics (by moving them up a grad, for example) will not succeed in engaging them as learners if the instruction is still at a slow pace. Research reveals that when gifted students are accelerated through the curriculum they are more likely to explore STEM (science, technology, engineering, and mathematics) fields (Sadler & Tai, 2007).

Depth

Enrichment activities go into depth beyond the topic of study to content that is not specifically a part of your grade-level curriculum but is an extension of the original mathematical tasks. For example, while studying place value both to very large numbers and decimals, mathematically gifted students can stretch their knowledge to study other bases such as base five, base eight, or base twelve. This provides an extended view of how our base-ten numeration system fits within the broader system of number theory. Other times the format of enrichment can involve studying the same topic as the rest of the class while differing on the means and outcomes of the work. Examples include group investigations, solving real problems in the community, writing data-based letters to outside audiences, or identifying applications of the mathematics learned.

Complexity

Another strategy is to increase the sophistication of a topic by raising the level of complexity or pursuing greater depth to content, possibly outside of the regular curriculum or by making interdisciplinary connections. For example, while studying a unit on place value, mathematically gifted students can deepen their knowledge to study other numeration systems such as Roman, Mayan, Egyptian, Babylonian, Chinese, and Zulu. This provides a multicultural view of how our numeration system fits within the historical number systems (Mack 2011). In the algebra strand, when studying sequences or patterns of numbers, mathematically gifted students can learn about Fibonacci sequences and their appearances in the natural world in shells and plant life. See the **Mathematics Integration Plan** that can be used to help plan ways to integrate core content for a gifted student or depending on the student's ability can be used by students to create independent explorations or research. Using this approach, students can think about a mathematics topic through another perspective or through an historic or even futuristic viewpoint.

Creativity

By presenting open-ended problems and investigations students can use divergent thinking to examine mathematical ideas—often in collaboration with others. These collaborative experiences could include students from a variety of grades and classes volunteering for special mathematics projects, with a classroom teacher, principal, or resource teacher taking the lead. Their creativity can be stimulated through the exploration of mathematical "tricks" using binary numbers to guess classmates' birthdays or design large-scale investigations of the amount of food thrown away at lunchtime (Karp, K. & Ronau, R., 1997). A group might create tetrahedron kites or find mathematics in art using proportional thinking and measurement (Bush, Karp, Popelka, Miller-Bennett, & Nadler, 2013). Another aspect of creativity provides different options for students in culminating performances of their understanding, such as demonstrating their knowledge through inventions, experiments, simulations, dramatizations, visual displays, and oral presentations.

Noted researcher on the mathematically gifted, Benbow (Read, 2014), states that acceleration combined with depth through enrichment is best practice. Then learning is not only sped up but the learning is deeper and at more complex levels.

Strategies to Avoid

There are a number of ineffective approaches for gifted students, including the following:

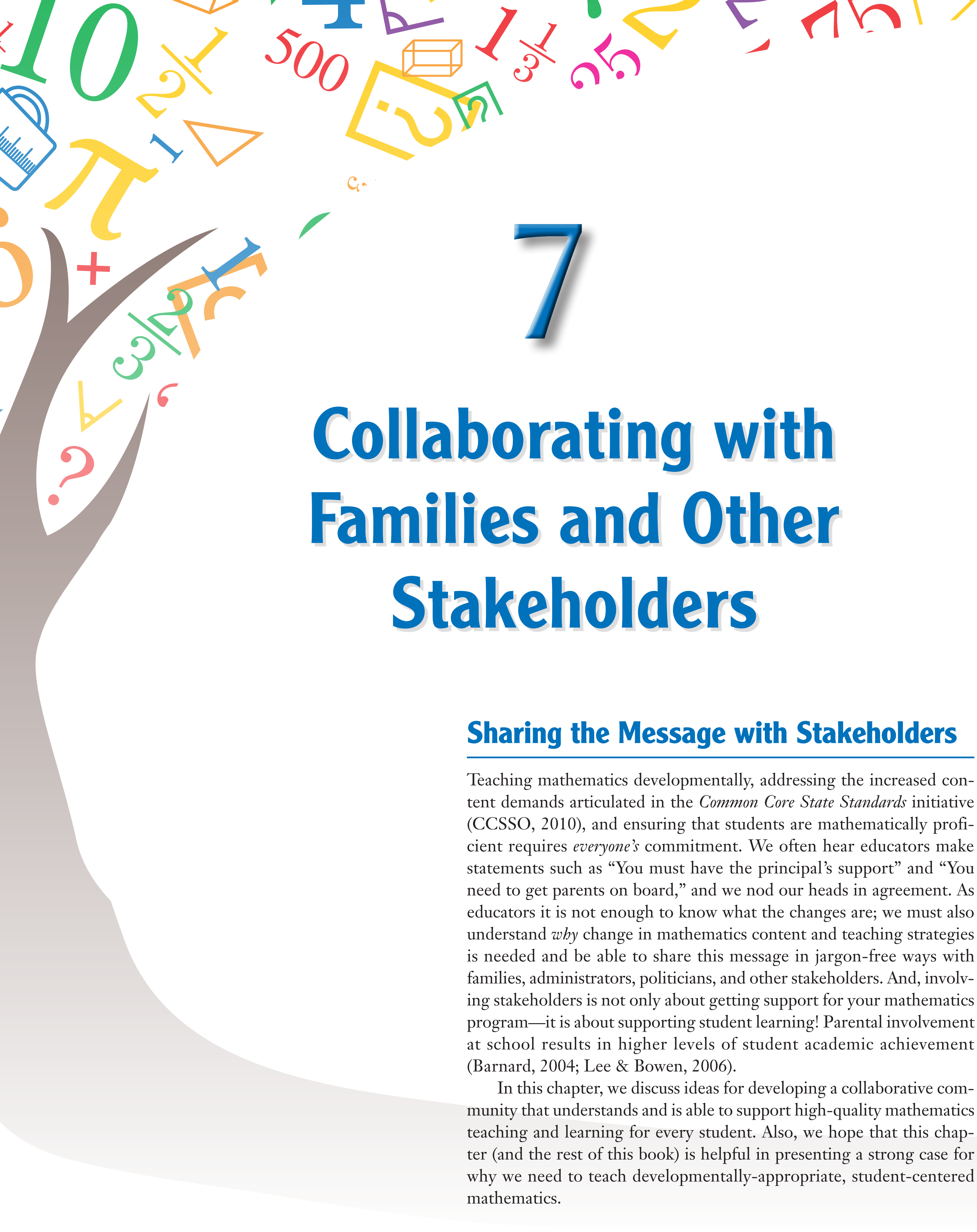
1. *Assigning more of the same work.* This is the least appropriate way to respond to mathematically gifted students and the most likely to result in students hiding their ability.
2. *Giving free time to early finishers.* Although students find this rewarding, it does not maximize their intellectual growth and can lead to students hurrying to finish a task.

3. *Assigning gifted students to help struggling learners.* Routinely assigning gifted students to teach other students who are not meeting expectations does not stimulate their intellectual growth and can place them in socially uncomfortable or sometimes undesirable situations.
4. *Providing gifted pull-out opportunities.* Unfortunately, generalized programs are often unrelated to the regular mathematics curriculum (Assouline & Lupkowski-Shoplik, 2011). Disconnected, add-on experiences are not enough to build more complex and sophisticated understanding of mathematics.
5. *Offering independent enrichment on the computer.* Although there are excellent enrichment opportunities to be found on the Internet and terrific apps, the practice of having gifted students use a computer program that focuses on skills does not engage them in a way that will enhance conceptual understanding, critical thinking or support students' ability to justify their thinking.

Sheffield writes that gifted students should be introduced to the "joys and frustrations of thinking deeply about a wide range of original, open-ended, or complex problems that encourage them to respond creatively in ways that are original, fluent, flexible, and elegant" (1999, p. 46). Accommodations, modifications, and interventions for mathematically gifted students must strive for this goal.

Stop and Reflect

How is equity in the classroom different from teaching all students equitably?



7

Collaborating with Families and Other Stakeholders

Sharing the Message with Stakeholders

Teaching mathematics developmentally, addressing the increased content demands articulated in the *Common Core State Standards* initiative (CCSSO, 2010), and ensuring that students are mathematically proficient requires *everyone's* commitment. We often hear educators make statements such as "You must have the principal's support" and "You need to get parents on board," and we nod our heads in agreement. As educators it is not enough to know what the changes are; we must also understand *why* change in mathematics content and teaching strategies is needed and be able to share this message in jargon-free ways with families, administrators, politicians, and other stakeholders. And, involving stakeholders is not only about getting support for your mathematics program—it is about supporting student learning! Parental involvement at school results in higher levels of student academic achievement (Barnard, 2004; Lee & Bowen, 2006).

In this chapter, we discuss ideas for developing a collaborative community that understands and is able to support high-quality mathematics teaching and learning for every student. Also, we hope that this chapter (and the rest of this book) is helpful in presenting a strong case for why we need to teach developmentally-appropriate, student-centered mathematics.

Changes to the mathematics curriculum—new textbooks, content topics, technologies, teaching philosophies, instructional strategies, and routines—warrant communication with parents, principals, and community leaders. Communicating with families is one of the most important components of successfully implementing a new mathematics curriculum (Bay, Reys, & Reys, 1999). Without such opportunities for communication, people may draw their own conclusions about the effectiveness of the mathematics curriculum, develop frustrations and negative opinions about what is happening in their child's classroom or school, and communicate this apprehension to other parents and community leaders. This has certainly been the case in states that initially adopted the *Common Core State Standards* (CCSS-M), and then had it voted out again. The reasons people opposed the standards (or support them) are sometimes not actually characteristics of the CCSS-M but, rather, characteristics that have been assigned to the CCSS-M.

Rather than hope that there will be support for changes in mathematics teaching and learning, it is important to anticipate possible questions and concerns and develop a plan to address them (Bay-Williams & Meyer, 2003). Table 7.1 highlights common questions parents and other stakeholders ask about mathematics programs.

Be proactive! Don't wait for concerns or questions to percolate. Providing a forum for parents, administrators, and community leaders around mathematics highlights the importance of the subject and gives stakeholders confidence that your school is a great place for preparing children for college and beyond. In the sections that follow, we share possible responses to the four areas of concerns in Table 7.1.

Table 7.1. Questions related to change in mathematics teaching and learning.

Category	Types of Questions
Why Change?	• Why is mathematics teaching changing? • Is there evidence that this approach or curriculum is effective? • What are the *Common Core State Standards* and why do we have national standards? • Where can I learn more about the *Common Core State Standards*?
Pedagogy	• Why isn't the teacher teaching? (And what is the point of reinventing the wheel?) • Why is my child struggling more than in previous years? • Are students doing their own work when they are in groups? Is my child doing the work of other students? • Are calculators and other technology interfering with my child's fluency?
Content	• Is my child learning the basic skills? • Why is my child learning different algorithms/strategies (than I learned) for doing the operations? • Why are there less skills and more story problems?
Student Learning and Outcomes	• Will these standards prepare my child for middle school, high school, college, and beyond (e.g., ready for ACT, SAT, Algebra I in eighth or ninth grade)? • Why is my child struggling more than in previous years? • How can I help my child with their homework; to be successful?

Why Change?

Change in many domains is considered a good thing. Why, then, is any change in mathematics teaching met with resistance? Additionally, the same people who claim to not be good at math, or to not like math, are often the ones most concerned about changes in mathematics teaching and learning. Navigating through change in mathematics requires strong justification for why the change is occurring, and the justification must resonate with the particular stakeholders' concerns and experiences.

Changes in Content

Reflecting on how other professions—from doctors to mechanics to dentists to bankers to hair stylists—have changed their practices over the past 25-plus years can make for an interesting comparison. Many fields have changed based on changes in society, different desired outcomes, available tools and technologies, research on what works, and new requirements within the job.

Another approach is to share research on the *ineffectiveness* of the traditional U.S. approach to teaching mathematics. The Trends in International Mathematics and Science Study (TIMSS), an international study conducted regularly that includes many countries, continues to find that U.S. students achieve at an average level in fourth grade, then below average in mathematics than international students in eighth grade and high school. Discuss the implications of unpreparedness for students who want to seek higher-paying jobs on what is now an international playing field.

Evidence for Change

Just as research and advancements have changed procedures doctors and mechanics use, research and advancements have informed alterations in procedures used in mathematics. In teaching mathematics, this can be what we have learned about teaching (e.g., to start with concrete tools or to make connections explicit) or what we have learned about specific content (e.g., that writing equations in nontraditional ways improves student understanding of equivalence [McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011]). Also, in nearly all careers, certain mathematical proficiencies are essential: being able to select appropriate tools, determine and implement a strategy or algorithm, communicate and compare approaches, and reflect on the result of a procedure or solution. In other words, preparing students to be college- and career-ready means addressing the mathematical proficiencies described in the CCSS Mathematical Practices and NCTM Process Standards.

CCSS Mathematics

Chapter 1 addressed the CCSS. Here we focus on advocating with stakeholders related to the standards. If you are teaching in a "Common Core" state, then you may have encountered a number of questions about the CCSS and/or heard various incorrect facts communicated about the CCSS-Mathematics. CCSSI provides a list of myths and facts (see **http://www.corestandards.org/about-the-standards/myths-vs-facts/**). Here we summarize six common (and detrimental) myths related to mathematics (with facts in *italics* afterward):

Myth 1. CCSS are national standards. *Fact: CCSS was designed by Governors and State School Officers. They were adopted by states, and states can add to the core as they see fit. States are in charge of implementation and assessment of students.*

Myth 2. CCSS-Mathematics lowers existing state standards. *Fact: An analysis of state standards shows that the common core sets higher expectations than any individual state standards (and states can add more rigor if they choose).*

Myth 3. CCSS are not internationally benchmarked. *Fact: This was a significant purpose in creating the standards.*

Myth 4. CCSS do not prepare students for algebra in grade 8. *Fact: Those that complete CCSS-M through grade 7 can take algebra in grade 8.*

Myth 5. CCSS content is not in the right place. *Fact: Content was placed based on learning trajectories and research on student learning.*

Myth 6. CCSS dictates how teachers should teach. *Fact: The standards simply list what mathematics should be learned at what level. The Mathematical Practices describe what a mathematically proficient student can do. How to reach these outcomes for what a student should know and do are not addressed in the standards.*

Whether your state is or is not using the CCSS-M, there are no doubt times when changes in mathematics teaching and learning need to be communicated to various stakeholders. These messages must be carefully composed! We sometimes say things that, although well intentioned, increase the concerns of stakeholders rather than reassure them. Table 7.2 provides three such examples.

Stop and Reflect

Consider a statement you have heard or used with families. Ask yourself, "How might a parent (or other stakeholder) respond if he or she heard this statement?" "What might the parent misinterpret?" "How might my principal respond?" Then, read the responses in the table to see whether they represent stakeholders like those in your setting.

Table 7.2. Statements and possible (unintended) interpretations of the statements.

Original Statement	What a Stakeholder Might Think	A Stronger, Carefully Composed Statement
"The [mathematics program] still addresses skills, but it also includes concepts."	"Why are they bringing skills up? They must be taking those away. My child/U.S. children have to know basics. How can I put a stop to this?"	"The skills in the [mathematics program] are expanding from what we once learned and now include . . . "
"It is important for students to learn from one another, so I will be more in the role of facilitator."	"The teacher is not teaching? My child does better when things are explained clearly. When I come to see you teach, what am I looking for if you are just letting the kids learn on their own?"	"In our classroom, we learn from one another. I give carefully selected tasks for students to discuss, and then we talk about them together so that everyone has a chance to learn the mathematics we are doing, and that approach gives me the chance to work one-on-one as needed."
"This year, we are doing a whole new mathematics program that the state has adopted."	"My worst nightmare—an experiment of something new during the years my child is in middle school. This will cause problems for the rest of his life."	"We are doing some new things in order to make sure your child is well prepared for . . . [or that our program is the best available]. You might have noticed that last year we [added writing as a component to our math program]. This year, here are the big things we hope to accomplish . . . ."

Initially, these statements may not seem harmful, but they can set off alarms from the lens of a stakeholder. Consider these reactions, and then review the shifted language in the third column, which communicates a stronger (and less potentially disconcerting) message. It is very important to convey to stakeholders an excitement for and pride in your mathematics program. Being tentative, reserved, vague, or silent on the mathematics program can only raise concerns in the community. Help parents and administrators to understand that the mathematics program students are experiencing aligns with best practices in education, represents what students need to know in today's world, and prepares students for mathematics at the next level as well as the mathematics they need for life.

Pedagogy

When stakeholders ask questions that point to their belief that mathematics is best learned through direct instruction—just as they learned it—it is important to provide a rationale for why mathematics teaching and learning might be different now.

Teacher as Facilitator

Recall that two important findings about how children develop conceptual understanding is through (1) engaging in productive struggle and (2) making connections explicit (Hiebert & Grouws, 2007). Related to these two important research findings, compare the difference between being *shown* how to do something (e.g., this is how you divide fractions, now practice this) and *developing* an understanding of something (e.g., what it means to divide fractions, and how you might solve this division problem). As students explore carefully selected tasks they engage in productive struggle and have the opportunity to make connections among mathematical ideas and strategies.

Address the role of the teacher as *organizer* (organizes a worthwhile mathematical task), *facilitator* (facilitates student interaction), and *questioner* (asks questions to help students make connections or to deepen their understanding). Remind parents that just because the teacher is not *telling* their child what to do does not mean that the teacher is not teaching. The teacher is orchestrating the class so that each student develops the appropriate connections, understands the mathematics, has the ability to solve problems, and is developing a disposition that they can do mathematics.

Cooperative Groups

Parents and other stakeholders may also wonder about how frequently their child works in cooperative groups because this may differ from their own mathematics learning experiences. Help parents see the role of others in their learning as they solved the problems and as they heard solutions from those who were working at other tables. Connect that experience to the value of cooperative learning. You can do this in a variety of ways:

1. *Share the one-page parent overview from the NCTM Families Ask department titled "Cooperative Learning" (Coates & Mayfield, 2009).* "Families Ask," a feature posted on the NCTM website and published in *Mathematics Teaching in the Middle School*, provides over 20 excellent, written-for-parents discussions on a range of topics relevant to middle school.
2. *Include a feature in your parent newsletter.* Early in the year, you can feature cooperative learning and address its importance across content areas. In mathematics, this can include the following benefits: hearing different strategies, building meaning, designing solution strategies, and justifying approaches—all of which are essential to building a strong understanding of mathematics and important life skills.
3. *Send home letters introducing math units.* If you are about to teach a unit on functions, a letter can help parents know the important aspects of the content. This is a great time to mention that students will work in groups so that they can see different ways to illustrate functions with tables, graphs, equations, and story situations.
4. *Do a cooperative learning mathematics activity at a family math night or back-to-school event.* Use a task that lends itself to assigning roles to different members of the group and won't take long to solve. Have parents work with two to three others to solve the task.

Teaching Tip

Being proactive about communicating the *benefits* of cooperative learning, as well as how you build in *individual accountability* and *shared responsibility,* will go a long way toward converting parent concerns into parent support.

Parents may initially worry that students working in groups are simply copying from other students and not learning. Share strategies you use to build in individual accountability and shared responsibility. For example, teachers may ask each student to record explanations in his or her notebook. At other times, you may assign specific roles to each member of the group.

Use of Technology

Parents may be avid users of technology yet still have concerns about their child's use of calculators and computers in middle school when they haven't yet mastered certain algorithms or learned how to graph by hand. Even though research overwhelmingly finds that students using calculators achieve at least as much as those not using calculators, calculators are widely blamed for students' lack of reasoning and sense making. Reassure parents that students will learn to use calculators when it is appropriate. For example, students should notice that $x + 14 = 34$ can be solved by mentally thinking, "What number plus 14 is 34? 20," more quickly than it can be calculated on a calculator. Calculators support learning, and presenting examples to make this point is important. For example, consider sharing a measurement lesson (sixth grade) in which students are asked to measure the dimensions of boxes to the nearest centimeter to calculate surface area and volume. Without a calculator, such a lesson would get bogged down in calculations that take time and attention away from the goal of the lesson—measurement.

An important message to parents is that mastery of basic facts should *not* be a prerequisite to using a calculator. Instead, children (and teachers) should be making good decisions about whether a calculator supports or detracts from solving a particular problem (and learning the intended mathematics).

Content

Basic Facts and Standard Algorithms

A common concern of parents is that their children are not learning standard algorithms or the procedures they remember using when they were in middle school. You must address (at least) two points related to this critical issue. First, the skills that parents are looking for (e.g., invert and multiply for dividing fractions) are still in the curriculum—they just may look different because they are presented in a way based on understanding, not just memorization. Standard algorithms are still taught but they are taught *along with* alternative (or invented) strategies that build on students' number sense and reasoning. Let parents experience that both invented and standard algorithms are important in being mathematically proficient by inviting them to solve the following problems:

$$1399 + 547 = \underline{\hspace{3em}} \qquad 5009 - 998 = \underline{\hspace{3em}}$$

$$487 + 345 = \underline{\hspace{3em}} + 355$$

Ask for volunteers to share the ways that they thought about the problems. For the subtraction problem, for example, the following might be shared:

5000 take away 1000 is 4000, then add the 9 and 2 back on to get 4011.

998 up to 1000 is 2, up to 5000 is 4002, and up 9 more is 4011.

5009 to 5000 is 9, then down to 1000 is 4000 more (4009), and then down to 998 is 2 more (4011).

These invented strategies, over numerous problems, reinforce place-value concepts and the relationship between addition and subtraction. Noticing that these values are both near 1000 helps to select a strategy. The standard algorithm for this problem is very messy, and one that frequently results in computational errors. The best choice for solving this problem is one of the ways described above. The key to procedural fluency is to first assess the values in the problem and then decide how to solve it. This bird's-eye view of the problem is important in doing mathematics—rather than always doing the same thing regardless of the numbers. This is very evident in the third example, which can be solved with no computation if the relationships among the numbers are first noticed.

Second, what is "basic" in the 21st century is much more than computation. Many topics in middle school were not a part of the curriculum a generation ago (e.g., functions). Looking together through the essential concepts in the *Common Core State Standards* or the NCTM *Curriculum Focal Points* helps parents see that the curriculum is not just an idea generated at their child's school but the national consensus on what middle-school students need to learn.

Practice and Problem Solving

Parents may also wonder why there are fewer skill/practice problems and more story problems in the curriculum. Effective mathematics learning environments are rich in language. Real mathematics involves more "word" problems and far fewer "naked number" skill problems. In contrast to when the parents went to school, skills are now less needed in the workplace because of available technology, but the importance of number sense, reasoning, and being able to solve real problems has increased. Because some students struggle with reading and/or writing, share strategies you use to help them understand and solve story problems (Figure 7.1).

Figure 7.1

Share with parents how you support reading and problem solving.

Reading Strategies for Mathematics Problems

- Read aloud (whole class)
- Read a math problem with a friend
- Find and write the question
- Draw a picture of the problem
- Act out the problem
- Use a graphic organizer (recording page with problem-solving prompts)
- Discuss math vocabulary
- Play math vocabulary games

Student Learning and Outcomes

At the heart of parents' interest in school mathematics is wanting their child to be successful, not only in the current classroom but also at the next level of school and later on in the high-stakes assessments like the ACT or SAT for college entrance or to enter the workforce.

Preparation for College and Career

If your state has implemented the *Common Core State Standards*, you can share that the standards are for K–12 and designed to prepare students for college and future careers. The *Common Core State Standards* website has an increasing number of resources for parents (http://corestandards.org) to help them ensure that their child is college- and career-ready. Because the standards are at least as rigorous as prior state standards, and have an increased focus on student reasoning and sense making, they better prepare students for high-stakes assessments such as the ACT or SAT, which over the years have adapted to reflect changes in content expectations in high school curriculum, such as the increased focus on statistics (Jaschik, 2014; Peterson's, 2015). As noted in the CCSS myths and facts, these standards are more rigorous, and therefore better prepare students for high-stakes assessments, and still prepare students to take algebra in eighth or ninth grade.

Parents may be more interested in how your specific school is doing in preparing students for the future. Share evidence from your school of mathematics success, including stories about an individual student (no name given) or the success of a particular classroom, like the following one received by a middle-school principal:

> I was worried at the start of the year because my son has never liked math and was coming home with pretty complicated problems to solve. I wondered why the teacher hadn't shown him *how* to add and subtract integers, for example. But, now I can really see his number sense—he uses various ways to add, including that he can do it in his head and use a number line! He is also doing very well with problem solving and writing his own story problems! As an aside, I am also learning a lot—I didn't learn this way, but I am finding the homework problems are really interesting. Best of all, his confidence has skyrocketed. Is this something that will be continued next year and when he takes algebra?

Such communications help parents see that there is a transition period and that in the end a standards-based approach helps engage students and build their understanding over time.

Productive Struggle

Parents may worry when they see their child struggle with a single mathematics problem because they may believe that fast means successful. But faster isn't smarter. The book *Faster isn't Smarter* (Seeley, 2009) is a great read on this topic written for families, educators, and policy makers. Seeley offers 41 brief messages, many of which can address parent questions about mathematics (e.g., "A Math Message to Families: Helping Students Prepare for the Future," "Putting Calculators in their Place: The Role of Calculators and Computation in the Classroom," and "Do It in Your Head: The Power of Mental Math"). As noted above, engaging students in productive struggle is one of the two most effective ways teachers can help students develop conceptual understanding (the other is making connections among mathematical ideas) (Bay-Williams, 2010; Hiebert & Grouws, 2007). Rather than presenting a series of simpler problems for students to practice, standards-based curricula characteristically focus on fewer tasks, each of which provides students with an opportunity for higher-level thinking, multiple strategy solutions, and more time focused on math learning.

Share the first Standard for Mathematical Practice (Figure 7.2), and ask the parents what they notice. Focus on the importance of *perseverance.* This is true in mathematics and in life. Reassure parents that some tasks take longer because of the nature of the tasks, not because their child lacks understanding. Mathematics is not nearly as much about speed and memorization as it is about being able to grapple with a novel problem, try various approaches from a variety of options, and finally reach an accurate answer.

Figure 7.2

Standard 1 from the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs, or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Source: From Common Core State Standards for Mathematics. Published by *Common Core State Standards Initiative,* © 2010.

Administrator Engagement and Support

Teachers cite a supportive principal as one of the most essential components in successfully implementing a standards-based curriculum (Bay, Reys, & Reys, 1999). Principals play pivotal roles in establishing a shared vision for a problem-based mathematics program. Principals, however, often cannot take the time to attend the professional development workshops that are designed for teachers who will be teaching the mathematics program. And what they need to know is qualitatively different from what a classroom teacher needs to know.

Since the launch of the *Common Core State Standards*, school administrators, parents, and community members are more aware than ever about mathematics standards. If your state has not adopted the *Common Core State Standards*, there are still state-level standards that are the focus of mathematics goals and assessments. Even though principals are hearing more about mathematics standards, higher standards, and the need to ensure that all students are successful, it does not mean they understand what standards-based mathematics curriculum *is* in terms of the content or the related CCSS Standards for Mathematical Practice or NCTM Process Standards.

Administrators are likely to get bombarded with broad or specific questions from parents: "Is 'New Math' back?," "Why isn't the teacher teaching the procedures for multiplying and dividing?," "What are the Standards for Mathematical Practice?," or any of the other questions offered earlier in this chapter. When a principal is asked these questions, he or she needs to give a convincing response that is accurate and that also addresses the heart of the parents' concerns—that their child is going to get a sound math experience that will prepare him or her for college and career.

Meyer and Arbaugh (2008) suggest professional development specifically for principals. Although their focus is on the adoption of standards-based textbooks, the plan they outline applies to all principals who are seeking to be knowledgeable and effective advocates for implementing new standards or mathematics curricula. The following ideas are adapted from their suggested professional development to focus on one-on-one conversations.

1. *Contrast old and new curriculum.* As a first step, it is important to know what is new and different in the mathematics program. One way to start is to provide a set of materials that represents typical *Common Core State Standards*–aligned tasks alongside the previous curriculum. Point out the noticeable similarities and differences or the key features of the curriculum. (Note: It is important to focus on *both* similarities *and* differences—not *everything* is being replaced, and this is an important message.)
2. *Discuss how parents and students will respond.* Anticipate what will be noticed by parents (or their children): Which changes might be welcomed? Which changes might be worrisome? How will the welcome aspects be promoted and the worrisome aspects be explained?
3. *Experience the curriculum.* Invite the principal to visit your classroom or other classrooms where the Standards for Mathematical Practice or the NCTM Process Standards are being infused. Ask the principal to join a group of students and listen to their discussion of how they are solving a problem. Or organize a lesson when, in the *After* phase, the students actually present their solutions to the principal. For example, in the sixth grade, have students use visuals and explanations to show their different ways of dividing fractions. If possible, ask the principal to solve one of the problems the students are doing and share his or her strategy with the class. This firsthand experience can provide the principal with a wonderful story to share with parents and with insights that won't be gained from reviewing standards documents.

4. *Discuss emerging issues.* Plan a regular time to meet with the principal to discuss what he or she has heard from families about the mathematics program. Discuss what you might do to respond to questions (some of the anticipated issues may already have been described in the preceding section on parents' concerns). If there is a question about a problem-based approach, Chapter 2 should be a great read for a principal, and contains talking points to share with others.

Finally, keep your principal apprised of successes and breakthroughs. These stories provide the principal with stories and evidence to share when pressed by parents or community members. Principals are very often your strongest advocates and are in a position to serve as buffers between school mathematics and the community.

Stop and Reflect

What do you think the parents of your students would most value about "teaching mathematics *through* problem solving," and how will you use your response to this question to build strong family support and engagement? (Repeat the question for other stakeholders, such as your principal.)

Family Engagement

Parents know the importance of mathematics for their child's future. They participate in their child's learning by supporting homework, attending back-to-school nights or PTA meetings, and by meeting with teachers, even if they may recall unpleasant experiences or difficulties with school mathematics from their own schooling. Understanding that memories of mathematics classes are not always pleasant for parents and appreciating parental support prepare us to suitably identify for parents the mathematics goals that students should be experiencing in the 21st century.

Communication with families is key to encouraging their support and involves using one-way, two-way, and three-way communication strategies (Figure 7.3).

Figure 7.3

Ways to communicate with families.

One-way communication strategies	Letters sharing the goals of a unit	Websites where resources and curriculum information are posted	Newsletters
Two-way communication strategies	Log of student work (signed or commented on by parent)	PTA meetings/open houses	One-on-one meetings, class or home visits
Three (or more)–way communication strategies	Family math nights	Conferences (with parent and child)	Log/journal of student learning with input from student, parent, and teacher

Numerous studies have found a positive relationship between the level of parental involvement and their child's achievement in school (e.g., Aspiazu, Bauer, & Spillett, 1998; Henderson et al., 2002). Parents need frequent opportunities to get information directly from the school leaders and teachers about their child's mathematics program, including the kind of instruction that might differ from what they experienced in their own schooling. Even if your school has been engaged in implementing a mathematics program for a decade that reflects the NCTM *Principles and Standards for School Mathematics* and now the *Common Core State Standards*, the program will still be new to the parents of your students.

Family Math Nights

There are many ways to conduct a family or community mathematics event, such as including a math component in a back-to-school night, discussing it in a PTA meeting, or hosting a showcase for a new mathematics program. One idea is to host a Math Orientation Workshop (Ernst & Ryan, 2014). The purpose of this event is to develop consistency between the way math is taught in school and the way parents help at home. Beyond a focus on the content, parents can learn about *dispositions* of effective problem solvers, including the importance of asking questions; developing persistence; using multiple ways to solve problems; learning from mistakes; and reflecting on whether solutions make sense (see Ernst and Ryan [2014] for more details on designing and implementing this event). Providing opportunities to parents to learn about specific mathematics topics prior to their children learning the topics can lead to increased relationships with parents and increased student achievement (Knapp, Jefferson, & Landers, 2013).

Teaching Tip

Welcome back or family math nights are a great time to have parent–child teams experience doing math tasks together.

Critical to any plan is providing opportunities for parents to be learners of mathematics so that they can experience what it means to *do mathematics* (just like their children). When choosing mathematical tasks to use with parents, be sure the tasks focus on content that really matters to them and relates to what they know is a part of the grades 6–8 curriculum, such as ratios and proportions or algebraic thinking. Tasks throughout this book are ideal for a math night. Figure 7.4 contrasts two sixth-grade problems for learning about ratios—one is straightforward and lends to a single procedure (set up a proportion and find the missing value) and one is designed for a teaching-through-problem-solving experience (explore and determine a way to compare ratios).

Figure 7.4

Problems to explore at a parent or community night.

Problem 1: For every point Angie scores, Erica scores 4. What is the ratio? If Erica scored 20 points, how many did Angie score?

Problem 2: You are painting your room and want a blue–green shade but are not sure which choice to make from the many colors offered. You want the one that will turn out the bluest of the following choices. Which one should you pick?

Ratio of blue to green is 9:4.
Ratio of blue to green is 3:2.

Ratio of blue to green is 15:6.
Ratio of blue to green is 12:5.

Stop and Reflect

What distinctions do you notice between the two tasks? What is valued as "doing mathematics" in both of the problems?

The contrasting ratio problems are ideal for discussing with parents what it means to do mathematics because they (1) offer a familiar context, (2) require minimal prior knowledge, (3) have multiple solution strategies, (4) can involve manipulatives (color tiles and/or grid paper), and (5) have the potential to connect the mathematical ideas of fractions, ratios, rates, and proportions, as well as algebra. The potential each of these problems has to support and challenge children in making sense of mathematics should be made explicit during a discussion with parents.

Ask parents to solve the first problem and share their strategies and answers. Repeat for the second problem. Be sure to ask for multiple approaches. Then, ask participants to consider the learning opportunities in the two contrasting tasks. Ask questions such as these:

- What skills are being developed in each problem?
- Which problem gives more opportunity to make connections between mathematics and the real world?
- Which task would your child be more motivated to solve? Why?

Teaching Tip

Provide copies of the appropriate *Common Core State Standards* Introduction and Overview pages (the first two pages for each grade), and allow parents time to think about each "Critical Area."

Help parents identify the depth of the mathematics in the teaching-through-problem-solving task. Remind parents that in grades 6 through 8, students are building important foundations for algebraic thinking—looking for patterns, reasoning, and generalizing. Help parents see these aspects in this ratio problem. Share the *Common Core State Standards* and/or the NCTM standards (in parent-friendly language), and focus on the goal of having students become mathematically proficient, as described in those standards. Ask parents, "Where do you see these proficiencies being supported in the two tasks we did?"

Revisit the first problem in Figure 7.2 and consider how it *could* be a problem in which reasoning is involved (e.g., using different approaches, creating visuals to illustrate the relationships among the values). Invite parents to see whether they can solve the following proportion in a way *other than* the cross-product method or if they can solve it in more than one way.

Alicia is reading a book. She read 40 pages in 1 hour. If she continues to read at that rate, how long will it take her to read the 200-page book?

Parents may recall their own confusion in how to set up a proportion—for example, not knowing where to put the x or not knowing if they should cross-multiply. Reasoning through problems like this and hearing other ways to reason about the problems help students understand proportional situations, and then they are able to understand and use cross-multiplication when needed, or use reasoning strategies when such strategies are more meaningful.

Address any or all of the questions in Table 7.1 that apply to your setting. One way to do this is to have parents write their questions on note cards and collect them so you can identify common questions and decide the order in which to discuss each one.

Classroom Visits

Some parents who may have been very engaged in their child's elementary education are not as involved in middle school. They may feel their involvement is not as critical in middle school as it was in elementary school, or that their child may not want to see them at school. In any case, an invitation to come to a mathematics lesson or a math event (e.g., family math night) gives parents the chance to witness firsthand such things as how you ask questions, how problems can be solved in many ways, and how calculators can be used to support reasoning. They may notice that you encourage children to select their own strategy and explain how they know it works. Parents will also pick up on the language that you are using and will be able to reinforce that language at home. You can even provide a note-taking template that includes categories such as the following:

- What is the big idea of the lesson?
- What illustrations or tools are being used to help students (your child) understand?
- What are some questions the teacher is asking that I could also ask?
- What does the teacher do when a student (my child) is stuck or needs to be challenged?

Involving ALL Families

Some families are at all school events and conferences; others rarely participate. However, all families want their children to be successful in school. Parents who do not come to school events may have anxiety related to their own school experiences, or they may feel completely confident that the school and its teachers are doing well by their child and that they do not need to participate. In some cultures, questioning a teacher may be perceived as disrespectful. Rodríguez-Brown (2010, p. 352), a researcher on Hispanic families, writes, "It is not that Latino parents do not want to support their children's learning. . . . [They] believe that it is disrespectful to usurp the teacher's role."

Try to find ways to build a strong rapport with all families. Some strategies to consider include the following:

1. *Honor different strategies for doing mathematics.* Although this is a recommendation in standards documents, it is particularly important for students from other countries because they may have learned different ways to do the operations (Civil & Planas, 2010).
2. *Communicate with positive notes and phone calls.* Be sure to find a way to compliment each student's mathematical thinking (not just a good score on quiz) at some point early in the school year.
3. *Host informal gatherings to discuss mathematics teaching and learning.* Having regular opportunities to meet with the parents allows the development of rapport and trust. Consider hosting events in out-of-school facilities. Schools in communities with a high level of poverty have found that having parent events at a community center or religious institution brings in families that are reluctant to come into a school.
4. *Incorporate homework that involves the family.* When a student brings in homework that tells about his or her family and you provide positive feedback or a personal comment, then you are establishing a two-way communication with the family via homework.
5. *Translate letters that are sent home.* If you are doing a class newsletter (for families) or a letter describing the next mathematics unit, make an effort to translate the letter into the native language of the families represented in your class. If you cannot do this, consider having the first class session include a component in which students write to

their families about what they are about to do. Ask them to write in their parents' first language and to include visuals to support their writing. Ask parents to respond (in their language of choice). This is a great practice for helping students know what they need to learn, and it communicates to families that they are an important part of that learning.

6. *Post homework on your webpage.* For parents who are not native speakers of English, posting problems on your site makes it easier to take advantage of online translations. While these translations may not be perfectly accurate, they can help parents and students understand the language in the problems.

For more suggestions on ensuring that your mathematics tasks and homework are meeting the needs of culturally and linguistically diverse students, see Chapter 5 and read "NCTM Research Brief: Involving Latino and Latina Parents in Their Children's Mathematics Education" (Civil & Menéndez, 2010). For suggestions on students with special needs, see Chapter 6.

Homework Practices and Parent Coaching

You may have heard parents say, "I am not good at math" or "I don't like solving math problems." While parents may feel this way, such messages to their child can impede their success in mathematics. In fact, a parent's emotions are connected to the student's emotions, and these positive emotions are connected to better performance (Else-Quest, Hyde, & Hejmadi, 2008). It is our responsibility as educators to figure out ways to redirect parents to portray mathematics in a positive light.

Tips for Helping Parents Help their Child

Explicitly teaching parents *how to* help their children has also been found to make a difference in supporting student achievement and student attitudes (Cooper, 2007; Else-Quest et al., 2008; Patall, Cooper, & Robinson, 2008). Take the following recommendations into consideration when thinking about the homework that you will assign to your students.

1. *Mimic the three-phase lesson model.* Table 2.5 describes the three-phase lesson model. Homework can reflect these general phases. Complete a brief version of the *Before* phase of a lesson to be sure the homework is understood before students go home. At home, students complete the *During* phase. When they return with the work completed, apply the sharing techniques of the *After* phase of the homework. Students can even practice the *After* phase with their family if you encourage this through parent or guardian communications. Some form of written work must be required so that students are held responsible for the task and are prepared for the class discussion.
2. *Use a distributed-content approach*. Homework can address content that has been taught earlier in the year as practice, that day's content as reinforcement, or upcoming content as groundwork. Interestingly, research has found that distributed homework (homework that combines all three components) is more effective in supporting student learning (Cooper, 2007). The exception is students with learning disabilities, who perform better when homework focuses on reinforcement of skills and current class lessons.
3. *Promote an "ask-before-tell" approach with parents.* Parents may not know how best to support their child when he or she is stuck or has gotten a wrong answer. One important thing you can do is to ask parents to implement an "ask-before-tell" approach (Kliman, 1999). This means that before parents explain something, they should ask their child to explain how he or she did it. The child may self-correct (a life skill), and if not, at least

the parents can use what they heard from their child to provide targeted assistance. Teach successful homework strategies to students, and share these strategies with parents. For example, the following ideas, suggested by Wieman and Arbaugh (2014), can be posted in your classroom and sent home:

- Look for examples in our notes or daily work. Try those problems again.
- If you are stuck, take a break, then come back and try again.
- If you are confused, write a statement or question describing what is confusing.
- Ask for help using specific questions (from parents, peers, or online support sites).

4. *Provide good questioning prompts for parents.* Providing guiding questions for parents or guardians supports a problem-based approach to instruction as they help their children. Figure 7.5 provides guiding questions that can be included in the students' notebooks and shared with families. Translating questions for parents who are not native speakers of English is important. Often, a student can help you with this task.

Teaching Tip

Providing specific guidance to families makes a big difference in what to do (and *not* do) to help their children learn mathematics and be confident in doing mathematics.

5. *Use Games and Interactives.* Find opportunities to assign games or interactives for homework. The intent is that the student plays with family members (e.g., parent, sibling). The game can be played as part of a lesson and then sent home to play two to three times, with a written summary of what happened as they played due the next day. Games can be assigned to develop fluency with such concepts as reinforcing basic facts, order of operations, using symbolic notation for expressions and equations, and so on. Bowl-a-Fact (Shoecraft, 1982) is one such engaging activity that can be offered as a homework option at various times throughout the year.

   Several features of this game make it a great choice for homework: It only requires paper and three dice (which can also be substituted with playing cards, or just selected by the parent or student), it is a different challenge with every roll, it can be differentiated (students can use 2 of the 3 values instead of all three values), and it is fun. Students can choose to compete with a family member (knocking down more pins than their opponent) or collaborate.

Figure 7.5

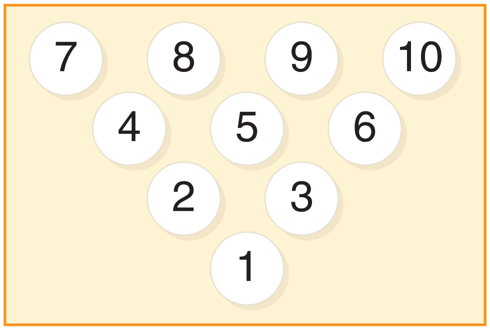
Questions for families to help their children with homework.

These guiding questions are designed to help your child think through his or her math homework problems. When your child gets stuck, ask the following:

- What do you need to figure out? What is the problem about?
- What words are confusing? What words are familiar?
- Do you have similar problems to look at?
- What math words or steps do you use in class?
- Where do you think you should begin?
- What have you tried so far? What else can you try?
- Can you describe where you are stuck or what is confusing?
- Can you make a drawing, table, or diagram to help you think about the problem?
- Does your answer make sense?
- Is there more than one answer?

GAME: BOWL-A-FACT

Download the Bowl-a-Fact Activity Page or draw circles placed in triangular fashion to look like bowling pins, with the front circle labeled 1 and the others labeled consecutively through 10.



Take three dice and roll them to get three numbers. Students use all three numbers to design and record an equation that results in an answer of 1–10 (thereby knocking down that pin). For example, if you roll 4, 2, and 3, they can "knock down" the 5 pin with 432–3, recording 4(2)–3=5 or 432–3=5. If they can produce equations to knock down all 10 pins, they get a strike. If not, roll again and see whether they can knock the rest down for a spare.

Finally, an important aspect of this game, which should be true for other games selected for homework, is that it can lead to a mathematical discussion. In this game, good questions include, "When are parentheses needed?" and "What strategies are you using to get some of the larger smaller numbers?"

Many interactives and applets are available across content strands (some of which are shared throughout Part 2 of this book). Asking students to explore, play, or solve on an applet can be another way to engage families with students. A favorite is the *Factor Game* or *Factor Dazzle*, which can be found in Classroom Resources on the NCTM website, and other places. This is a fun strategy game that helps students practice factors and multiples, a skill for which they need to be fluent in order to solve operations involving rational numbers and simplify rational expressions.

Homework of this nature communicates to families the problem-based or sense-making nature of your classroom and might help them see the value in this approach. A final note: A little bit goes a long way—if students are to spend time solving meaningful problems, then just a few engaging problems a night can accomplish more than a long set of practice problems.

Resources for Families

Parents will be better able to help their child if they know where to find resources. The Internet can either provide a wealth of information or be an overwhelming distraction. Help parents locate the good places to find math support. First, check whether your textbook provides websites with online resources for homework, including tutorials, video tutoring, videos, connections to careers and real-world applications, multilingual glossaries, audio podcasts, and more. Second, post websites that are good general resources. Here are some examples:

- *Figure This! Math Challenges for Families.* This NCTM site has a teacher corner and a family corner. It offers outstanding resources to help parents understand standards-based mathematics, help with homework, and engage in *doing* mathematics with their children. It is also available in Spanish.

- *Math Forum@NCTM.* This very popular and useful site includes many features for teachers and families. For example, "Ask Dr. Math" is a great homework resource because students can write in their questions and get answers fairly quickly. Parents may also want to read or participate in math discussion groups, read about key issues for the mathematics community, or download some of the very interesting problems posted here.
- *National Library of Virtual Manipulatives (NLVM).* This site has numerous applets and virtual tools for learning about many mathematics topics, appropriate for the classroom and the home.

Teaching Tip

Don't forget the value of your own website as the first site for parents to visit for support. Post your unit letters to families, newsletters, access to homework assignments, possible strategies for doing the homework, and even successful student solutions.

There are also great websites for specific content. For example, Conceptua Math has excellent applets for exploring fraction operations. Finally, print books can be important resources for teachers. Here are a couple that we recommend:

- *It's Elementary: A Parent's Guide to K5 Mathematics* (Whitenack, Cavey, & Henney, 2015). This book explains current teaching practices and fundamental math concepts, with many examples and student work.
- *What's Math Got to Do with It? How Parents and Teachers Can Help Children Learn to Love Their Least Favorite Subject* (Boaler, 2009). Jo Boaler describes how math can be understandable and fun, how children can excel in math, and how parents and schools can help.

Seeing and Doing Mathematics at Home

In the same way that families support literacy by reading and talking about books with their children, families can and should support numeracy. Because this has not been the practice in many homes, it means you, as the teacher, have the responsibility to help parents see the connection between numeracy and everyday life. Consider asking families to make the *Math Promise* (Legnard & Austin, 2014). Family members make this promise to one another, which means they explicitly agree they will do math together—get to know each other's mathematical reasoning, play math games, and notice mathematics in their daily lives. In her article "Beyond Helping with Homework: Parents and Children Doing Mathematics at Home," Kliman (1999) offers some excellent suggestions, which include asking parents to share anecdotes, find mathematics in the books they read, and create opportunities during household chores. Figure 7.6 provides a sample letter home that suggests these ideas to parents.

Trips in the car can include informal and fun mathematics explorations. For example, license plates can be noted and family members can try to use the numbers on the plate to create a true mathematics equation or someone can select a target number, and the values from the plate are used to try to reach that target number (Hildebrandt, Biglan, & Budd, 2013). Today's Date (Mistretta, 2013) is another activity that can be a part of informal family discussions. Today's Date involves taking the date (e.g., 18) and thinking of different expressions, ways to write it, and connections to personal interests (e.g., a favorite athlete's number or the age of a cousin). These tasks have many possible solutions and can be used repeatedly at any time of the year.

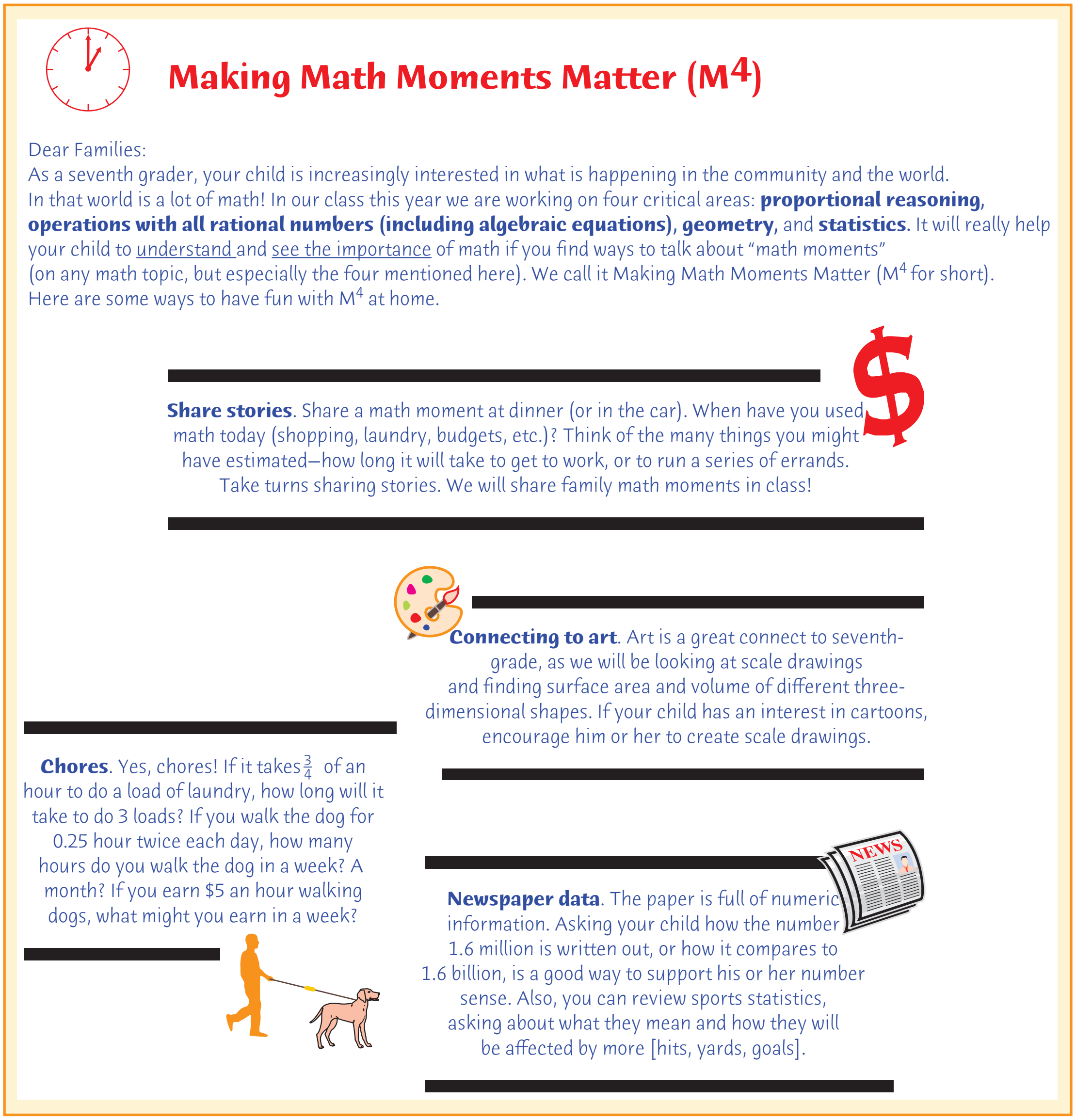


Figure 7.6

Sample letter to parents regarding ways to infuse mathematics into their interactions with their child.

Making Math Moments Matter (M^4)

Dear Families:

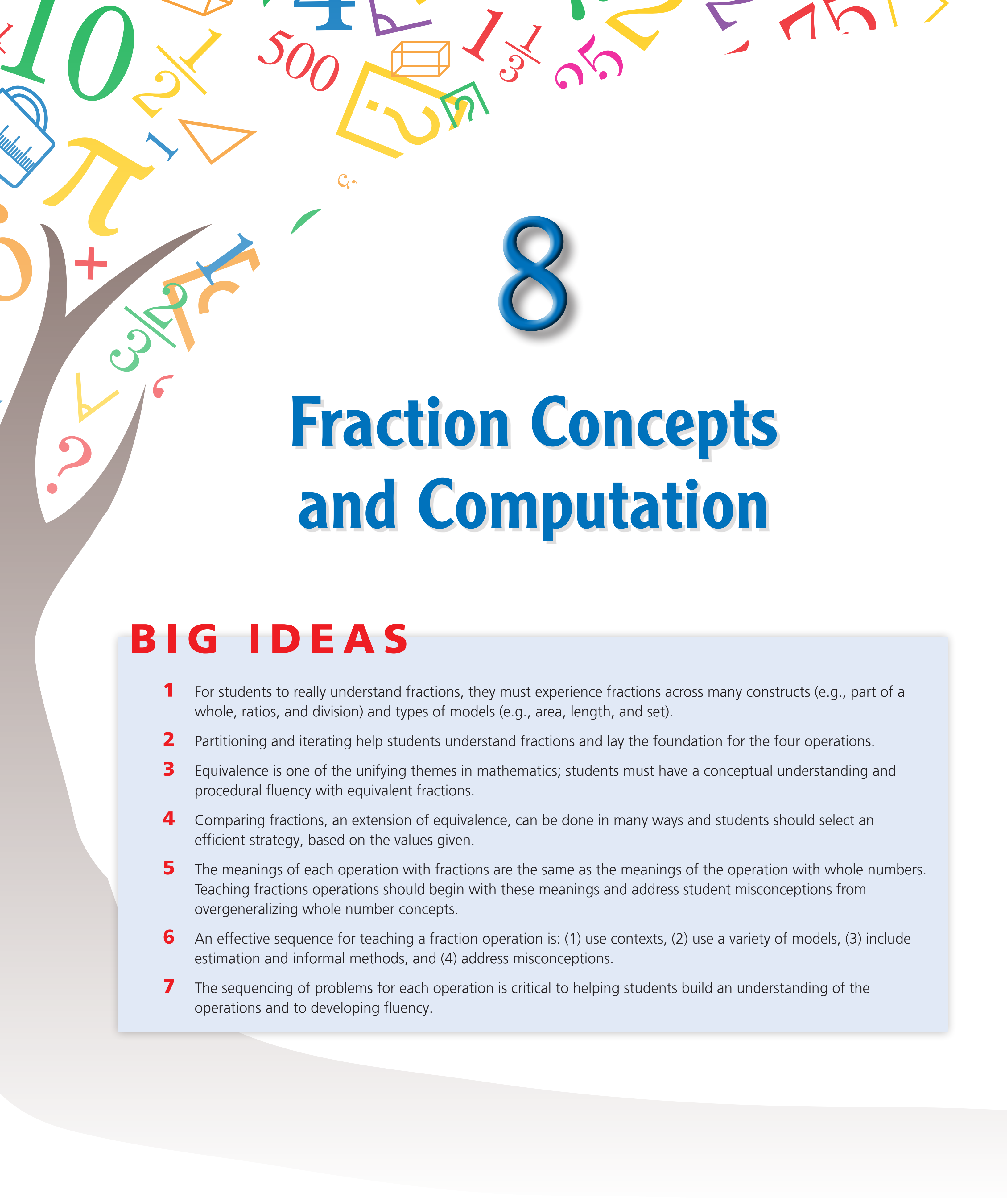
As a seventh grader, your child is increasingly interested in what is happening in the community and the world. In that world is a lot of math! In our class this year we are working on four critical areas: **proportional reasoning**, **operations with all rational numbers (including algebraic equations)**, **geometry**, and **statistics.** It will really help your child to understand and see the importance of math if you find ways to talk about "math moments" (on any math topic, but especially the four mentioned here). We call it Making Math Moments Matter (M^4 for short). Here are some ways to have fun with M^4 at home.

Share stories. Share a math moment at dinner (or in the car). When have you used math today (shopping, laundry, budgets, etc.)? Think of the many things you might have estimated—how long it will take to get to work, or to run a series of errands. Take turns sharing stories. We will share family math moments in class!

Connecting to art. Art is a great connect to seventh-grade, as we will be looking at scale drawings and finding surface area and volume of different three-dimensional shapes. If your child has an interest in cartoons, encourage him or her to create scale drawings.

Chores. Yes, chores! If it takes $\frac{3}{4}$ of an hour to do a load of laundry, how long will it take to do 3 loads? If you walk the dog for 0.25 hour twice each day, how many hours do you walk the dog in a week? A month? If you earn $5 an hour walking dogs, what might you earn in a week?

Newspaper data. The paper is full of numeric information. Asking your child how the number 1.6 million is written out, or how it compares to 1.6 billion, is a good way to support his or her number sense. Also, you can review sports statistics, asking about what they mean and how they will be affected by more [hits, yards, goals].



8

Fraction Concepts and Computation

BIG IDEAS

1. For students to really understand fractions, they must experience fractions across many constructs (e.g., part of a whole, ratios, and division) and types of models (e.g., area, length, and set).
2. Partitioning and iterating help students understand fractions and lay the foundation for the four operations.
3. Equivalence is one of the unifying themes in mathematics; students must have a conceptual understanding and procedural fluency with equivalent fractions.
4. Comparing fractions, an extension of equivalence, can be done in many ways and students should select an efficient strategy, based on the values given.
5. The meanings of each operation with fractions are the same as the meanings of the operation with whole numbers. Teaching fractions operations should begin with these meanings and address student misconceptions from overgeneralizing whole number concepts.
6. An effective sequence for teaching a fraction operation is: (1) use contexts, (2) use a variety of models, (3) include estimation and informal methods, and (4) address misconceptions.
7. The sequencing of problems for each operation is critical to helping students build an understanding of the operations and to developing fluency.

Meanings of Fractions

Fraction concepts are emphasized in the *Common Core State Standards* beginning in grade 3 and continuing through grade 7, with a focus on topics such as equivalence, the operations, and proportional reasoning. This emphasis over years is an indication of both the complexity and the importance of fraction concepts.

Deeper understanding and flexibility take time! This is recognized by research and the *Common Core State Standard*s, which suggest the following development across grades:

Grades 1 & 2: Partition regions into halves, fourths, thirds, etc.

Grade 3: Partition number lines, recognize fractions as composed of unit fractions, find equivalent fractions, and compare.

Grade 4: Extend strategies to compare, add and subtract fractions with like denominators, and multiply fractions by whole numbers.

Grade 5: Develop fluency with addition and subtraction of fractions, and begin multiplication and division of fractions in limited cases.

Grade 6: Apply and extend multiplication and division of fractions.

Grade 7: Solve real-world and mathematical problems involving the four operations with rational numbers (including complex fractions).

The National Assessment of Education Progress, commonly called the "Nation's Report Card," continues to show that middle-level students have a weak understanding of fraction concepts and operations (Sowder & Wearne, 2006; Wearne & Kouba, 2000). A lack of understanding of fractions is considered one of the major reasons students are not successful in algebra (Bailey, Hoard, Nugent, & Geary, 2012; Brown & Quinn, 2007; National Mathematics Advisory Panel, 2008). This means that even if students have not learned fractions, then you must work on developing an understanding of them.

Why Students Struggle with Fractions

If you think about the way in which students learn to add whole numbers and the way in which they learn to add fractions, you will see some stark differences that provide insights into why students struggle with fraction concepts and operations. Adding with whole numbers begins with counting, then counting on, then mental strategies, and as the numbers get bigger, students learn algorithms. All the while, they are using concrete objects, which include counters, ten-frames, hundreds charts, and number lines. In addition, story situations (concrete examples) are provided on a regular basis. Equations are written to model what has been illustrated with the manipulatives or the story. Yet, as students learn about fraction operations, far fewer manipulatives or visuals are used, mental strategies are rare, and story situations tend to be used only as an application problem after the algorithms have been learned. It is no surprise that students poorly understand the meaning behind fraction computation, and that leads to the many errors they make in performing fraction computations. Additionally, fractions are difficult because:

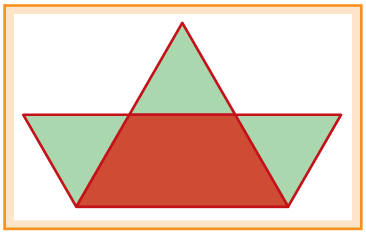
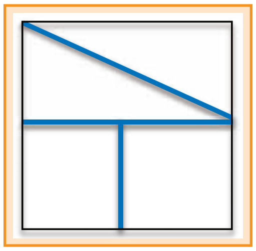
- There are many meanings of fractions (see later section "Fraction Constructs").
- Fractions are written in a unique way.
- Whole-number knowledge can be over-applied to related fraction concepts (McNamara & Shaughnessy, 2010).

Common misconceptions and how to help are presented in Table 8.1, as well as at the conclusion of each operation section in this chapter.

Teaching Tip

Anticipating student misconceptions is a critical part of planning—it can greatly influence task selection and how the lesson is structured.

Table 8.1. Common fraction misconceptions and how to help.

Misconception	What It Looks Like	How to Help
1. Numerator and Denominator are separate numbers.	$\frac{3}{4}$ is seen as a 3 over a 4. No concept of the relative size of the fraction.	• Find fraction values on a number line (e.g., a warm-up activity each day where students place values on a classroom number line). • Measure with inches to various levels of precision (e.g., to the nearest fourth, eighth, or sixteenth). • Avoid the phrase "three out of four" (unless you are talking about ratios or probability) or "three over four"; instead, say "three-fourths" (Siebert & Gaskin, 2006).
2. Fractional parts do not need to be equal-sized.	$\frac{3}{4}$ (three-fourths) of the figure is green. 	• Have students create their own representations of fractions across various types of models. • Provide problems like the one illustrated here, in which all the partitions are not already drawn and have students draw or show the equal-sized parts.
3. Fractional parts must be same shape.	This square is not showing fourths. 	• Provide examples and non-examples with partitioned shapes. (see Activity 8.3, for example) • Ask students to generate as many ways as they can to show fourths (or eighths).
4. Fractions with larger denominators are bigger.	$\frac{1}{5}$ is smaller than $\frac{1}{10}$ because 5 is less than 10.	• Use contexts, For example, ask students whether they would rather go outside for $\frac{1}{2}$ of an hour, $\frac{1}{4}$ of an hour, or $\frac{1}{10}$ of an hour and have them explain why. • Use visuals, such as paper strips or circles to visualize the approximate size of each fraction. • Teach estimation and benchmark strategies for comparing fractions.
5. Fractions with larger denominators are smaller.	$\frac{1}{5}$ is more than $\frac{7}{10}$ because fifths are bigger than tenths.	
6. Use whole number operation rules.	$\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$	• Tell stories with familiar contexts. • Ask students to create a story using a familiar context to make sense of the operation. • Use area and linear manipulatives and be sure students connect the visual to the steps in the algorithms.
7. Fractions represent a subtraction relationship.	Students report that the meaning of $\frac{2}{3}$ is that you took two pieces out of the pie.	• Avoid the use of the language "out of". • Show connections between area and length models to help students see the relationships between the parts and wholes.

Students who make the errors in column 2 of Table 8.1 do not understand fractions. Until they understand fractions meaningfully, they will continue to make errors by overapplying whole-number concepts (Cramer & Whitney, 2010; Siegler et al., 2010). The most effective way to help students reach higher levels of understanding is to use multiple representations, encourage multiple approaches, and elicit explanations and justifications (Harvey, 2012; Pantziara & Philippou, 2012).

Fraction Constructs

One of the commonly used meanings of fraction is part-whole. But, many who research fraction understanding believe students would understand fractions better (i.e., have fewer misconceptions) with more emphasis across other meanings of fractions (Clarke, Roche, & Mitchell, 2008; Lamon, 2012; Siebert & Gaskin, 2006).

Stop and Reflect

Beyond shading a region of a shape, how else are fractions used? How are these other ways alike and different from a part of a region or an area?

Part-Whole. Using the part-whole construct is an effective starting point for building meaning of fractions (Cramer & Whitney, 2010). Part-whole can be shading a region, part of a group ($\frac{3}{5}$ of the class went on the field trip), or part of a length (we walked $3\frac{1}{2}$ miles).

Measurement. Measurement involves identifying a length and then using that length as a measurement piece to determine the length of an object. For example, in the fraction $\frac{5}{8}$, you can use the unit fraction $\frac{1}{8}$ as the selected length and then count or measure to show that it takes five of those to reach $\frac{5}{8}$. This concept focuses on how much rather than how many parts, which is the case in part-whole situations (Behr, Lesh, Post, & Silver, 1983; Martinie, 2007).

Division. Consider the idea of sharing $10 with 4 people. This is not a part-whole scenario, but it still means that each person will receive one-fourth ($\frac{1}{4}$) of the money, or $2\frac{1}{2}$ dollars. Division is often not connected to fractions, which is unfortunate. Students should understand and feel comfortable with the example here written as $\frac{10}{4}$, $4\overline{)10}$, $10 \div 4$, $2\frac{2}{4}$, and $2\frac{1}{2}$ (Flores, Samson, & Yanik, 2006).

Operator. Fractions can be used to indicate an operation, as in $\frac{4}{5}$ of 20 square feet, or $\frac{2}{3}$ of the audience was holding banners. These situations indicate a fraction of a whole number, and students may be able to use mental math to determine the answer. This construct is not emphasized enough in school curricula (Usiskin, 2007). Just knowing how to represent fractions doesn't mean students will know how to operate with fractions (Johanning, 2008).

Ratio. Ratios are part-part or part-whole fraction constructs. For example, the ratio $\frac{3}{4}$ could be the ratio of those wearing jackets (part) to those not wearing jackets (part), or it could be part-whole, meaning those wearing jackets (part) to those in the class (whole).

Fraction Models

There is substantial evidence to suggest that the effective use of visuals in fraction tasks can help students understand fractions (Cramer & Henry, 2002; Siebert & Gaskin, 2006). Unfortunately, textbooks rarely incorporate manipulatives, and when they do, they tend to use only area models (Hodges, Cady, & Collins, 2008). This means that students often do not explore fractions in a variety of situations or have sufficient time to connect the visuals to the related concepts. Yet, critical to learning fractions is the use of physical models, which leads to mental models and then an understanding of fractions (Cramer & Whitney, 2010; Petit, Laird, & Marsden, 2010). Table 8.2 illustrates three categories of fraction models: area, length, and set.

Area Models

Circular fraction pieces are the most commonly used area model. The circle emphasizes the part–whole concept of fractions and the meaning of the relative size of a part to the whole (Cramer, Wyberg, & Leavitt, 2008). But circles can be overdone and are difficult to partition into same-size sections. There are many area model manipulatives, including pattern blocks, geoboards, color tiles, and fraction pieces. Various grid or dot paper are flexible in selecting the size of the whole and the size of the parts.

Table 8.2. Models for fraction concepts and related visuals and contexts.

Model Type	Description	Sample Contexts	Sample Manipulatives and Visuals
Area	Fractions are determined based on how a part of a region or area relates to the whole area or region.	Quesadillas (circular food) Pan of brownies Garden plot or playground	Fraction Circles/ Rectangles Pattern Blocks Tangrams Geoboards Grid paper regions
Length	Fractions are represented as a subdivision of a length of a paper strip (representing a whole), or as a length/distance between 0 and a point on a number line, subdivided in relation to a given whole unit. (The number line can represent positive and negative fractions.)	Walking/distance travelled String lengths Music measures Measuring with inches or yards	Cuisenaire Rods Paper strips Number lines
Set	Fractions are determined based on how many discrete items are in the whole set, and how many items are in the part.	Students in the class, school, stadium Type of item in a bag of items	Objects (e.g., pencils, toys) Counters (e.g., two color counters, colored cubes, teddy bear, sea shells)

technology note

An increasing number of Web resources are available to help students model fractions. One excellent source is Conceptua Fractions, developed by Conceptua Math (www.conceptuamath.com). This site offers tools that help students explore various fraction concepts with area, set, and length models (including the number line).

(See Blackline Masters 5–8 for a selection). Drawing rectangles is especially flexible and allows you to see if students partition correctly.

Length Models

With length models, lengths or measurements are compared instead of areas. Music, for example, can be a length example if you imagine partitioning a measure into halves, fourths, eighths, and sixteenths (Goral & Wiest, 2007). And, connecting fractions to measures in music can significantly improve student understanding of fractions (Courey, Balogh, Siker, & Paik, 2012).

Length models include Cuisenaire rods, paper strips (e.g., adding machine tape), fraction strips, and the number line.

Cuisenaire rods or strips provide flexibility because any length can represent the whole. And virtual rods can be found online at various sites such as *nrich*. The number line is a significantly more sophisticated measurement model (Bright, Behr, Post, & Wachsmuth, 1988). Reviews of research on fractions (Petit et al., 2010; Siegler et al., 2010) report that the number line helps students to understand a fraction as a number (rather than as one number over another number). The Institute of Educational Sciences report on effective teaching of fractions states that number lines should be a "central representational tool" (Siegler et al., 2010, p. 1).

The *Common Core State Standards* emphasize the importance of the number line across the grades, beginning with whole-number operations in the early grades and continuing through algebra in high school. Specific to fractions and middle school, sixth-grade standard 6.NS6 states, "Understand a rational number as a point on the number line" (CCSSO, 2010, p. 43), and in eighth grade, this is expanded to finding rational number estimations of irrational numbers on a number line. The following activity (based on Bay-Williams & Martinie, 2003) is a fun way to use a familiar context to engage students in thinking about fractions through a linear model.

Teaching Tip

Linear models are closely connected to the real-world contexts in which fractions are commonly used for measuring. Ask students to think of examples of when they have seen or used fractions—most of these will be length situations.

Activity 8.1

CCSS-M: 3.NF.A.2a, b; 3.NF.A.3a, b, d; 6.NS.C.6c

Who Is Winning?

Use Who Is Winning? Activity Page and give students paper strips or ask them to draw a number line. This activity can be done two ways (depending on your lesson goals). First, ask students to use reasoning to answer the question "Who is winning?" Students can use reasoning strategies to compare and decide. Second, students can locate each person's position on a number line. Explain that the friends below are playing "Red Light–Green Light." The fractions tell how much of the distance they have already moved. Can you place these friends on a line to show where they are between the start and finish? Second, rather than place them, ask students to use reasoning to answer the question "Who is winning?"

Mary: $\frac{3}{4}$	Harry: $\frac{1}{2}$	Larry: $\frac{5}{6}$	Han: $\frac{5}{8}$	Miguel: $\frac{5}{9}$	Angela: $\frac{2}{3}$

This game can be differentiated by changing the value of the fractions or the number of friends (fractions). The game red light–green light may not be familiar to English language learners (ELLs). Role playing the game with people in the class and using estimation are good ways to build background and support students with disabilities.

Set Models

The whole in a set model is a set of objects, and subsets of the whole make up fractional parts. For example, 3 red counters are one-fourth of a set of 12 counters. Set models include counters, people, or any discrete objects. The idea of referring to a collection of counters as a single entity makes set models difficult for some students. A piece of yarn in a loop around the items in the set can help students "see" the whole. A common misconception with set models is to focus on the size of a subset rather than the number of equal sets in the whole. For example, if 12 counters make a whole, then a subset of 4 counters is one-*third*, not one-fourth, because 3 equal sets make the whole. However, the set model helps establish important connections with many real-world uses of fractions and with ratio concepts.

Counters can be flipped to change their color to model various fractional parts of a whole set. Any countable objects (e.g., a box of crayons) can be a set model (with one box being the unit or whole). The activity below uses your students as the whole set, and it can be done as an energizer, warm-up, or intervention lesson.

Activity 8.2

CCSS-M: 3.NF.A.1; 3.NF.A.3b

Class Fractions

Use a group of students as the whole—for example, use six students if you want to work on halves, thirds, and sixths. Ask students, "What fraction of our group [are wearing tennis shoes, have brown hair, etc.]?" Change the number of people over time.

When you are determining what students already know about fractions, it is important to assess across each of these model types. For example, give students a piece of paper, fold it into thirds, and at the top of each section write *area*, *length*, and *set*. Have them show you a picture and write a sentence describing a context or example for a selected fraction (e.g., $\frac{3}{4}$) for all three models. This can be done exactly for commonly used fractions or can be an estimation activity with fractions like $\frac{31}{58}$. Each of these models will be important in fraction operations, as well as operations with other rational numbers, so it is important to determine students' strengths and needs and plan experiences accordingly.

Virtual manipulatives are available for all three models. Virtual manipulatives have been found to positively impact student achievement, especially when thes are paired with using the actual manipulatives (Moyer-Packenham, Ulmer, & Anderson, 2012). Recommended sites include:

Illuminations (NCTM)-Fractions Model. Explore length, area, region, and set models of fractions including fractions greater than one, mixed numbers, decimals, and percentages.

Math Playground-Fraction Bars: On this site you can explore fractional parts, the concepts of numerator and denominator, and equivalence.

National Library of Virtual Manipulatives: Numerous models for exploring fractions, including fraction bars and fraction pieces, and an applet for comparing and visualizing fractions.

Partitioning and Iterating

These two terms may not be common in middle school, but they are two foundational (and underemphasized) ideas for fractions that can greatly support student understanding of fraction operations, especially multiplication and division of fractions, which involve both processes. Each term is briefly discussed here.

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Partitioning

When you ask students to show eighths on a rectangle, or on a number line, they are not *dividing* (a commonly used term that can lead to confusion with the operation division) the shape or line, they are *partitioning* it. This term emphasizes the concept of part–whole (part-ition) and avoids confusion with division. Students should partition regions or shapes, lengths, and sets. Using examples and nonexamples helps students strengthen their understanding of fractions.

Activity 8.3

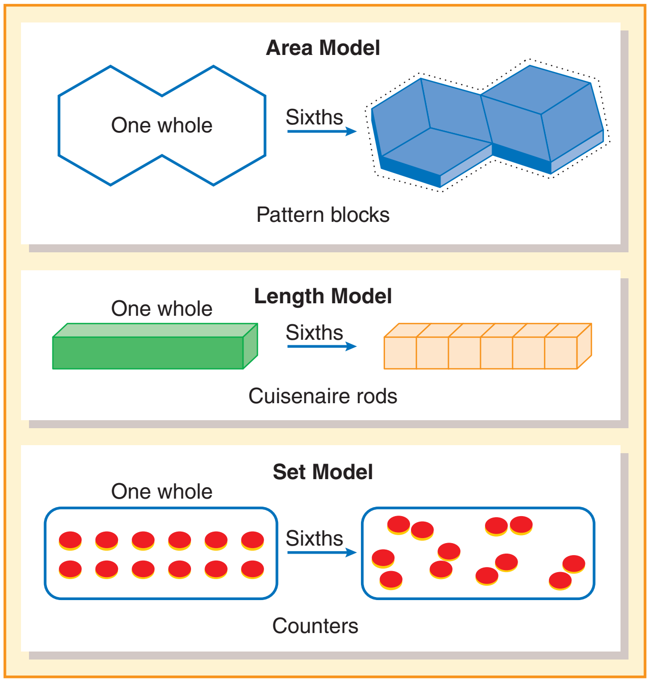
CCSS-M: 1.G.A.3; 2.G.A.3; 3.NF.A.1

Fourths or Not Fourths?

Use Fourths or Not Fourths Activity Page showing examples and nonexamples (which are very important to use with students with disabilities) of fourths. Ask students to identify the wholes that are correctly divided into fourths (equal shares) and those that are not. For each response, have students explain their reasoning. Repeat with other fractional parts, such as sixths. See Sixths or Not Sixths Activity Page (see Figure 8.1). To challenge students, ask them to draw shapes that fit each of the four categories (described below).

Figure 8.1

Which of these shapes are partitioned into sixths? Explain why or why not for each.



Students should be able to tell which of these figures are correctly partitioned in fourths, and tell why or why not. In the **Fourths or Not Fourths** Activity Page the shapes fall in each of the following:

1. Same shape, same size: (a) and (f) [equivalent]
2. Different shape, same size: (e) and (g) [equivalent]
3. Different shape, different size: (b) and (c) [not equivalent]
4. Same shape, different size: (d) [not equivalent].

Number lines are difficult, as students may ignore the size of the interval (McNamara & Shaughnessy, 2010; Petit et al., 2010; Shaughnessy, 2011). Activities 8.4 and 8.5 provide good activities for assessing if students understand fractional parts in a length context.

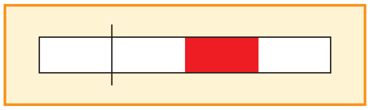
Activity 8.4

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

What Fraction Is Colored?

Prepare a set of paper strips prior to doing this activity (you can cut 1-inch wide pieces of 8.5" by 11" paper and shade, or cut pieces of adding machine tape. Color the strips so that they have a fractional amount shaded in various positions (not just left justified!) (Sarazen, 2012). Here are a few examples:

Explain that the strip represents one whole. Give each student a paper strip. Ask students to explain what fraction is partitioned and explain how they know. A common misconception is for students to count parts and call each of these one-third. If a student makes this error, ask if the parts are the same-sized and if not to partition to make same-sized parts. Use toothpicks or uncooked spaghetti to illustrate the partitions:

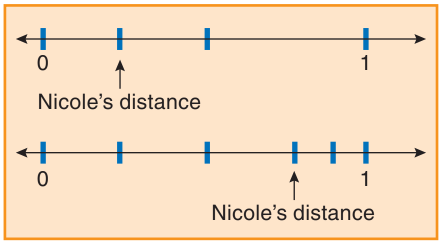


Students can also justify their reasoning by measuring the length of each partition.

Activity 8.5

CCSS-M: 3.NF.A.1; 3.NF.A.2a; 6.NS.C.6c

How Far Did Nicole Go?



Distribute How Far? Activity Page or give students number lines partitioned such that only some of the partitions are showing. Use a context such as walking to school. For each number line, ask, "How far has [Nicole] gone? How do you know?"

Students can justify their reasoning by measuring the size of the sections that have been partitioned.

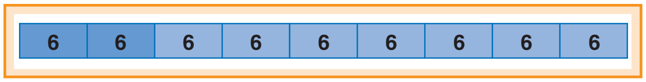
Formative Assessment Note

Activities 8.3 and 8.4 are good diagnostic assessments to see whether students understand that it is the size that matters, not the shape. If, for Activity 8.3, students identify all the wholes that are correctly partitioned into fourths except (e) and (g), they do not understand this concept, and you need to plan future tasks that focus on equivalence—for example, asking students to take a square and partition it in at least four different ways.

Partitioning is a strategy commonly used in Singapore as a method to solve story problems. Consider the following story problem (Englard, 2010):

A nurse has 54 bandages. Of those, $\frac{2}{9}$ have designs and the rest are just tan. How many of the bandages have designs?

Students who have learned the Singapore bar diagram model solve the problem by partitioning a strip into nine parts and figuring out how many bandages go into each part, and then how many are in two of the nine parts, as shown.



Set models can also be used to see if students understand fractional parts. Consider the following problem:

Eloise has 6 trading cards, André has 4 trading cards, and Lu has 2 trading cards. What fraction of the trading cards does Lu have?

Understanding that parts of a whole must be partitioned into equal-sized pieces across different models is an important step in conceptualizing fractions and provides a foundation for exploring sharing and equivalence tasks, which are prerequisite to performing fraction operations (Cramer & Whitney, 2010).

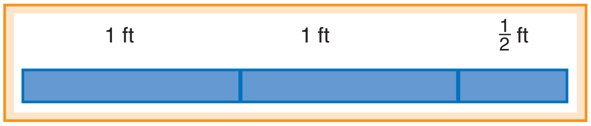
Iterating

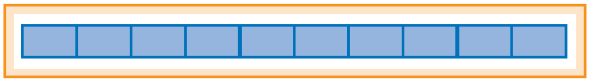
Counting fractional parts, or *iterating*, helps students understand the relationship between the parts (the numerator) and the whole (the denominator). The iterative concept is most clear when focusing on these two ideas about fraction symbols:

- The numerator *counts.*
- The denominator tells *what is being counted.*

Students need to understand that $\frac{3}{4}$, for example, can be thought of as a count of three parts called *fourths* (Post, Wachsmuth, Lesh, & Behr, 1985; Siebert & Gaskin, 2006; Tzur, 1999).

Iterating makes sense with length models because iteration is like measuring. Consider that you have $2\frac{1}{2}$ feet of ribbon and are trying to figure out how many fourths you have. You can draw a strip and start counting (iterating) the fourths, as shown.





Using a ribbon that is $\frac{1}{4}$ foot long as a measuring tool, a student counts off 10 fourths, as shown.

Students can participate in many tasks that involve iterating lengths (counting by fractional parts), progressing in increasing difficulty. Activity 8.6 involves strategy and reasoning, making it a good middle school task that reinforces fraction concepts.

Activity 8.6

CCSS-M: 3.NF.A.1; 3.NF.A.2a, b

A Whole Lot of Fun!

Use A Whole Lot Of Fun! Activity Page and a strip of paper like the one here:

Tell students that this strip is three-fourths of one whole (unit). Ask students to sketch strips of the other lengths on their paper (e.g., $\frac{5}{2}$). You can repeat this activity selecting other values for the starting amount and selecting different fractional values to sketch. A context, such as walking, is effective in helping students make sense of the situation. Be sure to use fractions less than and greater than 1 and mixed numbers.

Notice that to solve the task in Activity 8.6, students first partition the piece into three sections to find $\frac{1}{4}$ and then iterate the $\frac{1}{4}$ to find the other lengths.

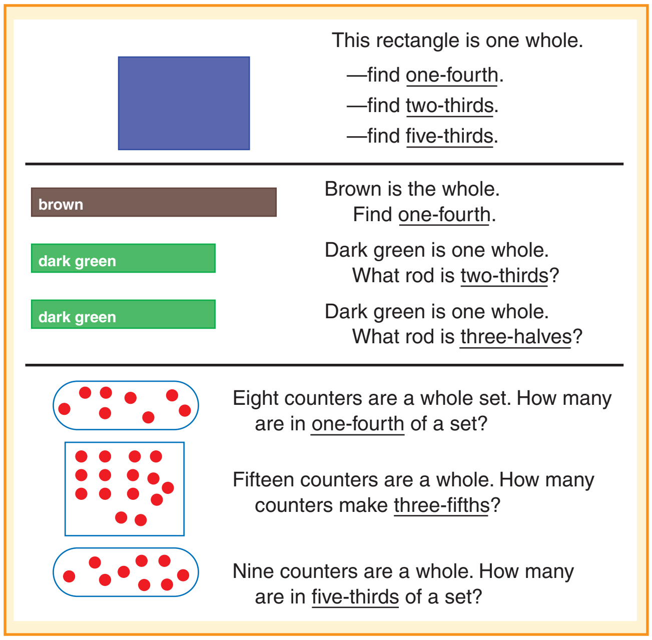
Iterating can be done with a variety of models. For example, you can display fraction pieces and simply count them together: "one-fourth, two-fourths, three-fourths, four-fourths, five-fourths." Figures 8.2, 8.3, and 8.4 provide examples across models of engaging students in iterating and partitioning in order to strengthen their understanding of fractions.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Figure 8.2

Given the whole and the fraction, find the part.



This rectangle is one whole.

—find one-fourth.

—find two-thirds.

—find five-thirds.

Brown is the whole. Find one-fourth.

Dark green is one whole. What rod is two-thirds?

Dark green is one whole. What rod is three-halves?

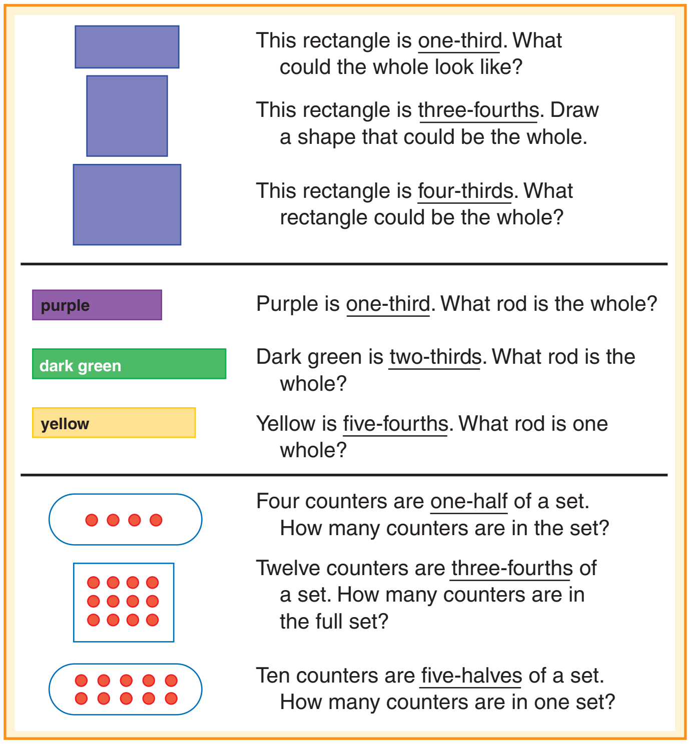
Eight counters are a whole set. How many are in one-fourth of a set?

Fifteen counters are a whole. How many counters make three-fifths?

Nine counters are a whole. How many are in five-thirds of a set?

Figure 8.3

Given the part and the fraction, find the whole.



This rectangle is one-third. What could the whole look like?

This rectangle is three-fourths. Draw a shape that could be the whole.

This rectangle is four-thirds. What rectangle could be the whole?

Purple is one-third. What rod is the whole?

Dark green is two-thirds. What rod is the whole?

Yellow is five-fourths. What rod is one whole?

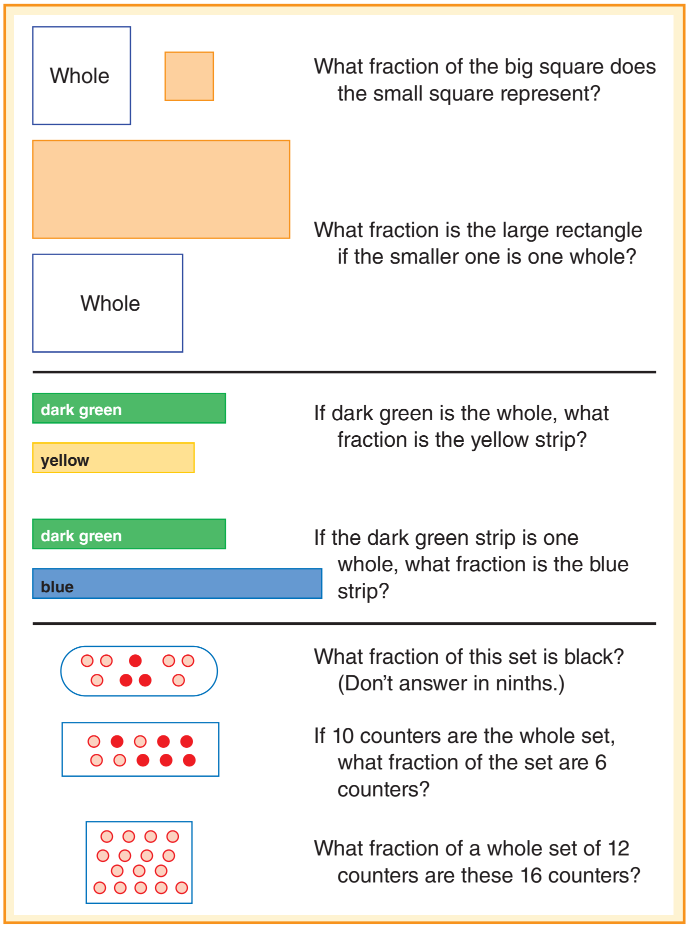
Four counters are one-half of a set. How many counters are in the set?

Twelve counters are three-fourths of a set. How many counters are in the full set?

Ten counters are five-halves of a set. How many counters are in one set?

Figure 8.4

Given the whole and the part, find the fraction.



What fraction of the big square does the small square represent?

What fraction is the large rectangle if the smaller one is one whole?

If dark green is the whole, what fraction is the yellow strip?

If the dark green strip is one whole, what fraction is the blue strip?

What fraction of this set is black? (Don't answer in ninths.)

If 10 counters are the whole set, what fraction of the set are 6 counters?

What fraction of a whole set of 12 counters are these 16 counters?

Stop and Reflect

Explore the problems in these three figures. If you do not have access to rods or counters, just draw illustrations on paper or solve mentally. Which problems and which models do you find easier to solve? Which ones do you think your students will find easier? Harder?

Ask students puzzle-type questions such as, “If 5 counters is one-fourth of the whole, how much of the whole is 15 counters?” and “Three counters represent $\frac{1}{8}$ of my set; how big is my set?” If the fraction is not a unit fraction then students first partition and then iterate. For example, the problem “Twenty counters represents $\frac{2}{3}$ of my set; how big is my set?” first requires finding $\frac{1}{3}$ (10 counters), then iterating that three times to get 30 counters in three-thirds (one whole). **Cuisenaire Parts and Wholes** Activity Page or the two-color counters activities (**Counting Counters: Find the Part** Activity Page and **Counting Counters: Find the Whole** Activity Page) are examples of ways to engage students in exploring parts and wholes.

Calculators are an excellent tool for iteration activities.

Activity 8.7

CCSS-M: 3.NF.A.1; 5.NF.B.7b; 6.NS.A.1

Calculator Fraction Counting

Many calculators (e.g., TI-15), display fractions in correct fraction format and offer a choice of showing results as mixed numbers or simple fractions. Ask students to type in a fraction (e.g., $\frac{1}{4}$) and then + and the fraction again. To count, press OP1 , OP1 , OP1, repeating to get the number of fourths wanted. The display will show the counts by fourths and also the number of times that the Op1 key has been pressed. Ask students questions such as the following: “How many $\frac{1}{4}$ s to get to 3?” “How many $\frac{1}{5}$ s to get to 2?” These can get increasingly more challenging: “How many $\frac{1}{4}$ s to get to $4\frac{1}{2}$?” “How many $\frac{2}{3}$ s to get to 6? Estimate and then count by $\frac{2}{3}$ s on the calculator.” Students, particularly students with disabilities, should coordinate their counts with fraction models, adding a new fourths piece to the pile with each count.

Fraction calculators provide a powerful way to help students develop fraction meaning. A variation on Activity 8.7 is to show students a mixed number such as $3\frac{1}{8}$ and ask how many counts of $\frac{1}{8}$ on the calculator it will take to count that high. The students should try to stop at the correct number ($\frac{25}{8}$) before pressing the mixed-number key. This can be connected to the division problem: $3\frac{1}{8} \div \frac{1}{8} =$ ____ (How many one-eighths are in three and one-eighth?).

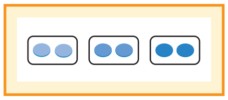
Fraction Equivalencies

Fraction operations involve a significant amount of working with fraction equivalences. If students know only rules for finding an equivalent fraction, they will struggle with all operations. Instead, equivalences must be well understood. Although this content is addressed in elementary school, any of the ideas in this section can be used as readiness lessons, warmup activities, or formative assessment, for example, ask:

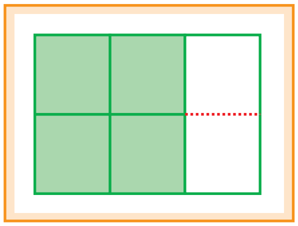
How do you know that $\frac{4}{6} = \frac{2}{3}$? Explain in two different ways.

Here are some possible student responses:

- They are the same because you can simplify $\frac{4}{6}$ and get $\frac{2}{3}$.
- If you have a set of 6 items, 4 of them would be $\frac{4}{6}$. But you can think of the 6 as 3 groups with two in each group, and the 4 would be 2 groups with two in each group. So, $\frac{2}{3}$ groups is equivalent to $\frac{4}{6}$ items.



- If you start with $\frac{2}{3}$, you can multiply the numerator and denominator by 2, and that will give you $\frac{4}{6}$, so they are equal.
- If you had a rectangle, partitioned it into 3 parts, and shaded 2 parts, that would be $\frac{2}{3}$ shaded. If you partitioned all 3 of these parts in half, that would be 6 parts in all, with 4 parts shaded. That's $\frac{4}{6}$, and it would be the same amount.



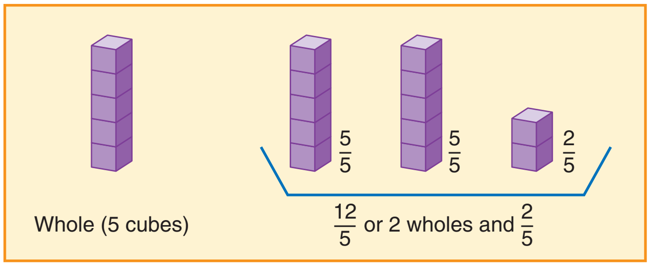
All of these answers are correct, but they vary in the connections to the concept of equivalence. Response 2 illustrates what is happening in response 1, and response 4 illustrates what is happening in response 3. You can reinforce a conceptual understanding of fraction equivalence by using language that focuses on the concept (responses 2 and 4).

Teaching Tip

The term *improper fraction* for fractions greater than one can be a source of confusion because it implies "not acceptable." In algebra, fractions are often preferred to mixed numbers. Replace this phrase with *fraction* or *fraction greater than one,* which is consistent with *Common Core State Standards* and National Council of Teachers of Mathematics (NCTM) terminology.

Figure 8.5

Stacking cubes illustrate the equivalence of $\frac{12}{5}$ and $2\frac{2}{5}$.



Fractions Greater Than One

Fractions less than and greater than one should be mixed together in fraction instruction. Too often, students aren't exposed to numbers greater than one (e.g., $\frac{5}{2}$ and $4\frac{1}{4}$), and then when they are added into the mix (no pun intended!), they find them confusing.

If students have had opportunities to iterate fractional parts beyond one whole, as in the previous section, they already know how to write $\frac{13}{6}$ or $\frac{13}{5}$. Ask students to select a manipulative or create a drawing to illustrate these values and find equivalent representations by using wholes and fractions (mixed numbers). Using connecting cubes is a very effective way to help students see both forms for recording fractions greater than one (Neumer, 2007) (Figure 8.5).

Students identify one cube as the unit fraction ($\frac{1}{5}$) for the problem ($\frac{12}{5}$). They count out 12 fifths and build wholes. Conversely, they can start with the mixed number, build it, and find out how many total cubes (or fifths) were used. Repeated experiences in building and solving these tasks will lead students to see a pattern of multiplication and division that closely resembles the algorithm for moving between these two forms.

By middle school, it is important that students use mental images. Challenge students to visualize equivalent forms. Even though middle-level students may already know the standard algorithm for moving between fractions and mixed numbers, it is important for them to be able to explain the relationship conceptually. This conceptual basis is critical for the reasoning needed in fraction operations and in algebra.

Estimating Fraction Size

Students must know "about" how big a particular fraction is if they are going to be able to estimate sums, differences, products, and quotients. As with whole numbers, students are less confident and less capable of estimating than they are at computing exact answers, and a focus on estimation can strengthen their understanding of fractions (Clarke & Roche, 2009). Here we share a few activities to reinforce (and formatively assess) student understanding of the size of a fraction. First, see if students can identify approximate fractional amounts from visuals, as in Activity 8.8.

Standards for Mathematical Practice

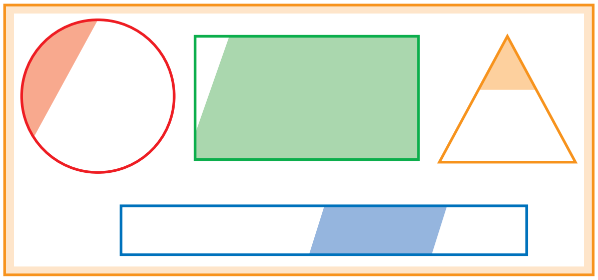
8 Look for and express regularity in repeated reasoning.

Activity 8.8

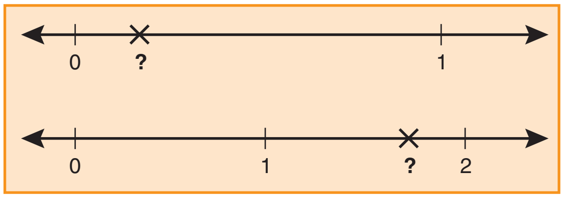
CCSS-M: 3.NF.A.1; 3.NF.A.2a, b; 6.NS.C.6c

About How Much?

Draw or project shaded regions like the ones shaded here.



Ask students to write down a fraction that they think is a good estimate of the amount shown. Listen to the responses of several students, and ask the class whether these fractions are a reasonable match. There is no single correct answer, but estimates should be "in the ballpark." Repeat with the number line, as illustrated here.



If students have difficulty coming up with an estimate, ask whether they think the amount is closer to 0, $\frac{1}{2}$, or 1. You may want to give students with disabilities a set of cards with possible options for estimates and have them match cards to the pictures.

The number line is a good model for helping students develop a better understanding of the relative size of a fraction (Petit, Laird, & Marsden, 2010). Remember also to estimate with values greater than one—for example, $3\frac{3}{7}$ or $\frac{19}{5}$ to the nearest benchmark, 3, $3\frac{1}{2}$, or 4.

Equivalence across Fraction Models

Note that the two examples above were area and set models. The following discussion and Activities 8.10, 8.11, and 8.12 similarly include attention to area and set models, along with length models.

Activity 8.9

CCSS-M: 3.NF.A.1; 3.NF.3a, b, c

Making Stacks

Select a manipulative that is designed for exploring fractions (e.g., pattern blocks, tangrams, fraction strips, or fraction circles). Use **Fraction Cards** Activity Page with different fractional amounts, such as $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $\frac{3}{2}$, $\frac{4}{3}$, and 2. Note: you may want to begin with fractions less than 1, then move to fractions equal to and greater than 1. Working individually or with a partner, ask students to first identify the whole. Then on top of the whole, ask them to see how many stacks they can make. A stack must use the same-sized piece. Ask students to record all the possibilities they find (they can color the shapes and write the fraction). After completing several examples, have students look at the fractions they wrote for a stack and describe or write about the patterns they notice.

In a classroom discussion following "Making Stacks," you can help students reason about equivalent fractions by asking them to consider what other equivalencies are possible (and justify their thinking). For example, ask, "What equivalent fractions could you find if we had sixteenths in our fraction kit? If you could have a piece of any size at all, what other fraction names are possible?"

Activity 8.10

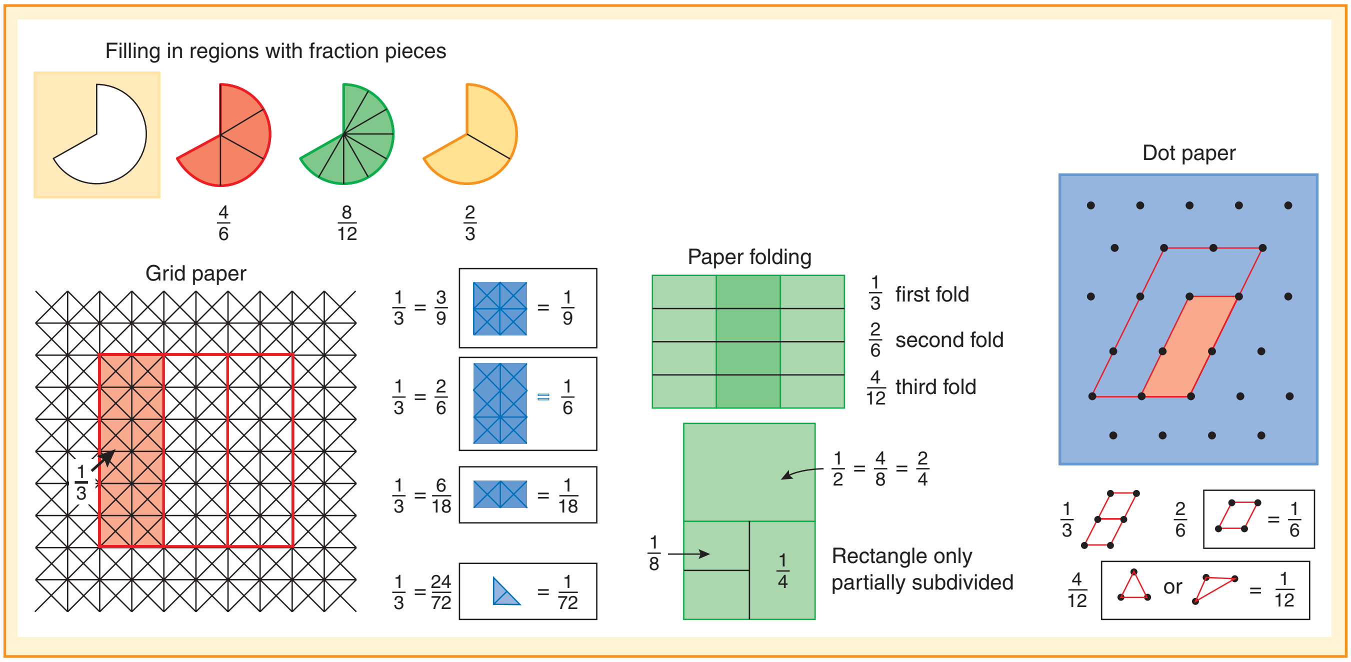
CCSS-M: 3.NF.A.1; 3.NF.3a, b, c

Dot-Paper Equivalancies

Use **Fraction Names** Activity Page, which includes three different grids with a fraction shaded (each enclosed area represents one whole). Ask students how many fraction names they think the first problem has. Then ask them to see how many they can find (working individually or in partners). Invite students to share and explain the fraction names they found for #1. Repeat for the next two problems. Alternatively, cut this page into three task cards, laminate the cards, and place each at a station along with an overhead pen. In partners have students rotate to a station and see how many fraction names they can find for that shape (using the pen as needed to show their ways). Rotate to the next station. You may also want to refer to the **Dot Paper Equivalencies Expanded Lesson**. To make additional pictures, create your own using your choice of **grid** or **dot paper** (or, choose from Blackline Masters 5–8). Figure 8.6 shows an example drawn on an isometric grid. Students should draw a picture of the unit fractional part that they use for each fraction name. The larger the size of the whole, the more names the activity will generate.

Figure 8.6

Area models for equivalent fractions.

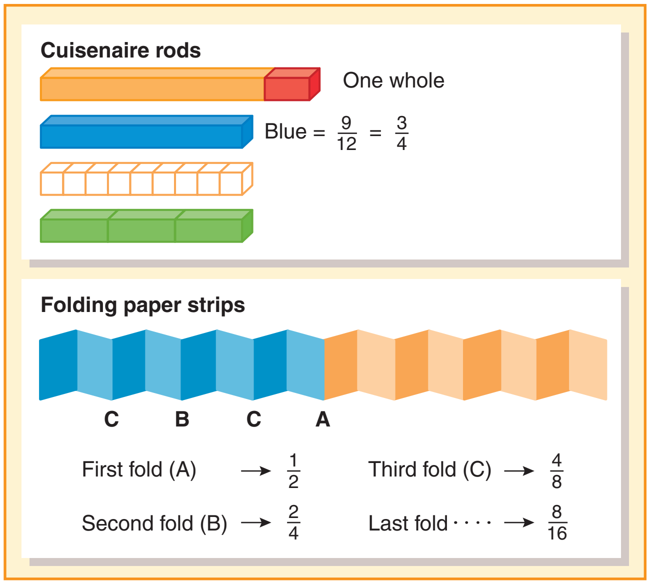


The “Dot-Paper Equivalences” activity involves what Lamon (2012) calls “unitizing”—that is, given a quantity, finding different ways to chunk the quantity into parts in order to name it. She points out that this is a key ability related not only to equivalent fractions but also to proportional reasoning, especially in the comparison of ratios.

Length models should be used in activities similar to the “Making Stacks” activity. Asking students to locate $\frac{2}{5}$ and $\frac{4}{10}$ on a number line, for example, can help them see that the two fractions are equivalent (Siegler et al., 2010). Measuring in inches is an excellent context for working on fraction equivalences with halves, fourths, eighths, and sixteenths. Using Cuisenaire rods, students find smaller rods to find fraction names for the given part (see the first example in Figure 8.7). To have larger wholes or values greater than one whole, use a train of two or three rods for the whole and the part. Folding paper strips is another method of creating fraction names. In the second example in Figure 8.7, one-half is partitioned by successive folding in half. Other folds would produce other equivalent fraction names.

Figure 8.7

Length models for equivalent fractions.



One excellent way to make a direct connection to algebra is to have students think of fractions as slope and find their locations on a coordinate axis (Cheng, 2010). For example, the fraction $\frac{3}{4}$ can represent the rise and run for slope (rise is the y-value, and run is the x-value). So, $\frac{3}{4}$ is plotted at (4,3). Similarly, the fraction $\frac{6}{8}$ is plotted at (8,6).

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Formative Assessment Note

Consider using a diagnostic interview to see if students are making the connection between equivalence and slope. Ask students to find other values that lie on this line. Ask what they know about the fraction they selected. Students should be able to explain that they are equivalent fractions and that they are fractions that represent the same slope. Conversely, you can give students fractions and ask, “Is this fraction equivalent to $\frac{3}{4}$?” and “Will this fraction be on the same line with $\frac{3}{4}$?”

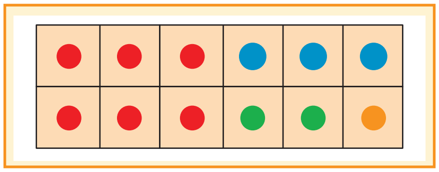
Legos, fun for all ages, can help students learn to write fraction equivalencies (see Gould, 2011 for elaboration). Lego bricks can be viewed as an area (array) or as a set (students can count the studs).

Activity 8.11

CCSS-M: 4.NF.B.3a, b; 5.NF.A.1

Lego Land: Building Options

Hand out one 2 by 6 Lego to each student. Ask students to imagine that a 12 piece of Legos represents one plot of land. It can be covered with various smaller pieces as shown here:



(*continued*)

Ask students to build the plot of land using different Lego pieces (1 by 2, 1 by 3, 2 by 2, 1 by 1, 2 by 6, etc). After they have completed their plot of land, ask students to tell the fraction of their land that is represented by a particular piece (e.g., the 2 by 6 is $\frac{6}{12}$ as well as $\frac{1}{2}$ and $\frac{2}{4}$).

To focus on iteration and to build connections to addition, students can write equations to describe their Lego Land. In the one pictured here, that would be $\frac{6}{12} + \frac{3}{12} + \frac{2}{12} + \frac{1}{12}$ OR $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$.

Note that students have misconceptions about how to name fractions parts, naming the blue part as $\frac{1}{3}$ rather than $\frac{1}{4}$ because they see three pieces (Wilkerson, Bryan, & Curry, 2012). This becomes a good topic for a classroom discussion: What is the fractional value of the blue pieces (simplified)?

The following activity is also a unitizing activity in which students look for different units or chunks of the whole in order to name a part of the whole in different ways.

Activity 8.12

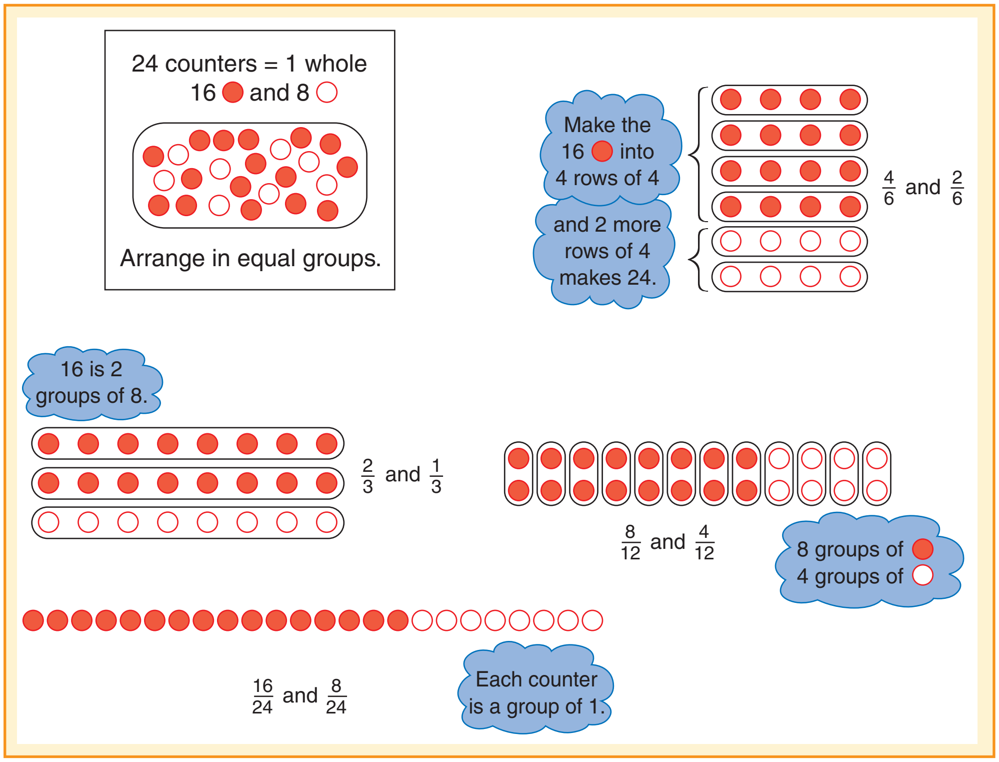
CCSS-M: 3.NF.A.1; 3.NF.3a, b, c

Apples and Bananas

Use Apples and Bananas Activity Page or just have students set out a specific number of counters in two colors—for example, 24 counters, with 16 of them red (apples) and 8 of them yellow (bananas). The 24 counters make up the whole. The task is to group the counters into different fractional parts of the whole and use the parts to create fraction names for the fractions that are apples and the fractions that are bananas. Ask questions such as, "If we make groups of four, what part of the set is red?" to encourage students to think of different ways to form equal-sized groups. In Figure 8.8, 24 counters (pieces of fruit) are arranged in different groups and arrays. ELLs may not know what is meant by the term *group* because this is often used in classrooms to arrange students. Spend time before the activity modeling what it means to group objects.

Figure 8.8

Set models for equivalent fractions.



Across the models, two instructional practices are critical in reinforcing the concept of equivalence:

- Notice that the phrase *reducing fractions* was not used. This terminology implies making the amount smaller and should be avoided. Fractions are simplified, *not* reduced.
- Do not tell students their answer is incorrect if not in the simplest or lowest terms. This also misinforms students about the equivalence of fractions. If you want the answer in a simplified form, provide feedback to the student that the answer is correct but must be simplified.

Equivalent-Fraction Algorithm

Middle-level students need to be able to use and understand the equivalent-fraction algorithm. Activity 8.14 is designed to build this connection. Adding a context can help struggling learners understand why the fractions are equivalent.

Activity 8.13

CCSS-M: 3.NF.3b; 4.NF.A.1

Garden Plots

Distribute the Garden Plots Activity Page or have students draw a square "garden" on blank paper, or hand students a square of paper (like origami paper). Begin by explaining that the garden is divided into rows of various vegetables. In the first example, you might illustrate four columns (fourths) and designate $\frac{3}{4}$ rows as corn. Then, explain that the garden is going to be shared with family and friends in such a way that each person has an equal share. Show how the garden can be partitioned horizontally to represent sharing with two people (Figure 8.9(a)). Ask, "What fraction of our newly partitioned garden is corn?" $\frac{6}{8}$. Next, tell students to come up with other ways to share the garden with friends. (They can choose how many, or you can structure it.) For each newly partitioned garden, ask students to record an equation showing the equivalent fractions.

After students have prepared their own examples, provide time for them to look at their fractions and gardens and notice patterns about the fractions and the diagrams. To help students see the connection of the garden plot to the algorithm, cover a part of the plot so that they cannot see all the sections (Figure 8.9(b)).

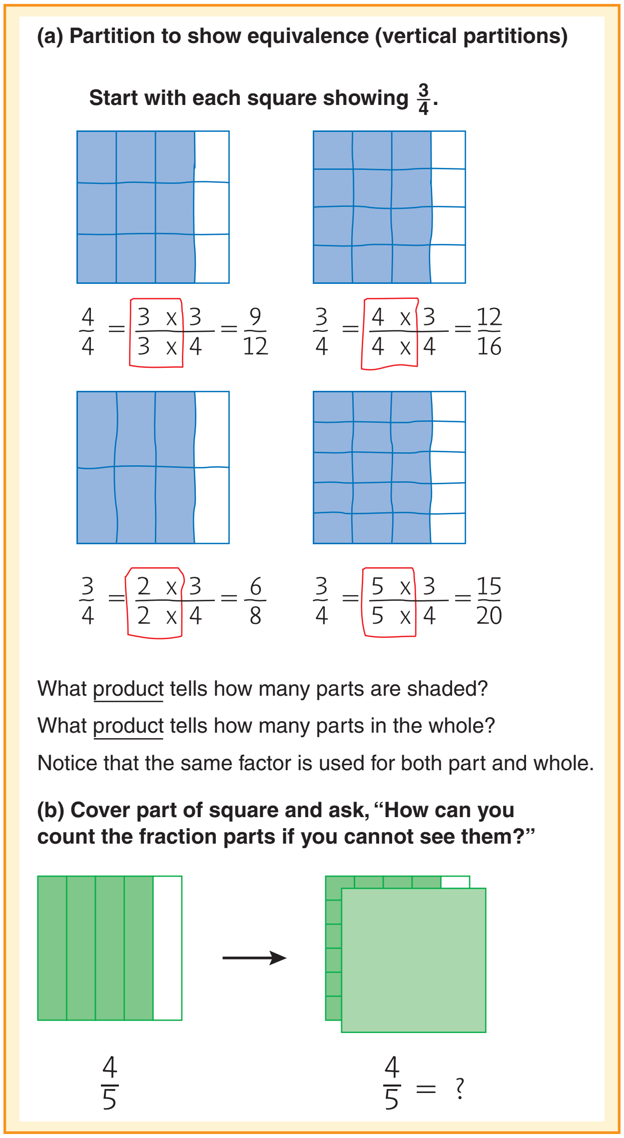
technology note

NCTM's Illuminations has three units called "Fun with Fractions." Each unit uses one of the model types (area, length, or set) and focuses on comparing and ordering fractions and equivalence.

- Set model unit: http://illuminations.nctm.org/LessonDetail.aspx?id=U112
- Region model unit: http://illuminations.nctm.org/LessonDetail.aspx?id=U113
- Length model unit: http://illuminations.nctm.org/LessonDetail.aspx?id=U152.

Figure 8.9

Conceptual illustrations of equivalent fraction algorithm.



(a) Partition to show equivalence (vertical partitions)

Start with each square showing $\frac{3}{4}$.

$\frac{4}{4} = \boxed{\frac{3\ \times}{3\ \times}}\frac{3}{4} = \frac{9}{12}$ $\frac{3}{4} = \boxed{\frac{4\ \times}{4\ \times}}\frac{3}{4} = \frac{12}{16}$

$\frac{3}{4} = \boxed{\frac{2\ \times}{2\ \times}}\frac{3}{4} = \frac{6}{8}$ $\frac{3}{4} = \boxed{\frac{5\ \times}{5\ \times}}\frac{3}{4} = \frac{15}{20}$

What <u>product</u> tells how many parts are shaded?

What <u>product</u> tells how many parts in the whole?

Notice that the same factor is used for both part and whole.

(b) Cover part of square and ask, "How can you count the fraction parts if you cannot see them?"

$\frac{4}{5}$ $\frac{4}{5} = ?$

Comparing Fractions

Comparing is an extension of equivalence. Rather than ask the comparison question, "Which of these fractions is greater?" we can ask, "Which of these fractions is greater, or are they equal?" (Smith, 2002). This question leaves open the possibility that two fractions that may look different can, in fact, be equal.

Being able to reason about the size of fractions is helpful when comparing fractions. Too often, middle-level students use an algorithm (cross-products) for comparing two fractions, which they do not understand. Comparison activities (Which fraction is more? Which is less?) can play a significant role in helping students develop an understanding of the relative size of fractions.

Activity 8.14

CCSS-M: 4.A.NF.2

Which Fraction Is Greater?

Use the "Which Is Greater?" Activity Page. Ask students to use a reasoning strategy to determine which fraction is greater. For each problem, ask students to explain how they determined their answer. Encourage students to use different reasoning strategies, such as benchmarks, across the problems (and not an algorithm like cross products).

Which fraction in each pair is greater?

A. $\frac{4}{5}$ or $\frac{4}{9}$
B. $\frac{4}{7}$ or $\frac{5}{7}$
C. $\frac{3}{8}$ or $\frac{4}{10}$
D. $\frac{5}{3}$ or $\frac{5}{8}$
E. $\frac{3}{4}$ or $\frac{9}{10}$
F. $\frac{3}{8}$ or $\frac{4}{7}$
G. $\frac{7}{12}$ or $\frac{5}{12}$
H. $\frac{3}{5}$ or $\frac{3}{7}$
I. $\frac{5}{8}$ or $\frac{6}{10}$
J. $\frac{9}{8}$ or $\frac{4}{3}$
K. $\frac{4}{6}$ or $\frac{7}{12}$
L. $\frac{8}{9}$ or $\frac{7}{8}$

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Here we share four different ways to compare. The first two rely on the meanings of the numerator and denominators in fractions and on the relative sizes of unit fractional parts. The third and fourth use the additional ideas of 0, $\frac{1}{2}$, and 1 as convenient anchors or benchmarks for thinking about the size of fractions. Students fluent in comparing fractions choose the best method, based on the fractions given.

1. *Same-size whole (same denominators).* To compare $\frac{3}{8}$ and $\frac{5}{8}$, think about having 3 parts of something and also 5 parts of the same thing. (This method can be used for problems B and G.)

2. *Same number of parts (same numerators) but different-sized parts.* Consider the case of $\frac{3}{4}$ and $\frac{3}{7}$. If a whole is partitioned into 7 parts, the parts will certainly be smaller than if partitioned into only 4 parts. (This strategy can be used with problems A, D, and H.)
3. *More than/less than one-half or one.* The fraction pairs $\frac{3}{7}$ versus $\frac{5}{8}$ and $\frac{5}{4}$ versus $\frac{7}{8}$ do not lend themselves to either of the previous thought processes. In the first pair, $\frac{3}{7}$ is less than $\frac{1}{2}$ of the number of sevenths needed to make a whole, and so $\frac{3}{7}$ is less than $\frac{1}{2}$. Similarly, $\frac{5}{8}$ is more than $\frac{1}{2}$. Therefore, $\frac{5}{8}$ is the larger fraction. The second pair is determined by noting that one fraction is greater than 1 and the other is less than 1. (This method could be used on problems A, D, F, G, and H.)
4. *Closeness to one-half or one.* Why is $\frac{9}{10}$ greater than $\frac{3}{4}$? Each is one fractional part away from one whole, and tenths are smaller than fourths. Similarly, notice that $\frac{5}{8}$ is smaller than $\frac{4}{6}$ because it is only one-eighth more than $\frac{1}{2}$, while $\frac{4}{6}$ is a sixth more than $\frac{1}{2}$. Can you use this basic idea to compare $\frac{3}{5}$ and $\frac{5}{9}$? (*Hint:* Each is half of a fractional part more than $\frac{1}{2}$.) Also try $\frac{5}{7}$ and $\frac{7}{9}$. (This is a good strategy for problems C, E, I, J, K, and L.)

Stop and Reflect

If students do not have all of these reasoning strategies, what strategies might you use to showcase the ones that are not as familiar?

Understanding Fraction Operations

Students must be able to compute with fractions flexibly and accurately. Success with fractions, in particular computation, is closely related to success in Algebra I. If students enter formal algebra with a weak understanding of fraction computation (in other words, they have memorized the four procedures but do not understand them), they are at risk, which in turn can limit college and career opportunities.

While teaching one algorithm for each operation seems quicker, it is not effective! First, the algorithms don't help students think conceptually about the operations. When students follow a procedure they do not understand, they have no means of knowing when to use it, and no way of assessing whether their answers make sense. Second, mastery of poorly understood algorithms is quickly forgotten, particularly by students who struggle in mathematics. All too soon the different algorithms become a meaningless jumble. Students ask, "Do I need a common denominator, or do I just add or multiply the bottom numbers?" Third, students can't adapt the strategy to slightly different values, such as decimals or mixed numbers.

Effective Teaching Process

In a report that summarizes what works for teaching fraction operations (Siegler et al., 2010), four steps are suggested: (1) use contexts, (2) use a variety of models, (3) include estimation and informal methods, and (4) address misconceptions. These four steps are elaborated for each operation in the sections below.

Ongoing connections must be made between the contexts, models, and processes. It takes multiple experiences and time for these relationships to be well understood. Although this chapter addresses fraction operations in separate sections, students need opportunities to solve story problems to determine *which* operation fits the story. Too often students are given subtraction story problems on a day they are learning how to subtract, which removes the central question of "Which operation is a match to this story situation?"

Stop and Reflect

Which operation can be used to solve the story situations in Figure 8.10?

Figure 8.10

Mixed story problems involving fractions.

a. Jeremy's friends eat $\frac{3}{4}$ of his birthday cake. Jeremy eats $\frac{1}{8}$ of the cake. How much did Jeremy eat?

b. Jeremy's friends eat $\frac{3}{4}$ of his birthday cake. Jeremy ate $\frac{1}{8}$ of the leftover cake. How much of the cake did Jeremy eat?

c. Jeremy's friends ate $\frac{3}{4}$ of one birthday cake and $\frac{1}{8}$ of another cake. How much did they eat?

d. Jessica walked $\frac{1}{8}$ of a mile. Her goal is $\frac{3}{4}$ of a mile. How much further to reach her goal?

e. Jessica walks $\frac{1}{8}$ of a mile to school. How many trips to and from school are needed for her to reach her goal of walking $\frac{3}{4}$ of a mile?

Let's look at Jeremy's situations (area situation). In (a) the answer is in the problem Jeremy ate $\frac{1}{8}$ of the cake. In the second case, you must figure out $\frac{1}{8}$ of $\frac{1}{4}$, which is $\frac{1}{32}$ of the cake. In the third case, it is not clear if the cakes are the same-sized whole, so there is no way to solve this problem. If it is adapted to clarify that the cakes are the same size, then it could be solved using addition. In Jessica's first situation (linear), the difference is needed, comparing how far she has walked to how much she needs to walk. Subtraction is needed. In the latter case, the question is "How many trips?" or "How many eighths are in three-fourths?" This can be solved by counting eighths or by division: $\frac{3}{4} \div \frac{1}{8}$. A story problem like any one of these can be posted as a warm-up problem throughout year with the focus on what is happening in the story and then discussing which operation makes sense (and why).

Addition and Subtraction

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Students develop fluency with fraction addition and subtraction in grade 5, and then these concepts are revisited again in grade 7 as fraction operations are expanded to include negative numbers (CCSSO, 2010).

Contextual Examples and Models

Just like with whole numbers, invented strategies are important for students because they build on student understanding of fractions and fraction equivalence, and they can eventually be connected to the standard algorithm in such a way that the standard algorithm makes sense. Recall also that the CCSS-M content standards outline addition situations, which are connected to whole number operations, but that also should be applied to fraction addition and subtraction. Those situations are: join, separate, part–part–whole, and compare (CCSSO, 2010; Chval, Lannin, & Jones, 2013).

Consider the example of measuring something in inches (e.g., sewing, cutting molding for a doorway, hanging a picture). One inch is one unit or one whole. Measurements include halves, fourths, eighths, and/or sixteenths—fractions that can be added mentally by considering the relationship between the sizes of the parts (e.g., $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$). If you can easily find the equivalent fraction, (e.g., that one-fourth of an inch is also two-eighths), then you can add like-sized parts mentally.

Contexts should vary and be interesting to students. Here are several good examples. As you read, think about how they differ from each other (beyond the story context):

Jacob ordered 3 pizzas. But, before his guests arrived he got hungry and ate $\frac{3}{8}$ of one pizza. How much was left for the party?

On Friday, Lydia ran $1\frac{1}{2}$ miles, on Saturday she ran $2\frac{1}{8}$ miles and on Sunday she ran $2\frac{3}{4}$ miles. How many miles did she run over the weekend?

Sammy gathered $\frac{3}{4}$ pounds of walnuts and Chala gathered $\frac{7}{8}$ pounds. Who gathered the most? How much more?

In measuring the wood needed for a picture frame, Elizabeth figured that she needed two pieces that were $5\frac{1}{4}$ inches and two pieces that were $7\frac{3}{4}$ inches. What length of wood does she need to buy to build her picture frame?

Notice that these story problems (1) incorporate different addition situations (join, compare, etc.), (2) use a mix of area and linear contexts; (3) use a mix of whole numbers, mixed numbers, and fractions; (4) include both addition and subtraction situations; and (5) sometimes involve more than two addends.

Teaching Tip

When solving story problems, ask students to select a picture or tool to illustrate it, and write the symbols that accurately model the situation.

Formative Assessment Notes

A simple problem like any of the ones above (e.g., Gathering Walnuts) can be used as a formative assessment and assessed using an **observation checklist**. A checklist might include such concepts as: (1) Can determine a reasonable estimate; (2) Selects and accurately uses a manipulative or picture; (3) Recognizes equivalences between fourths and eighths; and (4) Can connect symbols and visuals.

Models

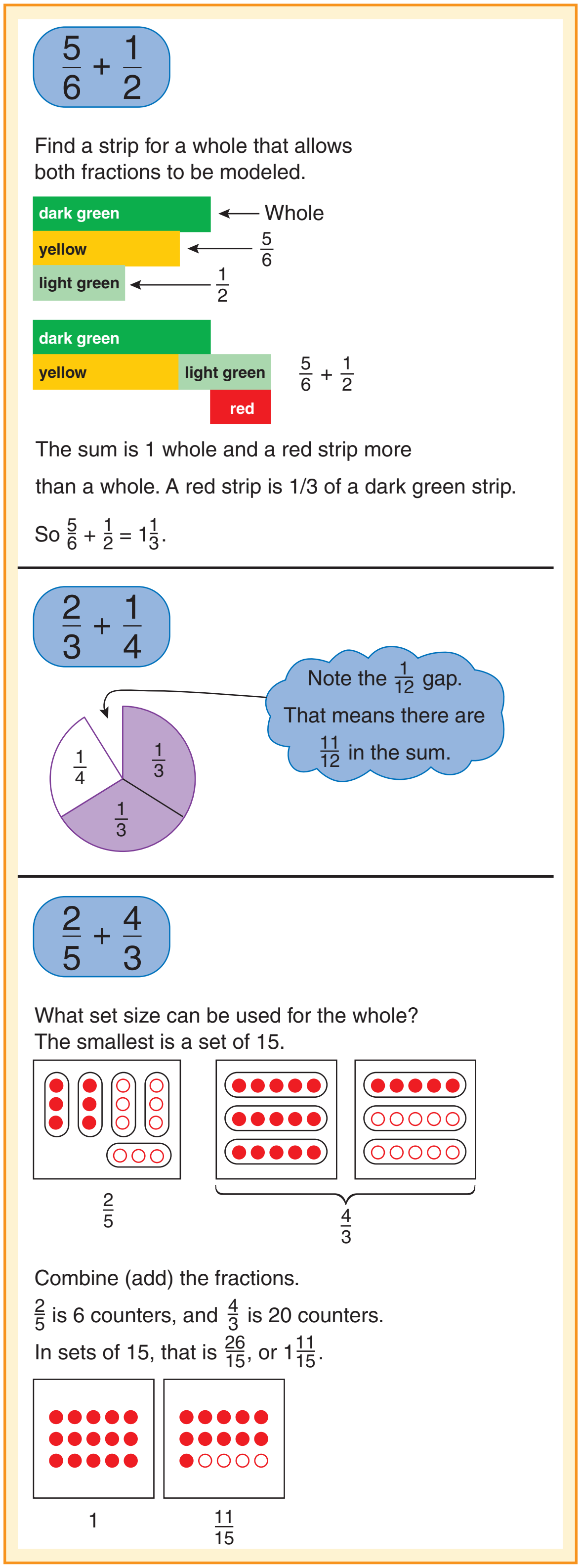
Figure 8.11 illustrates addition across all three models. Area and length models are very effective in illustrating the meaning of fraction addition and connecting the concepts to the procedures. Set models can also be used, but they can be more confusing for students.

Teaching Tip

The key to which manipulative to use lies in the context—if a context is round, fraction circles are a good tool. If the context is distance, then a paper strip or number line is a better tool.

Figure 8.11

Using models to add fractions.



Find a strip for a whole that allows both fractions to be modeled.

The sum is 1 whole and a red strip more than a whole. A red strip is 1/3 of a dark green strip.

So $\frac{5}{6}+\frac{1}{2}=1\frac{1}{3}$.

What set size can be used for the whole? The smallest is a set of 15.

Combine (add) the fractions.

$\frac{2}{5}$ is 6 counters, and $\frac{4}{3}$ is 20 counters.

In sets of 15, that is $\frac{26}{15}$, or $1\frac{11}{15}$.

Area Models

Cramer et al. (2008) suggest that circles are the best model for adding and subtracting fractions because circles allow students to develop mental images of the sizes of different pieces (fractions) of the circle. Consider this problem, in which the context is circular:

Jack and Jill ordered two medium pizzas, one cheese and one pepperoni. Jack ate $\frac{5}{6}$ of a pizza and Jill ate $\frac{1}{2}$ of a pizza. How much pizza did they eat together?

There are many other area models that can be used. Pattern blocks, for example, have pieces such that the hexagon can be one whole, the blue parallelogram $\frac{1}{3}$, the green triangle $\frac{1}{6}$, and the red trapezoid $\frac{1}{2}$. Like circles, rectangles can be drawn for any fractional value, depending on how they are partitioned. In fact, partitioning a rectangle can be easier and more accurate.

Activity 8.15

CCSS-M: 4.NF.B.3a, d; 5.NF.A.1; 5.NF.A.2

Gardening Together

Give each student a blank piece of paper (or a piece of paper with a large empty rectangle on it) or use the Empty Garden Activity Page. Explain the situation and ask each student to design a garden to illustrate these quantities:

Al $= \frac{1}{4}$ Bill $= \frac{1}{8}$ Carrie $= \frac{3}{16}$

Danielle $= \frac{1}{16}$ Enrique $= \frac{1}{4}$ Fabio $= \frac{1}{8}$

They decide to pair up to share the work. What fraction of the garden will each of the following pairs or groups have if they combine their portions of the garden? Show your work.

- **Bill and Danielle**
- **Al and Carrie**
- **Fabio and Enrique**
- **Carrie, Fabio, and Al**

Linear Models

Suppose that you had asked the students to solve the Jack and Jill pizza problem but changed the context to submarine sandwiches (linear context), suggesting students use Cuisenaire rods or fraction strips to model the problem. The first decision that must be made is what rod or strip to use as the whole. That decision is not required with a circular model, where the whole is already established as the circle. With Cuisenaire Rods, the 6-rod or the dark green strip works, because it can be partitioned into sixths (1 rod/white) and into halves (3 rod/light green).

What if you instead asked students to compare the quantity that Jack and Jill ate? Recall that subtraction can be thought of as "separate" where the total is known and a part is removed, "comparison" as two amounts being compared to find the difference, and "how many more are needed" as starting with a smaller value and asking how much more to get to the higher value (think addition). This sandwich example is a comparison—be sure to include more than "take away" examples in the stories and examples you create.

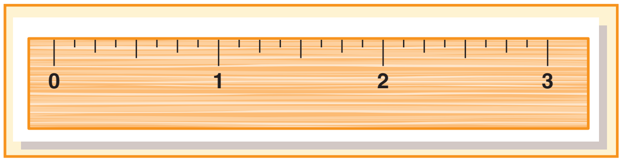
An important model for adding or subtracting fractions is the number line (Siegler et al., 2010). The number line is also a more challenging model than the circle model because it requires that the student understand $\frac{3}{4}$ as 3 parts of 4, and also as a value between 0 and 1 (Izsák, Tillema, & Tunc-Pekkam, 2008). Using the number line in addition to area representations can strengthen students' understanding of fractions (Clarke et al., 2008; Cramer et al., 2008; Petit et al., 2010). One advantage of the number line is that it can be connected to the ruler, which is the most common real context for adding or subtracting fractions.

Activity 8.16

CCSS-M: 4.NF.B.3a, d; 5.NF.A.1; 5.NF.A.2

Jumps on the Ruler

Use **Jumps on the Ruler** Activity Page (or just provide a ruler and problems such as these:



$$\frac{3}{4} + \frac{1}{2} \qquad 2\frac{1}{2} - 1\frac{1}{4} \qquad 1\frac{1}{8} + 1\frac{1}{2}$$

You can also add a linear context, such as length of grass in the yard, hair growing or being cut. Have students use jumps on the ruler to add or subtract. These jumps encourage students to invent strategies without having to find a common denominator (Taber, 2009). When sharing strategies, listen for students iterating (counting) and using fraction equivalencies. In the first problem, students might use 1 as a benchmark. They count up $\frac{1}{4}$ to get to a whole, then have $\frac{1}{4}$ more to add on to $1\frac{1}{4}$. Or, they could use $\frac{1}{2}$ from the $\frac{3}{4}$ to make a whole with the $\frac{1}{2}$, then add on the $\frac{1}{4}$. ELLs may not be as familiar with inches because most countries measure in metric units. In this case, be sure to spend time before the activity discussing how the inch is partitioned, and/or add labels for fourths as a reminder that the inch is different from metric system units.

Estimation and Invented Strategies

Estimation is one of the most effective ways to build understanding and procedural fluency with fractions. A frequently quoted result from the Second National Assessment (Post, 1981) concerns the following item:

Estimate the answer to $\frac{12}{13} + \frac{7}{8} =$ ___

You will not have time to solve the problem with paper and pencil.

Here is how 13-year-olds answered:

Response	Percentage of 13-Year-Olds
1	7
2	24
19	28
21	27
Don't know	14

There are different ways to estimate for fraction computation (Siegler et al., 2010):

- *Benchmarks.* Decide whether the fractions are closest to 0, $\frac{1}{2}$, or 1 (or to 3, $3\frac{1}{2}$, or 4—the closest whole numbers and the half in between them for mixed numbers). After making the determination for each fraction, mentally add or subtract.

Addition example: $\frac{7}{8} + \frac{1}{10}$. Think "$\frac{7}{8}$ is close to 1, and $\frac{1}{10}$ is close to 0, so the sum is about $1 + 0$, or close to 1."

- *Relative size of unit fractions.* Decide how big the fraction is based on its unit (denominator), and apply this information to the computation.

Addition example: $\frac{7}{8} + \frac{1}{10}$. Think "$\frac{7}{8}$ away from a whole (1), and $\frac{1}{8}$ is a close to (but bigger than) $\frac{1}{10}$, so the sum will be close to, but less than, 1."

The following activity can be done regularly as a warm-up or an intervention lesson.

Activity 8.17

CCSS-M: 4.NF.B.3a; 5.NF.A.2

Over or Under 1

Tell students that they are going to estimate sums and differences of fractions. Project problems from the **Estimating Addition and Subtraction of Fractions Problems**. For each, ask students to decide whether the exact answer is more than one or less than one (without solving it). Project a problem briefly, then hide or remove it. Use **Less Than One, More Than One** cards, a mini whiteboard their choice of "over" or "under" one, a "thumbs up" or "thumbs down", or clickers to quickly gather data (see www.iclicker.com). Then return to each problem and discuss how students decided on their estimates. Students, in particular students with disabilities, may benefit from having a number line as a tool to think about the problems. They may also need more time to think about the quantities. During the discussion, ask students to refer to a real-life example, a number line, or an area model (e.g., rectangular region) to justify their decision.

Activity 8.17 can include tasks that are more challenging, or it can be differentiated with different groups of students working on different "over" and "under" values. Consider the following variations:

- Use a target other than one. For example, estimate more or less than $\frac{1}{2}$, $1\frac{1}{2}$, 2, or 3.
- Choose fraction pairs in which the fractions are both less than one or both greater than one.
- Ask students to *create* equations that are slightly less than or slightly more than one (or other values). They can trade equations with other students, who in turn need to decide whether the sum, difference, product, or dividend is over one or under one (or other value).

An engaging estimation activity is shared in Activity 8.18, based on Fung & Latulippe (2010) and Hynes (1996).

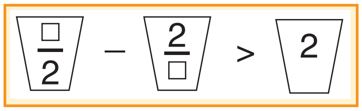
Activity 8.18

CCSS-M: 4.NF.B.3a; 5.NF.A.2

Cups of Milk

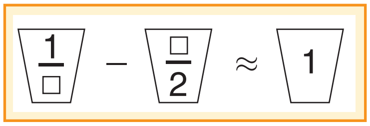
Ask students to fill in the missing numerator and denominator with something that makes the equation or inequality true. They cannot use any of the numbers already in the problem. The context of milk can help students reason about the quantities. Four examples are provided here:

$$\frac{\square}{4} + \frac{3}{\square} \approx \frac{1}{2}$$



$$\frac{\square}{2} - \frac{2}{\square} > 2$$

$$\frac{3}{\square} + \frac{\square}{4} \leq \frac{1}{2}$$



$$\frac{1}{\square} - \frac{\square}{2} \approx 1$$

Keep in mind that these are estimates, so many values are possible. Except in some cases, there are no solutions (can you identify the impossible one in this list?)

Developing the Algorithms

The algorithms develop side by side with the visuals and situations. The way fractions are notated can lead to errors when students compute fractions in isolation. The way to prevent this is to ensure students are connecting fraction symbols with contexts and visuals. When students are working on adding with like denominators, it is important to be sure that they are focusing on the key idea—the units are the same, so they can be combined (Mack, 2004).

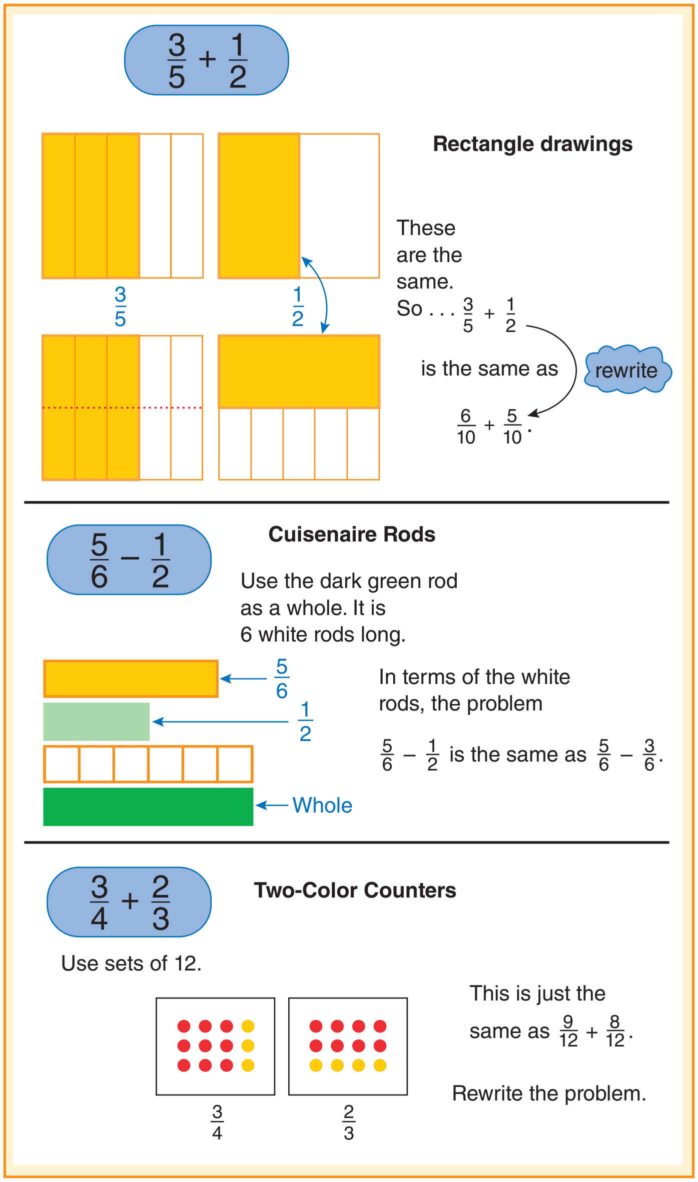
When working with unlike denominators, students may have memorized an algorithm that did not make sense. Reminders such as "you can't add apples and oranges" work against students' understanding why, and the statement is essentially false. A correct statement might be, "Remember that we must add equal-sized parts. The algorithm is designed to find common denominators because that means the parts that are being added or subtracted are the same size."

Students who have a level of fluency in moving between $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$ and $\frac{8}{16}$ or $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{12}{16}$ can adjust the fractions as needed to combine or subtract fractions. For example, ask students to think about how far a person has walked, if the following equation represents their walking: $\frac{3}{8} + \frac{1}{2}$. Given sufficient concrete experiences, a student should be able to readily trade out one-half for four-eighths to solve:

$$\frac{3}{8} + \frac{1}{2} = \frac{3}{8} + \frac{4}{8} = \frac{7}{8}$$

Figure 8.12

Rewriting addition and subtraction problems involving fractions.



If students are making errors, they likely do not understand why they are finding common denominators. To help, use models or drawings like those in Figure 8.12, and ask students to explain why they are finding the same denominator.

As you may recall in the discussion of whole numbers and in algebra, it is important to not always have the result as the unknown. The parts can also be unknown. This helps students relate addition to subtraction. Any of the stories provided earlier can be adapted to have a part unknown. For example:

Sammy gathered some walnuts and Chala gathered $\frac{7}{8}$ pounds. Together they gathered $1\frac{1}{2}$ pounds. How much did Sammy gather?

Activity 8.19 provides a set of tasks that involves reasoning about fractions when the result is known.

Activity 8.19

CCSS-M: 4.NF.B.3a; 5.NF.A.2

Can You Make It True?

Use the Can You Make It True? Activity Page, which includes a list of equations with missing values, such as the ones here:

$$\frac{\square}{6} + \frac{\square}{3} = 1 \text{ and } \frac{4}{\square} - \frac{\square}{2} = \frac{1}{2}$$

To begin, post one example and invite students to share possible solutions. If they have found one way, ask "Is there more than one way to make that equation true?" Include examples that are impossible, such as:

$$\frac{1}{\square} - \frac{1}{\square} = 1$$

As students work, ask them to explain the reasoning they are using. Encourage students to use visuals (e.g., number line or fraction pieces) to support their thinking.

Fractions Greater Than One

Include mixed numbers in all of your activities with addition and subtraction, and let students solve these problems in ways that make sense to them. Students are likely to add or subtract the whole numbers and then the fractions. Sometimes, this is all that needs to be done, but in other cases, regrouping across the whole number and fraction is needed.

Teaching Tip

To reinforce equivalences, ask students, "What equivalent fractions might you use so that you have equal-sized parts?" rather than "What is the common denominator?"

Dealing with the whole numbers first makes sense. Consider this problem: You had $5\frac{1}{8}$ yards of fabric and used $3\frac{5}{8}$ yards. How much do you have left? What equation describes this situation? $5\frac{1}{8} - 3\frac{5}{8}$. It is a linear situation, so a number line is a good way to think about this problem. Given the context students are likely to subtract 3, leaving $2\frac{1}{8}$, and then need to subtract $\frac{5}{8}$. Students may count down (iterate), stopping at $1\frac{1}{2}$. Another approach is to take $\frac{5}{8}$ from the whole part, 2, leaving $1\frac{3}{8}$, and then add the $\frac{1}{8}$ back on to get $1\frac{4}{8}$ or $1\frac{1}{2}$. The "standard" algorithm, which is not as intuitive, is to trade one of the wholes for $\frac{8}{8}$, add it to the $\frac{1}{8}$ to get $1\frac{9}{8}$ and then take away $\frac{5}{8}$.

An excellent alternative to regrouping is to convert mixed numbers to fractions. Let's revisit $5\frac{1}{8} - 3\frac{5}{8}$. This can be rewritten as $\frac{41}{8} - \frac{29}{8}$. Because 41 – 29 is 12, the solution is $\frac{12}{8}$ or $1\frac{1}{2}$. This is certainly efficient and will always work. Provide both options to students and encourage them to decide which strategy works the best in different situations. To develop mathematically proficient students, they must pick the strategy that is most efficient given the situation or the numbers in the problem.

technology note

The National Library of Virtual Manipulatives (NLVM) has three different activities that can help reinforce fraction addition and subtraction across different models:

Fractions—Adding: Two fractions and an area model for each are given. The user must find a common denominator to rename and add the fractions.

Fraction Bars: This applet places bars over a number line on which the step size can be adjusted, providing a flexible model that can be used to illustrate addition and subtraction.

Diffy: This is a puzzle where you have to find the differences between the numbers on the corners of a square, working to a desired difference in the center. This activity encourages students to consider equivalent forms of fractions to solve the puzzle.

Addressing Misconceptions

By middle school, students have had experiences with adding fractions. It is important to assess students' understanding and keep an eye on common misconceptions that will need to be addressed. Here are the most common misconceptions related to the addition and subtraction of fractions.

1. *Adding both numerators and denominators*

   The most common error in adding fractions is to add both the numerators and the denominators. Consider the following task:

Ms. Rodriguez baked a pan of brownies for the bake sale and cut the brownies into 8 equal-sized parts. In the morning, three of the brownies were sold; in the afternoon, two more were sold. What fractional part of the brownies had been sold? What fractional part was still for sale?

Students solving this task are able to effectively draw a rectangle partitioned into eighths and are able to shade $\frac{3}{8}$ and $\frac{2}{8}$, as shown here. However, about half of students will write $\frac{3}{8} + \frac{2}{8} = \frac{5}{16}$, even after drawing the model correctly. And they won't seem to be bothered that the two answers ($\frac{5}{8}$ and $\frac{5}{16}$) are different (Bamberger, Oberdorf, & Schultz-Ferrell, 2010). In such a case, ask students to decide whether both answers can be right. Ask them to defend which is right and why the other answer is not right. You cannot just tell students which is right—the key is for students to connect the situation to the fraction symbols.

2. *Failing to find common denominators*

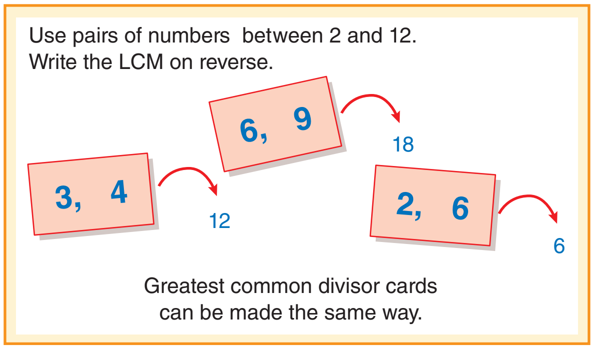
   Less common, but still prevalent, is the tendency to just ignore the denominator and add the numerators (Siegler et al., 2010). For example, $\frac{4}{5} + \frac{4}{10} = \frac{8}{10}$. This is an indication that students do not understand that the different denominators indicate different-size pieces. Using a number line or fraction strip, where students must pay attention to the relative sizes of the fractions, can help develop a stronger understanding of the role of the denominator in adding.

3. *Difficulty finding common multiples*

   Many students have trouble finding common denominators because they are not able to come up with common multiples of the denominators quickly. This skill requires having a good command of multiplication facts. Activity 8.20 is aimed at the skill of finding least common multiples or common denominators. Students benefit from knowing that *any* common denominator will work. As students' skills improve, they will see that finding the smallest multiple is more efficient.

Figure 8.13

Least common multiple (LCM) flash cards.


Use pairs of numbers between 2 and 12.
Write the LCM on reverse.

Greatest common divisor cards can be made the same way.

Activity 8.20

CCSS-M: 4.OA.B.4; 5.NF.A.1

Common Multiple Flash Cards

Use **Common Multiples Cards** make your own flash cards on notecards. Use pairs of numbers that are potential denominators. Most should be less than 12 (Figure 8.13). Place students with partners and give them a deck of cards. At a student's turn, he or she turns over a card and states a common multiple (e.g., for 6 and 8, a student might suggest 48). The partner gets a chance to suggest a smaller common multiple (e.g., 24). The student suggesting the least common multiple gets to keep the card. Be sure to include pairs that are prime, such as 9 and 5; pairs in which one is a multiple of the other, such as 2 and 8; and pairs that have a common divisor, such as 8 and 12. Start students with disabilities with the card on which one member of the pair is a multiple of the other.

A context can help in finding common multiples. The activity **Interference** engages students in determining how often two orbiting satellites will cross paths.

4. *Difficulty with mixed numbers*

   Too often, instruction with mixed numbers is not well integrated into fraction instruction, and therefore students find these values particularly troubling. Here are three misconceptions described in the research (Petit et al., 2010; Siegler, et al., 2010; Spangler, 2011).

1. When given a problem like $3\frac{1}{4} - 1\frac{3}{8}$, students subtract the smaller fraction from the larger ($\frac{3}{8} - \frac{1}{4}$). Although this occurs with whole-number subtraction, it is more prevalent with mixed numbers.
2. When given a problem like $4 - \frac{7}{8}$, students don't know what to do with the fact that one number is not a fraction. They will tend to place an 8 under the 4 ($\frac{4}{8} - \frac{7}{8}$) in order to find a solution.
3. When given a problem like $14\frac{1}{2} - 3\frac{1}{8}$, students focus only on the whole-number aspect of the problem and don't know what to do with the fractional part.

To overcome these misconceptions, use more mixed numbers and whole numbers with fractions less than one, using manipulatives and contexts.

Multiplication

Can you think of a situation that requires using multiplication of fractions? Do you use this algorithm outside teaching? Often, the answer to these questions is no. It is not that there are no situations involving fraction multiplication, but the connection between the concept and the procedure is not well understood, so the algorithm is never used. As you will see in the sections that follow, the foundational ideas of iterating (counting) fractional parts and partitioning are at the heart of understanding multiplication of fractions.

Contextual Examples and Models

When working with whole numbers, we would say that 3 × 5 means "3 sets of 5" (equal sets) or "3 rows of 5" (area or array), or "5 three times" (number line). Different visuals must be used and aligned with contexts so that students get a comprehensive understanding of multiplication of fractions. The multiplication story problems that you pose to students need not be elaborate, but it is important to think about the numbers and contexts that you use in the problems.

Multiplication of fractions is really about scaling. If you scale something by a factor of 2, you multiply it by 2. When you scale by 1 (1 times the size), the amount is unchanged (identity property of multiplication). Similarly, multiplying by $\frac{1}{2}$ means taking half of the original size, while multiplying by $1\frac{1}{2}$ means the original size plus half the original size. This scaling concept can enhance students' ability to decide whether their answers are reasonable. A possible progression of problem difficulty is developed in the sections that follow.

Multiply a Fraction by a Whole Number

These problems look like this: $5 \times \frac{1}{2}$, $6 \times \frac{1}{8}$, $10 \times \frac{3}{4}$, and $3 \times 2\frac{1}{3}$ and in the CCSS-M are introduced in grade 4. Look again at the wording of this subsection—because the word fraction preceded whole number, there is confusion over what this means, but to multiply a fraction *by a whole number*, means examples such as the ones provided here. Students' first experiences should be this type because conceptually it a close fit to multiplication of whole numbers using the idea of equal sets. Consider the following two situations:

Marvin ate 3 pounds of meat every day. How much meat did Marvin eat in one week?

Murphy ate $\frac{1}{3}$ pounds of meat every day. How much meat did Murphy eat in one week?

Stop and Reflect

What expressions represent each situation? What reasoning strategies would you use to solve each?

For Marvin, the expression is 7 (groups of) 3 pounds, or 7 × 3. You can solve this by skip counting $3 + 3 + 3 + 3 + 3 + 3 + 3$ to get the answer of 7 pounds (but if you know your facts, you just *multiplied*). Murphy similarly ate $7 \times \frac{1}{3}$ pounds, and it can similarly be solved by skip counting, this time by his portion size of thirds: $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$, which in total is seven-thirds ($\frac{7}{3}$). Notice the skip counting, called iterating, is the meaning behind a whole number times a fraction.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Activity 8.21

CCSS-M: 4.NF.B.4a, b

Hexagon Wholes

Distribute a set of pattern blocks to students. To start, designate the yellow hexagon as the whole. Ask students the fractional value of the green, blue, and red pieces. Ask students to find how many wholes for different quantities:

5 blue pieces?

10 green pieces?

Ask students to write a fraction equation to represent their problem. For the two examples here:

$$5 \times \frac{1}{3} = 1\frac{2}{3} \text{ or } \frac{5}{3} \qquad 10 \times \frac{1}{6} = \frac{10}{6} \text{ or } 1\frac{4}{6} \text{ or } 1\frac{2}{3}$$

Create a variety of tasks, for example on task cards. Other wholes can be used, for example using two hexagons as one whole to vary the types of fractions that can be used. ELLs may benefit from having the shapes labeled with their names and the fraction symbols written with fraction words (e.g., $\frac{1}{6}$ is "one-sixth")

Circular Fraction Pieces can be used in addition to, or instead of, pattern blocks to do Activity 8.21. Notice that in the answers above, the fraction shows a pattern that can be generalized as $a \times \frac{1}{b} = \frac{a}{b}$, an important pattern for students to discover from having explored many examples that follow this pattern. Linear examples are also important to include. Use **Jumps on the Ruler-Multiplication** Activity Page (See Activity 8.16 for addition and subtraction version) to explore multiplication as jumps of equal-length.

Multiply a Whole Number by a Fraction

Students' second experiences with multiplication should involve finding fractions of whole numbers. While multiplication is commutative, the thinking involved in this type of multiplication involves partitioning (not iterating). The fraction construct is fraction as operator (Lamon, 2012). Examples look like this: $\frac{1}{2} \times 8$, $\frac{1}{2} \times 5$, $\frac{1}{5} \times 8$, $\frac{3}{4} \times 24$, and $2\frac{1}{2} \times 3$. Notice this is a compare or *scaling* situation (think of creating a scale drawing that is $\frac{1}{5}$ of the actual size). In the CCSS-M, this type of fraction multiplication is introduced in grade 5.

These stories can be paired with manipulatives to help students understand this type of fraction operation:

1. The walk from school to the public library takes 25 minutes. When Anna asked her mom how far they had gone, her mom said that they had gone $\frac{1}{2}$ of the way. How many minutes have they walked? (Assume a constant walking rate.)
2. There are 15 cars in Michael's matchbox car collection. Two-thirds of the cars are red. How many red cars does Michael have?

For a full lesson using multiplication stories use **Expanded Lesson: Multiplication-of-Fraction Stories** and **Solving Problems with Fractions** Activity Page. Notice that the thinking in these situations is partitioning (finding a part of the whole). How might students think through each problem? For problem 2, students might partition 15 into three groups (or partition a line into three parts) and then see how many are in two parts. Recording this in symbols ($\frac{2}{3}$ of 15) gives the following result:$15 \div 3 \times 2$.

Counters (a set model) is an effective tool for finding parts of a whole. Recall that in Figure 8.2, 2-color counters were used to develop the concept of partitioning and iterating with prompts such as, "If the whole is 45, how much is $\frac{1}{5}$ of the whole?" and "If the whole is 24, what is $\frac{3}{8}$ of the whole?" These can be slightly adapted to make the connection to multiplication more explicit by having students write the equations that match the question. **Counting Counters: Fraction of a Whole** Activity Page is designed to do this.

An area model, such a rectangle, provides an excellent visual tool for illustrating and making generalizations (Witherspoon, 2014), as can be seen in Activity 8.22.

Activity 8.22

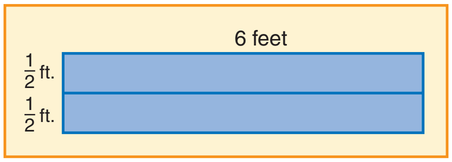
CCSS-M: 5.NF.B.4a, b; 5.NF.B.4a, b

How Big Is the Banner?

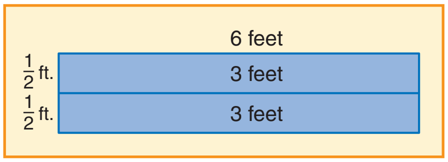
To begin, explain to (or show) students that you have a roll of paper you will be using to make banners. The roll is one-foot wide (you can also use one-yard or one-meter) and you are going to roll out several feet. The first banner you cut is 1 ft. by 6 ft.:

Ask students, "What is the area of this banner?" ($1 \times 6 = 6$ square feet). You can ask additional questions of banners of other lengths (with a width of 1 foot).

Then, explain to students that you want to cut the banners lengthwise to make banners. Ask students to use this rectangle to show banners are $\frac{1}{2}$ foot wide $\times$ 6 feet long. Then ask them to tell you the square feet of the new banner:



Students fill in the strips to show that there are 3 feet in each half-strip:



Repeat the process by asking how big the banners will be if the 6 feet is cut lengthwise into three strips. Repeat with fourths. After exploring this 6 foot banner, use a variety of lengths (e.g., 12 feet, 15 feet) and various widths (e.g., halves, fourths, thirds). Encourage students to notice patterns that help them determine the area of the banners, considering how the banner is scaled based on the values involved in the problem. For students with disabilities, or students who benefit from using physical materials, you can cut out paper strips in advance and have them fold the paper to show the partitions.

Fractions of Fractions—No Subdivisions

Once students have had experiences with fractions of a whole ($\frac{2}{3}$ of 15) or wholes of fractions (15 groups of $\frac{2}{3}$), a next step is to introduce finding a fraction of a fraction. Carefully pick tasks in which no additional partitioning is required. See if you can mentally answer the next three problems:

You have $\frac{3}{4}$ of a pizza left. If you give $\frac{1}{3}$ of the leftover pizza to your brother, how much of a whole pizza will your brother get?

Someone ate $\frac{1}{10}$ of a loaf of bread, leaving $\frac{9}{10}$. If you use $\frac{2}{3}$ of what is left of the loaf to make French toast, how much of the whole loaf will have been used?

Gloria used $2\frac{1}{2}$ tubes of blue paint to paint the sky in her picture. Each tube holds $\frac{4}{5}$ ounce of paint. How many ounces of blue paint did Gloria use?

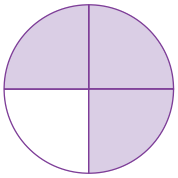
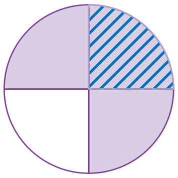
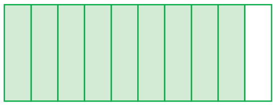
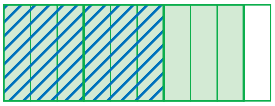
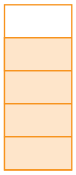
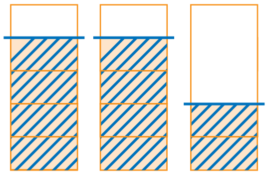
Standards for Mathematical Practice

5 Use appropriate tools.

Figure 8.14 shows how to use different manipulatives to illustrate multiplication. However, there is more than one way to partition. In $\frac{1}{3} \times \frac{3}{4}$, for example, you can find one-third of the three-fourths (as in Figure 8.14), or you can find one-third of *each* fourth and then combine the pieces (Izsák, 2008).

Figure 8.14

Connecting representation to the procedure for three problems involving multiplication.

Task	Finding the starting amount	Showing the fraction of the starting amount	Solution
Pizza Find $\frac{1}{3}$ of $\frac{3}{4}$ (of a pizza) or $\frac{1}{3} \times \frac{3}{4}$			$\frac{1}{3}$ of the $\frac{3}{4}$ is $\frac{1}{4}$ of the original pizza. $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$
Bread Find $\frac{2}{3}$ of $\frac{9}{10}$ (of a loaf of bread) or $\frac{2}{3} \times \frac{9}{10}$			$\frac{2}{3}$ of the $\frac{9}{10}$ is 6 slices of the loaf or $\frac{6}{10}$ of the whole. $\frac{2}{3} \times \frac{9}{10} = \frac{6}{10}$
Paint Find $2\frac{1}{2}$ of $\frac{4}{5}$ (ounces of paint) or $2\frac{1}{2} \times \frac{4}{5}$			$2\frac{1}{2}$ of the $\frac{4}{5}$ is $\frac{4}{5} + \frac{4}{5} + \frac{2}{5} = \frac{10}{5}$

Fractions of Fractions—Subdividing the Unit Parts

When the pieces must be subdivided into smaller unit parts, the problems become more challenging.

Zack had $\frac{2}{3}$ of the field left to cut. After lunch, he cut $\frac{3}{4}$ of the field that was left to cut. How much of the whole field did Zack cut after lunch?

The zookeeper had a huge bottle of the animals' favorite liquid treat, Zoo Cola. The monkey drank $\frac{1}{5}$ of the bottle. The zebra drank $\frac{2}{3}$ of what was left. How much of the bottle of Zoo Cola did the zebra drink?

In Zack's lawn problem, it is necessary to find fourths of two things, the two-thirds of the grass left to cut. In the Zoo Cola problem, you need thirds of four things, the four-fifths of the Zoo Cola that remain. Again, the concepts of the numerator counting and the denominator naming what is counted play an important role. Figure 8.15 shows a possible solution for Zack's lawn problem. Using a paper strip and partitioning is an effective way to solve multiplication problems, especially when they require additional partitioning (Siebert & Gaskin, 2006). A similar approach can be used for the Zoo Cola problem.

The NLVM website has a nice collection of fraction applets. *Number Line Bars—Fractions* allows the user to place bars of any fractional length along a number line. The number line can be adjusted to have increments from $\frac{1}{2}$ to $\frac{1}{15}$, but the user must decide. For example, if bars of $\frac{1}{4}$ and $\frac{1}{3}$ are placed end to end, the result cannot be read from the applet until the increments are in twelfths.

Using Rectangles to Illustrate

Rectangles provide a powerful visual to show that a result of a fraction times a fraction can be quite a bit smaller than either of the fractions, or that if the fractions are both close to 1, then the result is also close to 1. And it is a good model for connecting to the standard algorithm for multiplying fractions.

Provide students with a rectangle, and ask them to partition and shade to illustrate the fraction that is the initial value (See Figure 8.16). For example, in $\frac{3}{5} \times \frac{3}{4}$, you are finding $\frac{3}{5}$ of $\frac{3}{4}$, so you start with $\frac{3}{4}$ [step (a) in Figure 8.16]. To find fifths of the $\frac{3}{4}$, draw five horizontal lines through the $\frac{3}{4}$ [step (b)] or all the way across the square so that the whole is in same-size partitions, and shade three of the five [step (c)]. The overlap of the shading illustrates what is $\frac{3}{5}$ of $\frac{3}{4}$ of the original whole.

Figure 8.15

Solutions to a multiplication problem when the parts must be subdivided.

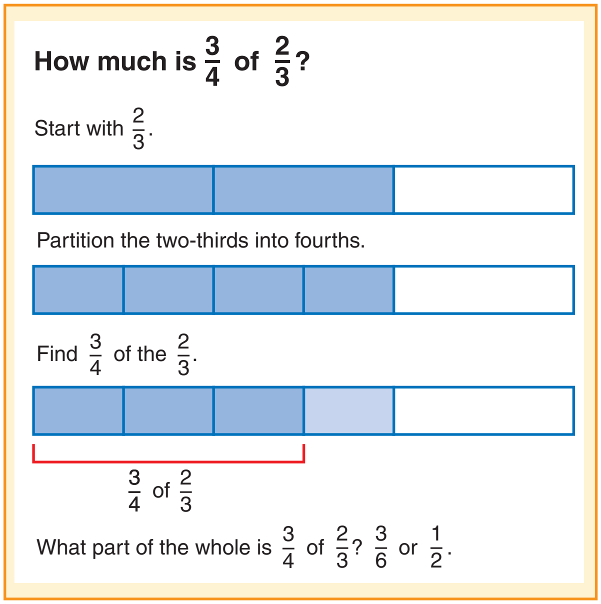
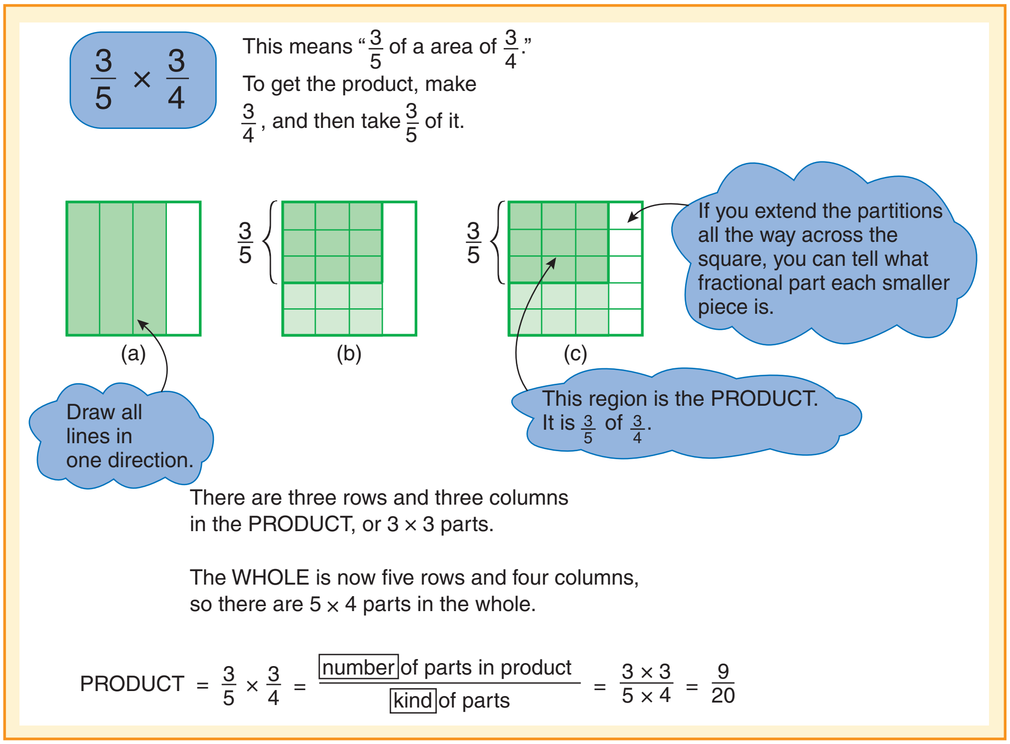


Figure 8.16

Modeling multiplication with rectangles (area model).



There are three rows and three columns in the PRODUCT, or 3 × 3 parts.

The WHOLE is now five rows and four columns, so there are 5 × 4 parts in the whole.

$$\text{PRODUCT} = \frac{3}{5} \times \frac{3}{4} = \frac{\boxed{\text{number}}\text{ of parts in product}}{\boxed{\text{kind}}\text{ of parts}} = \frac{3 \times 3}{5 \times 4} = \frac{9}{20}$$

Using a context and building on whole-number knowledge can support student reasoning about a fraction of a fraction. Quilting is a good context because quilts are rectangles and the individual rectangles (or squares) within the quilt are fractions of the whole quilt. Activity 8.23 is a two-step activity with quilts.

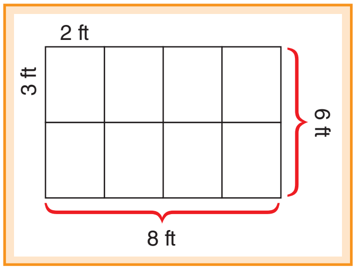
Activity 8.23

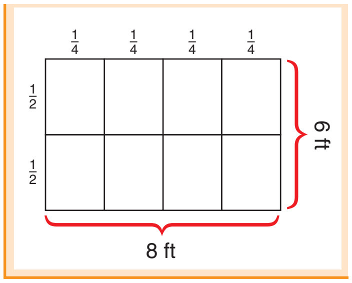
CCSS-M: 5.NF.B4b; 5.NF.B.5b

Quilting Pieces

Have students use grid paper to sketch a drawing of a quilt that will be 8 feet by 6 feet (or create a full-size one for your class!). Explain that each group will prepare a picture that is 3 feet by 2 feet for the quilt. Ask students to tell you what fraction of the quilt each group will provide.

Second, rephrase the task. Explain that each group is to prepare a section of the quilt that is $\frac{1}{4}$ of the length and $\frac{1}{2}$ of the width. Ask students to sketch the quilt and the portion that their group will prepare. Help students make the connection that $\frac{1}{4}$ the length $\times$ $\frac{1}{2}$ the width $= \frac{1}{8}$ the area $\left(\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}\right)$.





Source: Adapted from Tsankova, J. K., & Pjanic, K. (2009/2010). The area model of multiplication of fractions. *Mathematics Teaching in the Middle School, 15*(5), 281–285.

The following activity (based on Imm, Stylianou, & Chae, 2008) engages students in exploring the multiplication of fractions and the commutative property.

Activity 8.24

CCSS-M: 5.NF.B.4b; 5.NF.B.6

Playground Problem

Show students the following problem. Ask students to predict which community will have the bigger playground. Record the predictions. Place students with partners and ask one to solve the problem for community A and the other to solve for community B. Once they have completed their illustration and solution, ask students to compare their responses and to be ready to report to the class what they decided.

Two communities, A and B, are building playgrounds in grassy lots that are 50 yards by 100 yards. Community A has decided to convert $\frac{3}{4}$ of its lot to a playground and to cover $\frac{2}{5}$ of the playground with blacktop. Community B is building its playground on $\frac{2}{5}$ of the lot and covering $\frac{3}{4}$ of the playground with blacktop. In which park is the grassy playground bigger? In which lot is the blacktop bigger? Illustrate and explain.

---

technology note

The NLVM site has an applet called "Fractions Rectangle Multiplication" to explore multiplication of any two fractions up to 2 × 2.

Estimation and Invented Strategies

In the real world, there are many instances when whole numbers and fractions must be multiplied and mental estimates or even exact answers are quite useful. For example, sale items are frequently listed as "half off," or we read of a "one-third increase" in the number of registered voters. To get an estimate of 75 percent of $36.69, it is useful to think of 75 percent as $\frac{3}{4}$, finding one-fourth (about $9) and then three-fourths ($27).

When numbers are more complex, encourage students to use *compatible numbers*. To estimate $\frac{3}{5}$ of $36.69, for example, a useful compatible number is $35. One-fifth of 35 is 7, so three-fifths is 3 × 7, or 21. The two estimation strategies discussed in addition and subtraction also apply to multiplication:

- *Benchmarks:* $\frac{7}{8} \times \frac{5}{12} =$ __. Think, "This is about one times one-half, but a little less, so the answer will be less than the benchmark one-half."
- *Relative size of unit fractions:* $\frac{1}{3} \times 3\frac{4}{5} =$ __. Think "I need a third of this value. One third of 3 is 1, and $\frac{1}{3}$ of $\frac{4}{5}$ is going to be just over $\frac{1}{5}$ (since there are four parts), so about $1\frac{1}{5}$."

Developing the Algorithms

Students are introduced to multiplication of fractions algorithms before middle school, but if they lack an understanding of why they work, it will be important to connect visuals with the operations, as illustrated above. Ask students to solve three examples such as the following set, and to illustrate the solution by partitioning a rectangle that represents the whole:

$$\frac{5}{6} \times \frac{1}{2} \qquad \frac{3}{4} \times \frac{1}{5} \qquad \frac{1}{3} \times \frac{9}{10}$$

Ask questions that press students to tell how the computation connects to the illustration: "How did you figure out how what the unit of the fraction was for the answer?" Or, more specifically, "How did you figure out that the denominator would be twelfths?"

Factors Greater Than One

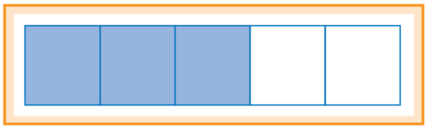
Multiplying fractions in which at least one factor is greater than one for example, $\frac{3}{4} \times 2\frac{1}{2}$ be integrated into multiplication with fractions less than one. When these problem types are mixed, students can see the impact of multiplying by a number less than one and a number more than one. Activity 8.25 is a way to focus on this reasoning (Thompson, 1995).

Activity 8.25

CCSS-M: 5.NF.B.3; 5.NF.B.5a, b

Can You See It?

Post a partially shaded illustration like the one shown here.



Ask students the following questions and have them explain or show how they see it.

Can you see $\frac{3}{5}$ of something? Can you see $\frac{5}{3}$ of $\frac{3}{5}$?
Can you see $\frac{5}{3}$ of something? Can you see $\frac{2}{3}$ of $\frac{3}{5}$?

The area model is a good way to illustrate the product of fractions greater than one (Figure 8.17). But they can be left as mixed numbers, as illustrated in Figure 8.18. Notice that the same four partial products in Figure 8.18 can be found in the rectangle in Figure 8.17. The partial-products process is conceptual and lends itself to estimation. Just as with whole numbers, various strategies for multiplying fractions should be encouraged.

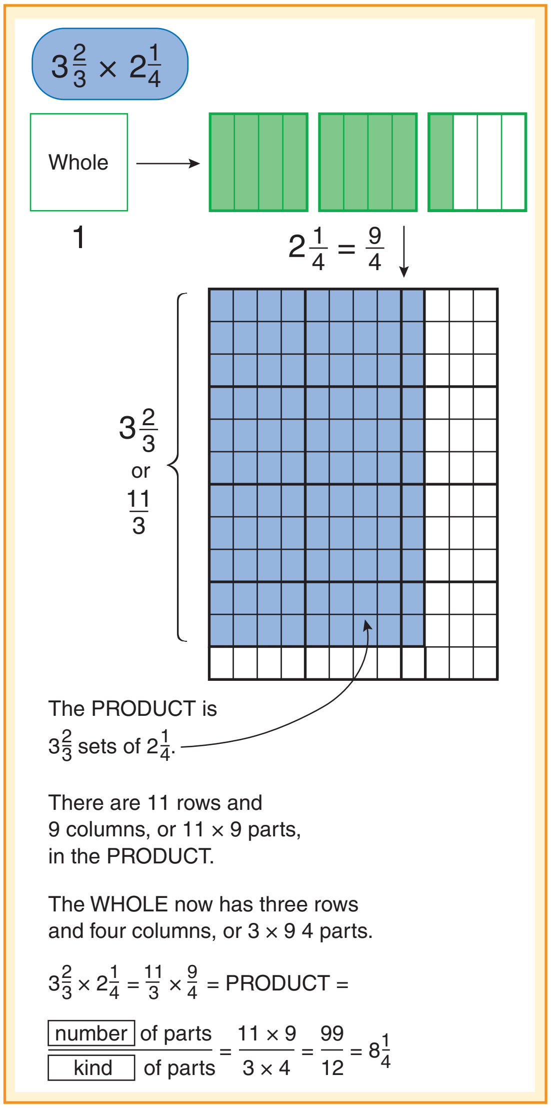
Addressing Misconceptions

Students' concepts of whole-number multiplication and fraction addition and subtraction can lead to confusion with fraction multiplication. This is exaggerated when students have not developed a conceptual understanding of multiplication of fractions and have only memorized an algorithm, such as "multiply both the denominators and the numerators."

The result of memorizing rules that don't make sense is an inability to solve multiplication problems. This becomes a significant barrier for solving proportions and algebraic expressions. Here are some common misconceptions and suggestions for helping students.

Figure 8.17

Multiplication of fractions with factors greater than one.



1. *Treating the denominator the same as in addition and subtraction problems*

   Why is it that the denominators stay the same when fractions are added but are multiplied when fractions are multiplied? (What is a conceptual explanation for this?) Compare the two operations with various visuals, such as a rectangle, circle, and number line to see how these are conceptually different.

2. *Inability to estimate the approximate size of the answer*

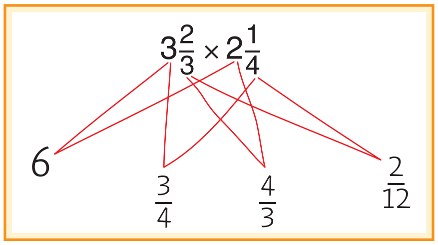
   Students often think multiplication makes bigger. So, they have much difficulty in deciding if their answers make sense. On the one hand, they may never even think about whether their answer makes sense, so any answer looks good to them (e.g., $\frac{1}{2} \times 6\frac{1}{4} = 12\frac{1}{8}$). On the other hand, they may actually notice the answer ($\frac{1}{2} \times 6\frac{1}{4} = 3\frac{1}{8}$) but become concerned that it can't be right because the answer should be bigger. Estimation and the use of contexts and visuals help students think about whether their answers are reasonable.

3. *Matching multiplication situations with multiplication (and not division)*

   Multiplication and division are closely related, and our language is sometimes not as precise as it needs to be. In the question, "What is $\frac{1}{3}$ of \$24?" students may think divide by 3 or multiply by $\frac{1}{3}$. But they may also divide by $\frac{1}{3}$, confusing the idea that they are finding a fraction of the whole. This becomes more evident when the numbers are more complex or the story is more involved. Estimation can help students. This is particularly true for ELLs, who become confused by language such as "divide it in half" and "divide it by half" (Carr et al., 2009). One way to help is to ask, "Should the result be larger or smaller than the original amount?" Also, having students rewrite a phrase to state the problem more clearly can help them determine whether the appropriate operation is multiplication or division.

Figure 8.18

Partial-product approach to multiplying



Division

Division of fractions is one of the least understood topics in the K–8 curriculum. Can you think of a real-life example of dividing by a fraction? Many people cannot, even though we conceptually use division by fractions in many real-life situations. We want to avoid this mystery at all costs and help students really understand when and how to divide fractions.

Partitioning and iterating are absolutely essential prior knowledge to begin exploring division involving fractions. Division should follow a developmental progression that focuses on four types of problems:

a. A whole number divided by a whole number: 14 ÷ 5 OR 22 ÷ 7
b. A fraction divided by a whole number: $\frac{1}{2} \div 4$ OR $\frac{7}{8} \div 2$
c. A whole number divided by a fraction: $4 \div \frac{1}{3}$ OR $6 \div \frac{2}{3}$
d. A fraction divided by a fraction: $\frac{7}{8} \div \frac{1}{8}$ OR $2\frac{1}{4} \div \frac{1}{2}$

These types of problems are each described in the Contextual Examples and Models section. In the CCSS-M, division involving fractions begins in 5th grade but is limited to types (a) – (c) above, and only with unit fractions. In 6th grade, all types are explored and in 7th grade, all types are explored using both positive and negative fractions.

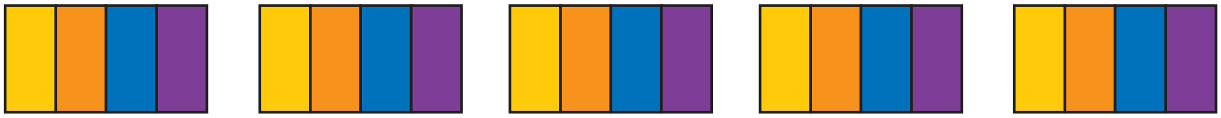
Contextual Examples and Models

As with whole number division, there are two meanings of division: partitive (sharing) and measurement (equal subtraction) (Gregg & Gregg, 2007; Kribs-Zaleta, 2008; Tirosh, 2000). These meanings are both needed to develop the different types of fraction problems, as you will see in these examples.

Whole Number Divided by Whole Number

A partitive or sharing context is helpful in interpreting division of a whole number by a whole number (Lamon, 2012). Very young children can understand sharing (e.g., three cookies shared with two people).

Sharing tasks can result in each person receiving a fractional part: 5 sandwiches shared with 4 friends (5 ÷ 4). If you partition each sandwich into fourths, you see that each friend will have five-fourths (friend 1 takes the yellow section from each sandwich, friend 2 takes the orange section, etc.):



Notice that $5 \div 4 = \frac{1}{4} \times 5 = \frac{5}{4}$. The first expression means five sandwiches shared with 4 friends; the second expression means find one-fourth of 5 sandwiches (that is one person's fair-share); and the final expression means five-fourths is one persons share a fourth from each of 5 sandwiches. Students must be able to see the connections and meanings of each of these equivalent expressions.

Standards for Mathematical Practice

7 Look for and make use of structure.

This concept applies, no matter how messy the numbers! Think of 92 sandwiches shared with 11 people (92 ÷ 11). Still, each person would receive $\frac{1}{11}$ of each of 92 sandwiches, and so have $\frac{92}{11}$ of the sandwiches.

Having seen that division of a whole number by a whole number, is the same as multiplying the number by a unit fraction, students are ready to extend the same reasoning to division of a unit fraction by a whole number.

Fraction Divided by a Whole Number

These problem types are introduced in the CCSS-M in grade 5 as unit fractions ($\frac{1}{4} \div 3$) and in sixth grade for nonunit fractions ($\frac{9}{10} \div 3$ *or* $2\frac{1}{2} \div 6$). Notice that in partitive (sharing) problems, you are asking, "How much is the share for *one* friend?" Questions could also be "How many miles are walked in *one* hour?" or "How much ribbon for *one* bow?"

There are many situations that can fit this equation. Activity 8.26 provides three different situations for exploring this situation.

Activity 8.26

CCSS-M: 5.NF.B.7a, c

Fractions Divided by Whole Number Stories

Provide students with different situations to explore the same problem. Here are three stories (one area, one linear, and one set):

- **Garden Plots**. Three gardeners are equally sharing $\frac{1}{4}$ of an acre for their plots. What part of an acre is each gardener's plot? (The Garden Plots Activity Page provides four 'plots' that students can use to illustrate garden sharing problems.)
- **Water Bottles**. There is $\frac{1}{4}$ of a gallon of water that is poured equally into three water bottles. How much is poured into each?
- **Cheese Sticks**. Arlo buys a bag of cheese sticks (24 in the bag). He takes $\frac{1}{4}$ of the cheese sticks to a picnic. At the picnic he decides to share the cheese sticks equally with 2 friends (and himself). What fraction of the bag does each person get?

After they finish, compare their different visuals and connect the meaning of the operation to the visuals. Ask students to write an expression for each (they should be the same!) Emphasize the notion of "How much for *one*?" After exploring this initial task, students can be challenged to create their own stories to match problems like $\frac{1}{4} \div 3$.

Once students have explored a unit fraction divided by a whole number, they need experiences dividing any fraction (or mixed number) by a whole number, using contexts. Ribbon provides a good linear model:

Cassie has $5\frac{1}{3}$ yards of ribbon to make four bows for birthday packages. How much ribbon should she use for each bow if she wants to use the same length of ribbon for each?

When the $5\frac{1}{3}$ is thought of as fractional parts, there are 16 thirds to share, or 4 thirds for each bow. Alternatively, one might think of allotting 1 yard per bow, leaving $1\frac{1}{3}$ yards. These 4 thirds are then shared, one per bow, for a total of $1\frac{1}{3}$ yards for each bow. The unit parts (thirds) required no further partitioning in order to do the division. Students need numerous experiences with sharing tasks that come out evenly before exploring tasks that do not.

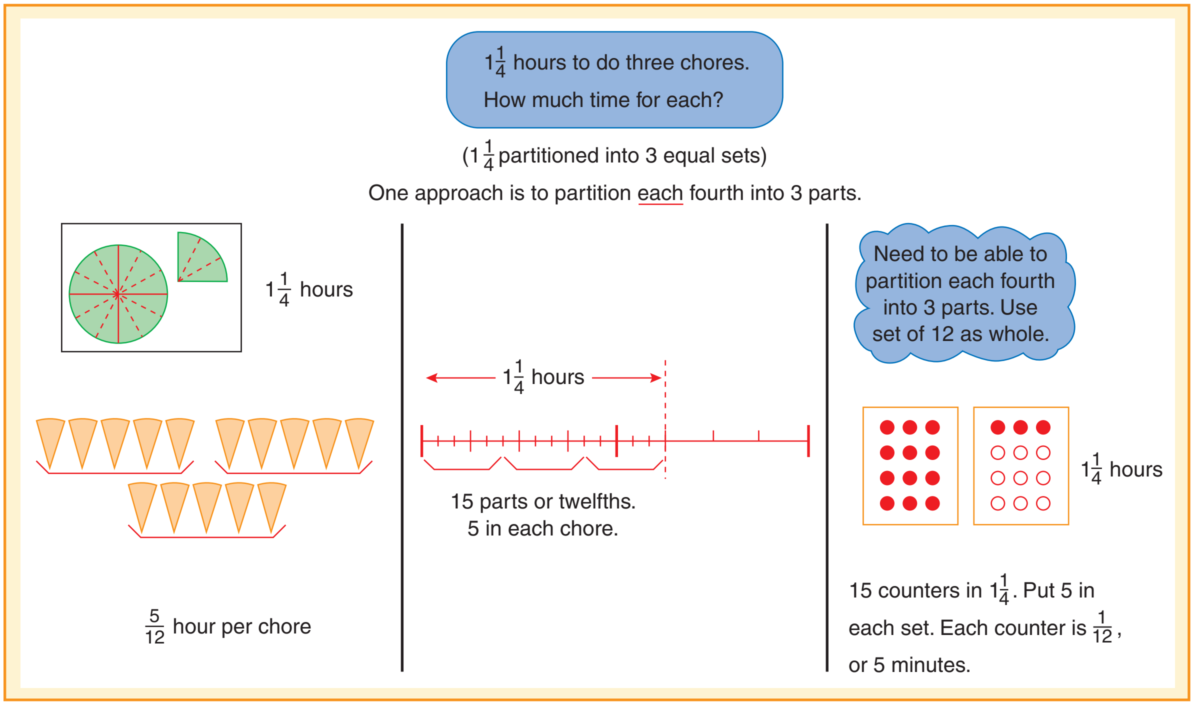
In the following problem, the parts must be split into smaller parts:

Mark has $1\frac{1}{4}$ hours to finish his three chores. If he divides his time evenly, how many hours can he give to each chore?

Note that the question is, "How many hours for one chore?" The 5 fourths of an hour that Mark has do not split neatly into three parts. So some or all of the parts must be partitioned. Figure 8.19 shows how to model these with each type of model (area, linear, and set). In each

Figure 8.19

Three models of partitive division with a whole-number divisor.



case, all of the fourths are subdivided into three equal parts, producing twelfths. There are a total of 15 twelfths, or $\frac{5}{12}$ hour for each chore. (Test this answer against the solution in minutes: $1\frac{1}{4}$ hours is 75 minutes, which divided among 3 chores is 25 minutes per chore. $\frac{25}{60} = \frac{5}{12}$.) See **Expanded Lesson: Division-of-Fraction Stories** for a complete lesson using Cassie and Mark's stories.

Whole Number Divided by Fractions

This problem type lends to a measurement interpretation (also called *repeated subtraction* or *equal groups*). In this case, the question we ask is, "How many 3s are in 13?"

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

The measurement interpretation is a good way to explore division by a fraction because students can draw illustrations to show the measures (Cramer et al., 2010). And measurement interpretation will be used to develop an algorithm for dividing fractions, so it is important for students to explore this idea in contextual situations. A good context for a measurement interpretation is counting servings of a particular size.

Activity 8.27

CCSS-M: 5.NF.B.7b, c

Sandwich Servings

Super Sub Sandwiches is starting a catering business. A child's serving is $\frac{1}{6}$ of a Super Sub and an adult serving can be either $\frac{1}{3}$ or $\frac{1}{2}$ of a Super Sub, depending on whether the catering customer requests small or medium. The employees must be quick at deciding

the number of subs for an event based on serving size. See how you do—make a decision without computing.

1. **Which portion size serves the most people—child size $\frac{1}{6}$, small $\frac{1}{3}$, or medium $\frac{1}{2}$? Why?**

$$6 \div \frac{1}{6} \qquad 6 \div \frac{1}{3} \qquad 6 \div \frac{1}{2}$$

2. **Describe what is happening in each sandwich situation below (how many sandwiches and which kind of serving size). Estimate: Which situation serves the most people? Explain your reasoning.**

$$8 \div \frac{1}{3} \qquad 5 \div \frac{1}{2} \qquad 6 \div \frac{1}{6}$$

Notice that #1 provides the opportunity to compare the impact of unit divisors and discuss *why* you get more servings when you have smaller fractions as divisors. This helps to build the relationship between multiplication and division and students may notice the general case: $1 \div \frac{1}{n} = n$ and therefore, $a \div \frac{1}{n} = a \times n$ (Cavey & Kinzel, 2014). The second prompt provides the opportunity for students to test their conjectures from #1 on new examples. Both prompts can be illustrated using various visuals.

After students have explored unit divisors, they are ready to explore measurement situations with non-unit fraction divisors, such as this task:

Les & Colin's Smoothies Shop has just bought a machine that blends 6 pints of smoothies for one batch. Their Smoothie Cups hold $\frac{3}{4}$ of a pint. How many smoothies can be served from one batch?

A visual that fits this context might be a vertical number line or bar graph, or it could be six rectangles partitioned into fourths. Students may be able to readily count the total number of fourths (24), but not be sure on how to count servings of *three*-fourths. Encourage students to use visuals and to group three of the fourths together as one serving.

A sharing or partitioning interpretation can (and should) be used in working with whole numbers divided by fractions. Remember that the focus question in sharing is "How much for one (e.g., person)?" or "How many for one?" Look back at Figure 8.3 "Given the Part, find the Whole." Essentially, these tasks are saying, "How much for one [whole]?" and are therefore a good way to build meaning for division by a fraction. In Activity 8.28 these tasks are revisited with a more explicit connection to division:

Activity 8.28

CCSS-M: 5.NF.B.7a, c

How Much in One Whole Set?

Use Counting Counters: How Much for One? **Activity Page or give students a collection of counters, such as two-color counters. Ask them to solve sets of tasks, such as the ones provided here:**

If 8 counters represents $\frac{1}{4}$ of the whole, how much is one whole set?

Expression: $8 \div \frac{1}{4}$

(continued)

If 15 counters represents $\frac{3}{5}$ of the whole, how much is one whole set?

Expression: 15 ÷ $\frac{3}{5}$

If 18 counters represents $2\frac{1}{4}$ sets, how much is one whole set?

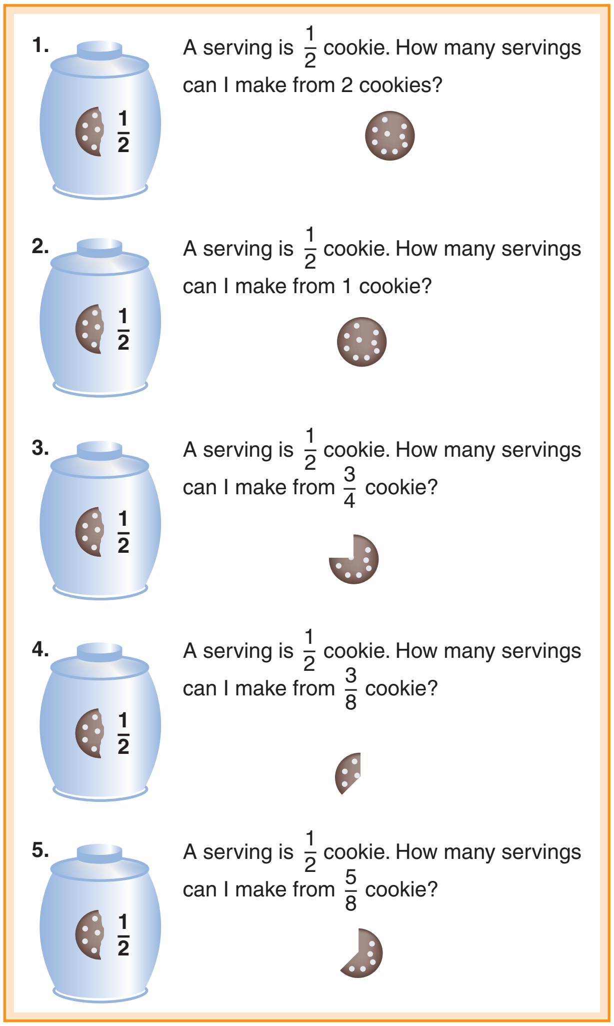
Expression: 18 ÷ $2\frac{1}{4}$

As students are working, ask them to describe their reasoning strategies. Help them to notice that they are finding how many in *one part* (e.g., one-fourth or one-fifth), and then iterating (multiplying) to find out how many in *one whole* (e.g. four-fourths or five-fifths).

Sharing tasks such as this are very closely connected to the meaning of the division by a fraction algorithm, as described in Activity 8.28.

Figure 8.20

Tasks that use the measurement interpretation of "How many servings?" to develop the concept of fractions.



Source: Gregg, J., & Gregg, D. W. (2007). "Measurement and Fair-Sharing Models for Dividing Fractions." *Mathematics Teaching in the Middle School, 12*(9), p. 491. Reprinted with permission. Copyright © 2007 by the National Council of Teachers of Mathematics. All rights reserved.

Fractions Divided by Fractions

Over time, using various contexts and numbers that vary in difficulty, students will be able to take on problems that are more complex both in the context and in the numbers involved. Using the measurement concept of serving size, Gregg and Gregg (2007) use cookie serving size of $\frac{1}{2}$ to bridge from a whole number divided by a fraction to a fraction divided by a fraction. Examples are illustrated in Figure 8.20, but many more examples can be inserted along this progression from whole number divided by a fraction, to fraction divided by a fraction (that has a remainder).

As the examples in the figure illustrate, moving slowly to more complex examples will enable students to use their whole-number concepts to build an understanding of division with fractions.

Although you may think that mixed numbers are more difficult, using mixed numbers early can help students make sense of division of a fraction by a fraction:

Farmer Brown found that he had $2\frac{1}{4}$ gallons of liquid fertilizer concentrate. It takes $\frac{3}{4}$ gallon to make a tank of mixed fertilizer. How many tankfuls can he mix?

Fractions divided by fractions can also be explained using partitioning/sharing, the focus of Activity 8.29.

Activity 8.29

CCSS-M: 5.NF.B.7a, c; 6.NS.A.1

How Much for One?

Pose contextual problems, like the ones here, in which the focus question is, "How much for one ________ ?"

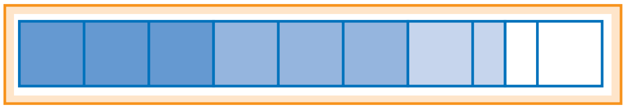
1. **Dan paid \$3.00 for a $\frac{3}{4}$-pound box of cereal. How much is that per pound?**
2. **Andrea found that if she walks quickly during her morning exercise, she can cover $2\frac{1}{2}$ miles in $\frac{3}{4}$ hour. She wonders how fast she is walking in miles per hour. Draw pictures or use models to illustrate your answer.**

Answers That Are Not Whole Numbers

Many problems are not going to result in a whole number answer and it becomes very important to make sense of the left over. If Cassie has 5 yards of ribbon to make bows and each bow needs $1\frac{1}{6}$ yards, she can make only four bows because a part of a bow does not make sense. But if Farmer Brown begins with 4 gallons of concentrate, after making five tanks of mix, he will have used $\frac{15}{4}$, or $3\frac{3}{4}$, gallons of the concentrate. With the $\frac{1}{4}$ gallon remaining, he can make a partial tank of mix. He can make $\frac{1}{3}$ tank of mix because it takes 3 fourths to make a whole, and he has 1 fourth of a gallon (he has one of the three parts he needs for a tank).

Here is another problem to try:

John is building a patio. Each patio section requires $\frac{1}{3}$ of a cubic yard of concrete. The concrete truck holds $2\frac{1}{2}$ cubic yards of concrete. If there is not enough for a full section at the end, John can put in a divider and make a partial section. How many patio sections can John make with the concrete in the truck?



One way to do this is counting how many thirds in $2\frac{1}{2}$? Here you can see that you get 3 patio sections from the dark whole, 3 more from the light whole, and then you get 1 more full section and $\frac{1}{2}$ of what you need for another patio section. The answer is $7\frac{1}{2}$. Students will want to write the "remainder" as $\frac{1}{3}$ because they were measuring in thirds, but the question is how many sections can be made—$7\frac{1}{2}$.

Estimation and Invented Strategies

Understanding division can be greatly supported by using estimation. What does $\frac{1}{6} \div 4$ mean? Will the answer be greater than 1? Greater than $\frac{1}{6}$? The answer to these last two questions is no. And, the answer should be obvious to someone who understands the meaning of this operation (that one-sixth is being shared four ways, resulting in even smaller parts). Conversely, consider what $12 \div \frac{1}{4}$ means. Will the answer be greater than 1? Greater than 12? The answer to these questions is yes — you are actually answering the question, "How many fourths in 12?" (there are 48 fourths in 12 wholes). The following Activity encourages estimation because an exact answer is not needed.

Activity 8.30

CCSS-M: 5.NF.B.7a, b, c; 6.NS.A.1

The Size Is Right: Division

Start with either fractions divided by whole numbers OR whole numbers divided by fractions, but then mix up the tasks. Use the Size Is Right Activity Page. Have each of these problems ready to flash and cover (on a document camera or Smartboard projection. Leave up for a few seconds, then remove. Ask students to pick one of the options from the Dashboard. Then invite students to paircompare their selections and decide if they are reasonable.

(continued)

Fractions divided by whole numbers

Less than $\frac{1}{8}$	Less than $\frac{1}{4}$	Less than $\frac{1}{2}$	Less than 1	More than 1	More than 2

Examples:

$\frac{1}{2} \div 3 \quad \frac{5}{6} \div 2 \quad \frac{7}{9} \div 3 \quad \frac{9}{2} \div 3 \quad \frac{15}{4} \div 3$

Whole numbers divided by fractions

Dashboard:

Less than 1	More than 1	More than 2	More than 4	More than 8

Examples:

$3 \div \frac{1}{3} \quad 1 \div \frac{2}{3} \quad 2 \div \frac{1}{3} \quad 4 \div \frac{7}{8} \quad 4 \div \frac{3}{8}$

For English language learners, provide sentence starters to support the speaking opportunities: "I think the answer is_ [dashboard choice]_ because . . . " For students with disabilities or students who benefit from visuals, have tools available such as Cuisenaire Rods or fraction circles so that they can more readily see the relative size of each fraction.

Activity 8.30 can also be used for fractions divided by fractions, with dashboards such as Quotient < 1, Quotient $= 1$ and Quotient > 1 (Johanning & Mamer, 2014). And, students can write division expressions that fit in each of these categories. The more estimation students do, the more they begin to develop a much needed number sense for fraction division.

Developing the Algorithms

There are two different algorithms for division of fractions. Both algorithms are discussed here.

Common-Denominator Algorithm

Common denominators are a great strategy for dividing, although not widely known or used in the United States. Let's revisit $2\frac{1}{2} \div \frac{1}{3}$. The problem would become $2\frac{3}{6} \div \frac{2}{6}$ or $\frac{15}{6} \div \frac{2}{6}$. The question becomes, "How many sets of 2 sixths are in a set of 15 sixths?" Or, "How many 2s in 15?" $7\frac{1}{2}$. Figure 8.21 shows how to illustrate this idea with an area model using the problem $\frac{5}{3} \div \frac{1}{2}$. Notice that once a common denominator is found, the thought process is the same as in the whole-number problem $10 \div 3$. The resulting algorithm, therefore, is as follows: To divide fractions, first get the common denominators, then divide the numerators. For example, $\frac{5}{3} \div \frac{1}{2} = \frac{20}{12} \div \frac{3}{12} = 20 \div 3 = \frac{20}{6} = 6\frac{2}{3}$.

Invert-and-Multiply Algorithm

Providing a series of tasks and having students look for patterns in how they are finding the answers can help them discover this poorly understood and commonly taught algorithm. For example, consider this first set, in which the divisor is a unit fraction. Pose the question that goes with each equation.

$$3 \div \frac{1}{2} = \text{(How many servings of } \frac{1}{2} \text{ in 3 containers?)}$$

$$5 \div \frac{1}{4} = (\text{How many servings of } \frac{1}{4} \text{ in 5 containers?})$$

$$8 \div \frac{1}{5} = (\text{How many servings of } \frac{1}{5} \text{ in 8 containers?})$$

$$3\frac{3}{4} \div \frac{1}{8} = (\text{How many servings of } \frac{1}{8} \text{ in } 3\frac{3}{4} \text{ containers?})$$

Ask students to look across these problems (and others) for a pattern. They will notice that they are multiplying by the denominator of the second fraction. For example, in the third example, a student might say, “You get five for every whole container, so 5 × 8 is 40.” You can also consider these expressions as partitive situations, which answers the question ”How much in one?” For example, Mary paid $3 for half a pound of coffee. How much did she pay for one pound? The computation is $3 \div \frac{1}{2}$. To find a whole pound, you would double what you have to get to one pound for $6.

Then move to similar problems, but with a second fraction that is not a unit fraction:

$$5 \div \frac{3}{4} =$$

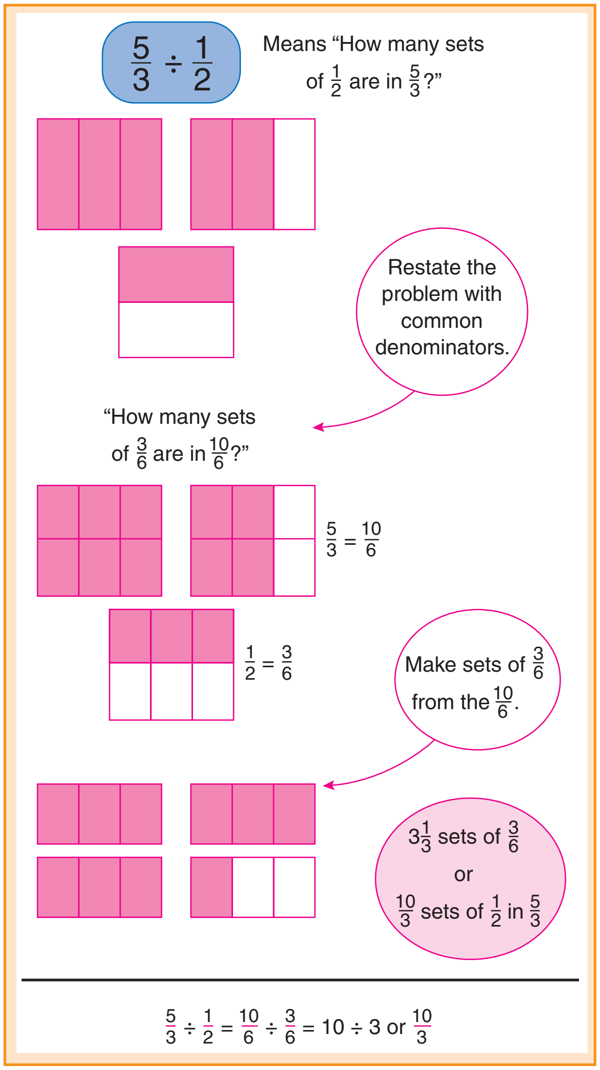
$$8 \div \frac{2}{5} =$$

$$3\frac{3}{4} \div \frac{3}{8} =$$

Have students solve these problems and compare their responses with those for the problems in the first set. For example, notice that if there are 40 one-fifths in 8, then when you group the fifths in pairs (two-fifths), you will have half as many—20. Stated in servings, if the serving is twice as big, you will have half the number of servings. Similarly, if the divisor is $\frac{3}{4}$, after finding how many fourths, you will group in threes, which means you will get $\frac{1}{3}$ the number of servings. You can see that this means you must divide by 3.

Figure 8.21

Common-denominator method for fraction division.



The examples given were measurements because the size of the group (serving) was known, but not the number of groups. Using partitive examples nicely illustrates the standard algorithm. Consider this example:

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning

You have $1\frac{1}{2}$ oranges, which is $\frac{3}{5}$ of an adult serving. How many oranges (and parts of oranges) make up 1 adult serving?

Source: Kribs-Zaleta, C. (2008). Oranges, posters, ribbons, and lemonade: Concrete computational strategies for dividing fractions. *Mathematics Teaching in the Middle School, 13*(8), 453–457.

You may be thinking that you first need to find what one-fifth would be—which would be one-third of the oranges you have—or $\frac{1}{2}$ of an orange (notice you are dividing by the numerator). Then, to get the whole serving, you multiply $\frac{1}{2}$ by 5 (the denominator) to get $2\frac{1}{2}$ oranges in 1 adult serving.

In either the measurement or the partitive interpretation, the denominator leads you to find out how many fourths, fifths, or eighths you have. So, we multiply by the denominator to determine total parts. The numerator tells you the size of the serving (or grouping). So, we divide by the numerator. At some point, someone thought, "Well, if they just flipped the divisor (fraction), then it would be more straightforward hence "invert and multiply."

Standards for Mathematical Practice

6 Attend to precision.

Addressing Misconceptions

The biggest misunderstanding for division of fractions is just not knowing what the algorithm means. Additional misconceptions are listed here.

1. *Thinking the answer should be smaller.* Based on their experiences with whole-number division, students think that when they divide by a fraction, the answer should be smaller. This is true if the divisor is a fraction greater than one (e.g., $\frac{5}{3}$), but it is not true if the fraction is less than one. Include estimating when you teach division of fractions.
2. *Connecting the illustration with the answer.* Students may understand that $1\frac{5}{3} \div \frac{1}{4}$ means, "How many fourths are in $1\frac{1}{2}$?" So, they may set out to count how many fourths—6. But in recording their answer, forget what the 6 refers to and record $\frac{6}{4}$ when actually it is 6 groups of one-fourths, not 6 fourths (Cramer et al., 2010).
3. *Knowing what the unit is.* Students might get an answer such as $\frac{3}{8}$ and when you say $\frac{3}{8}$ of what? They just don't know. To make sense of division, students must know what the unit is. Errors occur less frequently when units (e.g., servings, feet, or quesadillas) are emphasized (Dixon & Tobias, 2013).
4. *Writing remainders.* Knowing what the unit is (the divisor) must be understood in interpreting the remainder (Coughlin, 2010/2011; Lamon, 2012; Sharp & Welder, 2014). In the problem $3\frac{3}{8} \div \frac{1}{4}$, students are likely to count 4 fourths for each whole (12 fourths) and one more for $\frac{2}{8}$, but then not know what to do with the extra eighth. Ask, "How much of the next piece do you have?" Context can also help. In this case, if the problem was about pizza servings, there would be 13 full servings and $\frac{1}{2}$ of the next serving.

Teaching Fractions Effectively

One reason why fractions are not well understood is that there is a lot to know about them from part–whole relationships to division. In addition, building understanding means representing across area, length, and set models and including contexts that fit these models. Estimation activities can support student understanding of fractions and are an important skill in and of themselves. Equivalence, including comparisons, is a central idea for which students must have sound understanding and skill. Connecting visuals with the procedure and not rushing the algorithm too soon are important aspects of the process.

We close this chapter with a summary of research-based recommendations on effective fraction instruction (Clarke et al., 2008; Cramer & Whitney, 2010):

1. Give a greater emphasis to number sense and the meaning of fractions, rather than to rote procedures for manipulating them.
2. Provide a variety of models and contexts to represent fractions.
3. Emphasize that fractions are numbers, making extensive use of number lines in representing fractions and solving problems involving fractions.
4. Spend whatever time is needed for students to understand equivalences (concretely and symbolically), including flexible naming of fractions.
5. Link fractions to key benchmarks, and encourage estimation.

Literature Connections

The Man Who Counted: A Collection of Mathematical Adventures *Tahan, 1993*

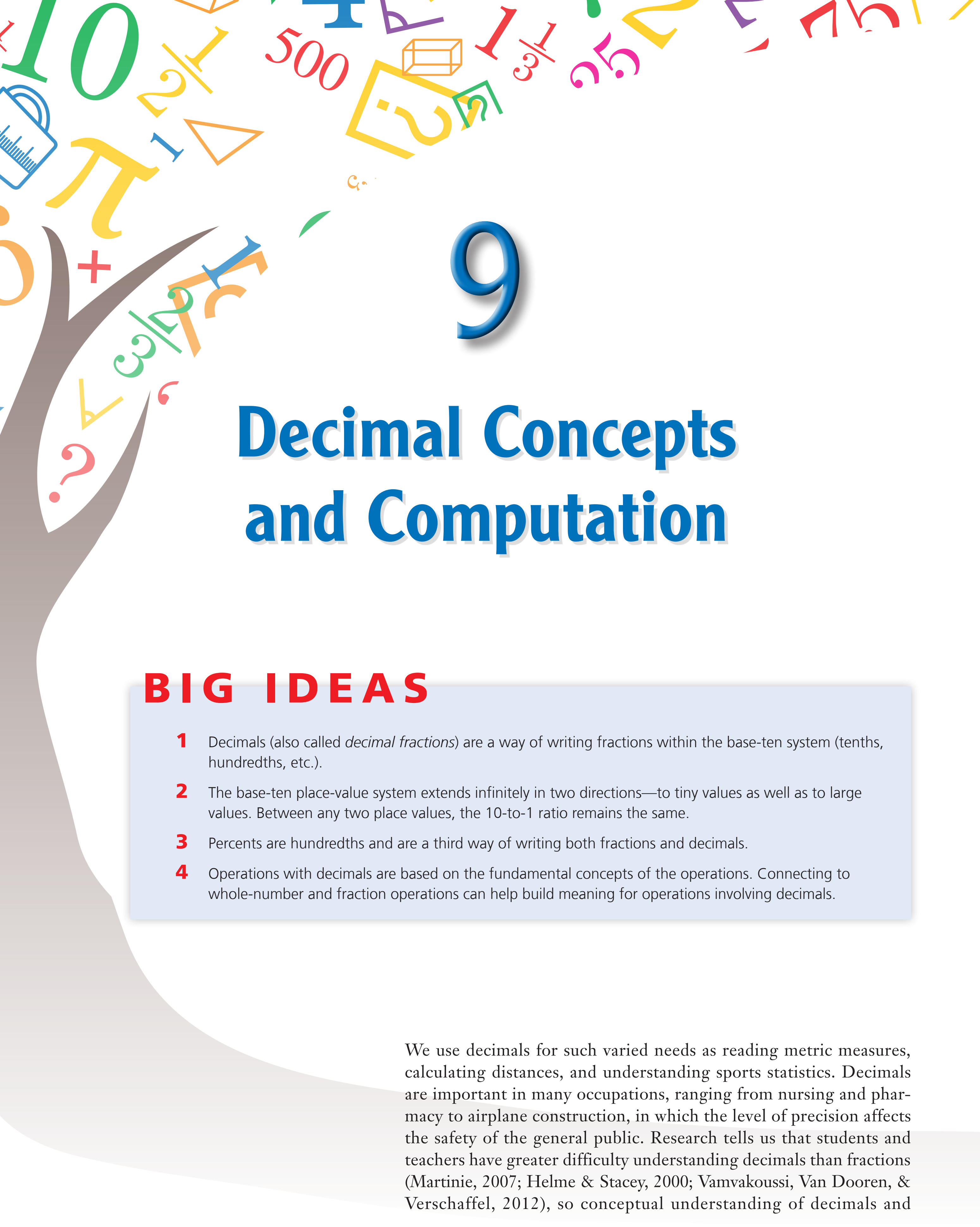
This book contains a story, “Beasts of Burden,” about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked by three brothers to solve an argument. Their father has left them 35 camels to divide among them: half to one brother, one-third to another, and one-ninth to the third brother. The story provides an excellent context for discussing fractional parts of sets and how fractional parts change as the whole changes. However, if the whole is changed from 35 to, say, 36 or 34, the problem of the indicated shares remains unresolved. The sum of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ will never be one whole, no matter how many camels are involved. Bresser (1995) describes three days of activities with his fifth graders.

Alice’s Adventures in Wonderland *Carroll and Gray, 1865/1992*

This well-known children’s story needs no introduction. Because Alice shrinks in the story, there is an opportunity to explore multiplication by fractions. Taber (2007) describes in detail how she used this story to engage students in understanding the meaning of multiplication of fractions. She begins by asking how tall Alice would be if she were originally 54 inches tall but was shrunk to $\frac{1}{9}$ of her height. What height will Alice be if she is later restored to only $\frac{5}{6}$ her original height? The students write their own Alice multiplication-of-fractions equations.

The Man Who Made Parks *Wishinsky and Zhang, 1999*

This nonfiction book explains the remarkable story of Frederick Olmsted, who designed Central Park in New York City. Creating a park design, students can be given fractional amounts for what needs to be included in the park—for example, $\frac{2}{5}$ gardens, $\frac{1}{10}$ playgrounds, $\frac{1}{2}$ natural habitat (streams and forest), and the rest special features (like a zoo or outdoor theater). Students can build the plan for their park on a rectangular grid. To include multiplication of fractions, include guidelines such as that $\frac{3}{4}$ of the park is natural habitat, with $\frac{1}{3}$ of that to be wooded and $\frac{1}{6}$ to be water features, and so on.



9

Decimal Concepts and Computation

BIG IDEAS

1. Decimals (also called *decimal fractions*) are a way of writing fractions within the base-ten system (tenths, hundredths, etc.).
2. The base-ten place-value system extends infinitely in two directions—to tiny values as well as to large values. Between any two place values, the 10-to-1 ratio remains the same.
3. Percents are hundredths and are a third way of writing both fractions and decimals.
4. Operations with decimals are based on the fundamental concepts of the operations. Connecting to whole-number and fraction operations can help build meaning for operations involving decimals.

We use decimals for such varied needs as reading metric measures, calculating distances, and understanding sports statistics. Decimals are important in many occupations, ranging from nursing and pharmacy to airplane construction, in which the level of precision affects the safety of the general public. Research tells us that students and teachers have greater difficulty understanding decimals than fractions (Martinie, 2007; Helme & Stacey, 2000; Vamvakoussi, Van Dooren, & Verschaffel, 2012), so conceptual understanding of decimals and their connections to fractions must be carefully developed. In the *Common Core State Standards*, the developmental progression for understanding decimals is:

> *Grade 4:* Develop an understanding of decimal notation (to hundredths).
>
> *Grade 5:* Advance understanding in this critical area, including operations with decimals (to hundredths).
>
> *Grade 6:* Extend decimal operations beyond tenths and hundredths to all decimals, and develop standard algorithms, as well as explore percent of a quantity as a rate.
>
> *Grade 7:* Develop a "unified understanding of number" so as to be able to move fluently among decimals, fractions, and percents.*

Fractions with denominators of 10, 100, 1000, and so on—for example, $\frac{7}{10}$ or $\frac{63}{100}$, which can also be written as 0.7 and 0.63—are decimal fractions. The phrase decimal fractions is often shortened to decimals, and in this chapter we will use these terms interchangeably.

Extending the Place-Value System

Decimals, like whole numbers, are written in a base-ten format, so connection to whole-number place value can help students understand place value of very small numbers. Here, we share three important ideas that apply to decimal fractions.

The 10-to-1 Relationship: Now in Two Directions!

One of the most basic of these ideas is the 10-to-1 multiplicative relationship between the values of any two adjacent positions. In terms of a base-ten model such as paper strips and squares, 10 of any one piece will make 1 of the next larger (to the immediate left) piece; for example, 10 tens makes 1 hundred, and 10 millions makes 1 ten-million. This is also true for decimal fractions; for example, 10 ten-thousandths makes 1 thousandth, and 10 thousandths makes 1 hundredth. And movement of a piece to the immediate right involves division by 10 (1 divided by 10 is one-tenth). The ratio can be stated as 1 to 10 as the place values move to the right: 1 hundred is 10 tens, 1 million is 10 hundred-thousands, and 1 thousandth is 10 ten-thousandths. These values continue *infinitely in two directions*, as illustrated in Figure 9.1. The 10-to-1 relationship can be explored meaningfully with the calculator.

Activity 9.1

CCSS-M: 4.NF.C.6; 5.NBT.A.3a

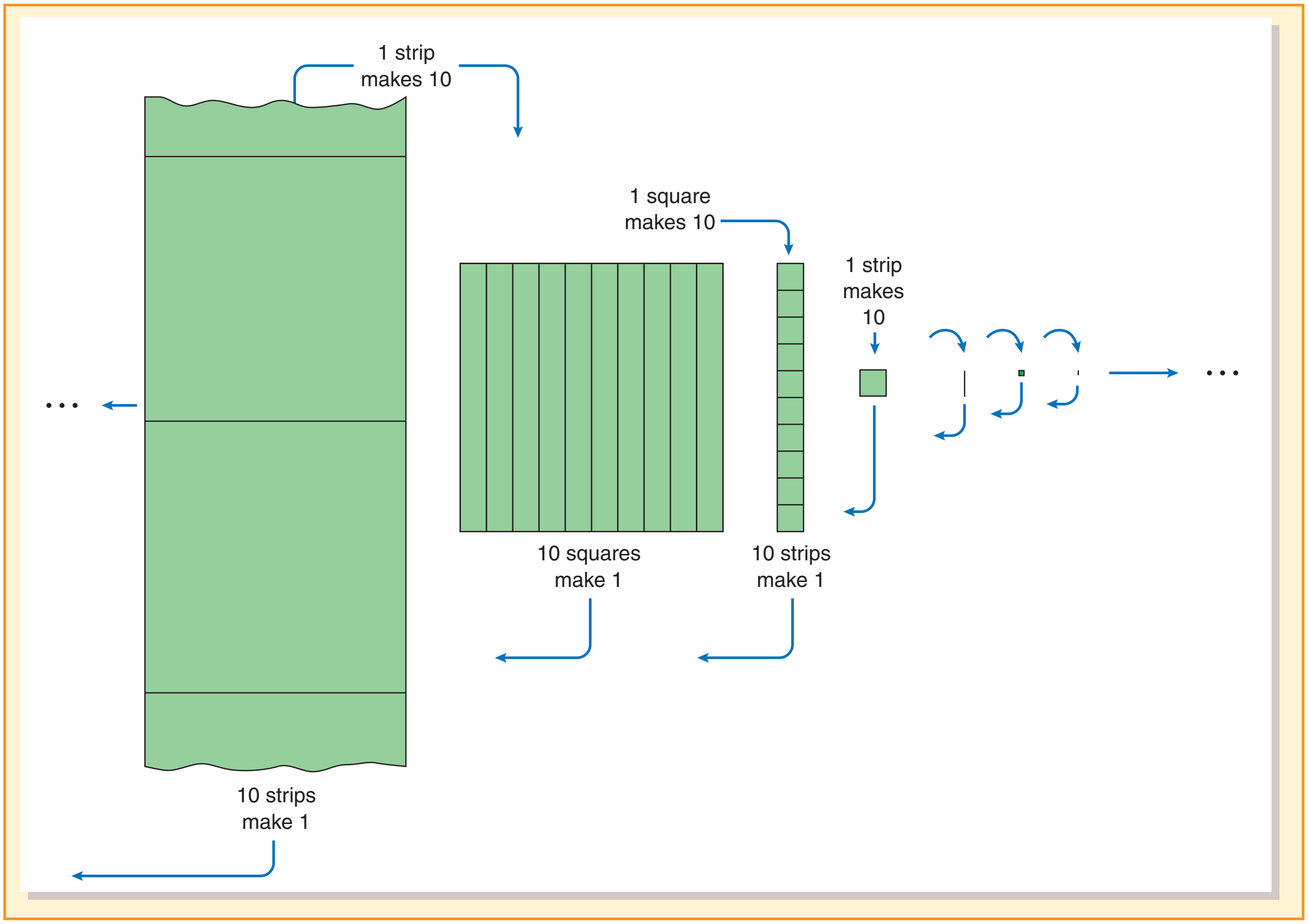
Calculator Decimal Counting

Many calculators are used to "count" by pressing $\boxed{+}$ [value] $\boxed{=}$ $\boxed{=}$ . . . To count by tenths, have students press $\boxed{+}$ 0.1 $\boxed{=}$ $\boxed{=}$ . . . When the display shows 0.9, stop and discuss what will happen with the next press (many students will say 0.10). When the tenth press produces a display of 1 (calculators are not usually set to display trailing zeros to the right of the decimal), you can discuss the 10-to-1 trade that just happened. Continue to count to 4 or 5. English language learners (ELLs) will benefit from counting aloud. Ask, "How many presses to get to the next whole number?" For students with disabilities or who are struggling with the decimal notation, pair the calculator counting with adding in base-ten strips for tenths. Ten presses equates to 10 strips (10 tenths). Repeat for other small values, like 0.001 and 0.0001. Ask, "How many [ten-thousandths] to make one-thousandth? How many to make one-hundredth? One-tenth? One?" Allow students to explore and respond to these questions.

**Common Core State Standards* was developed by the Council of Chief State School Officers. Copies may be downloaded at http://www.ccsso.org/.

Figure 9.1

The 10-to-1 place-value relationship extends infinitely in both directions.



Regrouping

Even middle-school students need to be reminded that regrouping really means that it takes 10 of a group to make one of the place value to the immediate left. Students can benefit from flexible regrouping—for example, regrouping 2451 into 24 hundreds, 5 tens, and 1 unit, or 245 tens and 1 unit or 2,451 units. This work with whole numbers prepares students to think about 0.6 as 6 tenths as well as 60 hundredths.

Measurement and Monetary Units

Decimal values often determine a measurement or an amount, and the unit must be stated in order to understand the number. For example, the metric system has seven place values with names (Figure 9.2). The decimal can be used to designate any of these places as the unit without changing the actual measure.

Our monetary system is also a decimal system. The amount $1,727,000.00 has ones as the unit. But this can also be written as $1.727 million, with the unit shifted to millions. And $5.72 can also be written as 572 cents, with hundredths designated as the unit.

Standards for Mathematical Practice

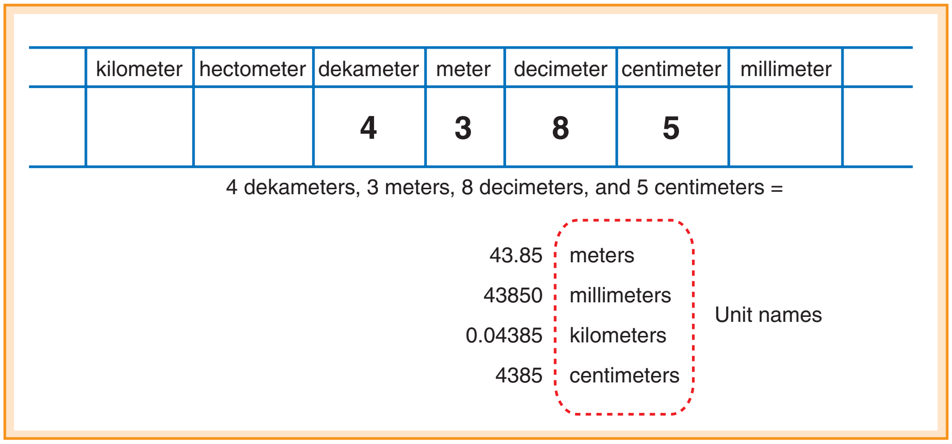
6 Attend to precision.

Precision and Equivalence

Common Core State Standards Mathematical Practice 6 states, “Mathematically proficient students . . . express numerical answers with a degree of precision appropriate for the problem context” (p. 7).

Figure 9.2

In the metric measurement system, each place-value position has a name. The decimal point follows the unit length. Any of the metric positions can be the unit length, as illustrated here.



	kilometer	hectometer	dekameter	meter	decimeter	centimeter	millimeter	
			4	3	8	5		

4 dekameters, 3 meters, 8 decimeters, and 5 centimeters =

43.85 meters
43850 millimeters
0.04385 kilometers
4385 centimeters

Consider the two values 0.06 and 0.060. They are equivalent in terms of numeric value, but the latter is more *precise*. It makes clear that the item was measured to the nearest thousandth and that there were 60 thousandths. In the first case, the measurement was completed only to the nearest hundredth, so the measure might have been 0.058 or 0.063, but not necessarily precisely at 0.060. Various sports measure to different levels of precision. For example, in basketball, a player's free throws are often measured in hundredths (and stated as a percent): *Reagan is an 87 percent free throw shooter.* In baseball, batting averages are measured in thousandths: *Willie has a batting average of .345 for the season.* Precision becomes important when very small items are measured. So, although 0.04500 may be equivalent to 0.045, it may be relevant to record the entire value to communicate the level of precision.

Connecting Fractions and Decimals

Did you know that students have greater difficulty understanding decimals than understanding fractions? This may be a surprise, given the computation for decimals may seem easier than the computation for fractions. Conceptual understanding of decimals and their connections to fractions must be carefully developed (Martinie, 2007). Decimals are sometimes referred to as decimal fractions, in particular in the research and in the *Common Core State Standards.* This is because, conceptually, a decimal is a fraction. It is a special case of a fraction in which the denominator is part of the base-ten number system, so it can be written in this way as a convention. For example, 0.03 is "three-hundredths," which can be written as $\frac{3}{100}$ or as 0.03. When the curriculum treats decimals and fractions separately, students are less likely to see this connection and realize that they are simply different ways to write the same value. The focus of this chapter is on how to ensure that middle-school students have a sound understanding of the different ways to represent rational numbers.

The symbols 3.75 and $3\frac{3}{4}$ represent the same quantity, yet on the surface the two appear quite different. For students, the world of fractions and the world of decimals are very distinct. Developing an understanding that 0.75 is the same as $\frac{3}{4}$ is difficult because the denominator (of 100) is hidden when decimals are written. By the end of middle school, students must see that both systems represent the same concepts and can be interchanged based on the demands of the problem. Here we share some ways to ensure that the connection between fractions and decimals is understood.

Say Decimal Fractions Correctly

Standards for Mathematical Practice

6 Attend to precision.

You must make sure you are reading and saying decimals in ways that support students' understanding. *Always* say "five and two-tenths" instead of "five point two." Using the "point" terminology results in a disconnect to the fractional part that exists in every decimal. This is not unlike the ill-advised reading of fractions as "two over ten" instead of the correct "two-tenths." This level of precision in language will provide your students with the opportunity to *hear* the connections between decimals and fractions, so that when they hear "two-tenths," they will think of both 0.2 and $\frac{2}{10}$.

Teaching Tip

Tell students that you want to hear the "ths" as they talk about decimals. Exaggerate "ths" in your own speaking. This is important for everyone, but especially for ELLs and students with disabilities, who may not notice the difference if it is not emphasized.

Use Select Fraction Models

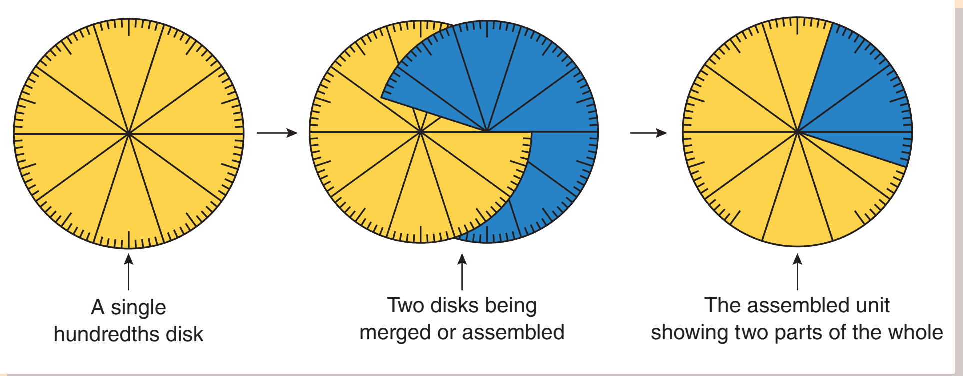
Recall that fractions can be represented in areas, lengths, and sets. When you are trying to make the connection to decimal fractions, the goal is to find a model for which tenths and hundredths (at least) are possible. Area models for this include circles partitioned into hundredths, **Base-Ten Materials**, **10 × 10 Geoboards**, and either **Base-Ten Grid Paper** or **10,000 Grid Paper**. Length models (a great connection to measurement) include Cuisenaire rods (at least for tenths), meter sticks, and the number line. Sets can be created in any number, so sets of 10, 100, and 1000 can be made, but typically these are best counted when in arrays with rows of 10, which then can be connected to an area model.

Area Models

As with fractions, area models are a good beginning. The **Rational Number Wheel** shown in Figure 9.3 shows two disks (each of a different color) marked with hundredths, cut along a radius and slipped together.

Figure 9.3

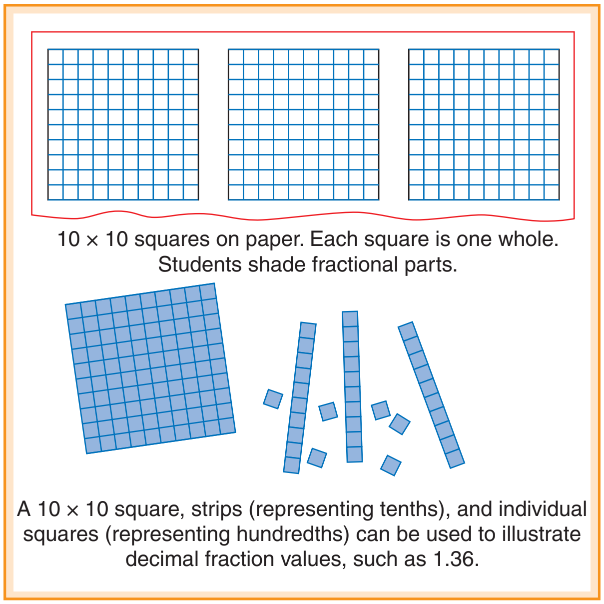
Rational Number Wheel. Turn the wheel to show 0.25, which is also $\frac{25}{100}$ or $\frac{1}{4}$ of the circle.



The most common area model for decimal fractions is a 10 × 10 square (Figure 9.4 and **10 × 10 Grids**. Base-ten blocks are often used for this, with the 10-cm square that was used as the "hundreds" now representing the whole, or 1. Each rod (strip) is then 1 tenth, and each small cube ("tiny") is 1 hundredth. With base-ten blocks, the thousands block can be the whole, and consequently the flats (squares) are then tenths, the rods hundredths, and small cubes thousandths.

Figure 9.4

These 10 × 10 squares show decimal fractions.



10 × 10 squares on paper. Each square is one whole. Students shade fractional parts.

A 10 × 10 square, strips (representing tenths), and individual squares (representing hundredths) can be used to illustrate decimal fraction values, such as 1.36.

10,000 Grid Paper provides a large square that is subdivided into 10,000 tiny parts. Students can identify how many squares are needed for 0.1, 0.01, 0.001, and 0.0001, using appropriate names for the values. Notice that any one of the pieces can be assigned the value of 1, and that affects the values of the other pieces (Figure 9.5).

Because students may be accustomed to a particular piece being used as the unit (e.g., the little square or cube being 1), they can benefit from activities in which the unit changes, such as Activity 9.2.

Teaching Tip

Call the pieces by their place value (e.g., hundredths) rather than by their shape (e.g., rod) to reinforce the precise language and the meaning.

Activity 9.2

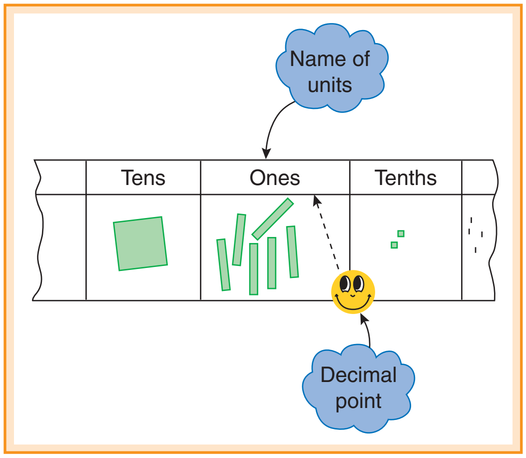
CCSS-M: 4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a

Shifting Units

Give students a collection of paper **Base-Ten Materials** or use base-ten blocks. Ask them to pull out a particular mix; for example, a student might have three squares, seven strips, and four "tinies." Tell students that you have the unit behind your back; when you show it to them, they are to figure out how much they have and to record the value. Hold up one of the units, like the strip. Observe students record their value. Ask students to say their quantity aloud while using appropriate terminology. For ELLs, it is particularly important that you write these labels with the visuals in a prominent place in the classroom (and in student notebooks) so that they can refer to the terminology and illustrations as they participate in the activity. This also supports the learning for students with disabilities. Repeat several times. Be sure to include examples in which a piece is not represented so that students will understand decimal values like 3.07. Continue playing in partners. One student selects a mix of base-ten pieces. The student's partner then tells him or her which one is the unit, and the student writes and says the number.

Figure 9.5

The decimal point indicates that the rods or strips are each one unit, so the 10 ×10 square is ten, and the little squares are tenths.

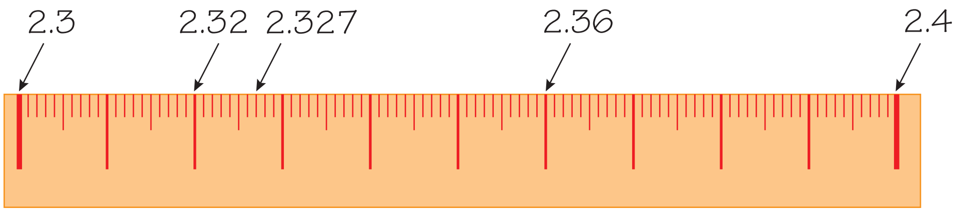


Length Models

As discussed in Chapter 8, length models are very important and often underutilized. In fact, linear models are particularly helpful in making the connection between fractions and decimals because students can see that values such as $\frac{3}{4}$ and 0.75 are the same length from zero. One of the best length models is a meter stick. Each decimeter is one-tenth of the whole stick, each centimeter is one-hundredth, and each millimeter is one-thousandth. Any number line model broken into 100 subparts is likewise a useful model for hundredths (see Figure 9.6). Empty number lines like those used with whole-number computation are also very useful in helping students compare decimals and think about scale and place value. Given two or more decimals, students can use the empty number line to position the values, revealing what they know about the size of these decimals by using zero, one, other whole numbers, or other decimal values as benchmarks. A large number line stretched across a wall or on the floor can be an excellent tool for exploring decimals conceptually.

Figure 9.6

Decimal fraction number line.



Cut poster board into long strips, tape it end to end, and mark decimals. This long number line can be placed on the board so that decimal values can be identified and labeled, as illustrated here.

Set Models

Money is a model for decimals, which has the advantage of being a familiar context. The disadvantage of money as a model is that it is always written to the hundredths place. Numbers like 3.2 and 12.1389 do not relate to money and can cause confusion (Martinie, 2007). The important thing is to use multiple representations, which will broaden students' understanding and their skill at decimal operations.

From Fractions to Decimals

Before learning about decimals, students should have developed a conceptual familiarity with fractions, such as halves, thirds, fourths, fifths, and eighths, across models. This familiarity must be extended to decimals. Students need to make concept-based translations—that is, translations based on understanding rather than on a rule or algorithm.

To understand decimals in terms of place value, a number must be understood in terms of its expanded form, in which the place values are written as individual parts (e.g., 0.45 as 4 tenths and 5 hundredths) and as one number (e.g., 45 hundredths). Base-ten pieces can help students see the expanded form, which is the focus of Activity 9.3.

Activity 9.3

CCSS-M: 4.NF.C.6; 5.NBT.A.1

Build It, Name It

For this activity, have students use their Base-Ten Materials. Assign the large square to be one unit. Ask students to use their pieces to show the fraction that you are about to display. Display a mixed number like $2\frac{35}{100}$ and wait until students have collected the matching pieces. Ask students to tell you how many units, how many tenths, and how many hundredths they used. Ask students to write and say their amount as a decimal while using the terms *tenths*, *hundredths*, and *thousandths* (not *point*!). Repeat the activity by starting with the decimal (e.g., 4.6 or 3.712) and ask students to determine the fraction.

technology note

Calculators that permit entry of fractions have a conversion key. Some calculators will convert a decimal such as 0.25 to the fraction $\frac{25}{100}$ and allow either manual or automatic simplification. Graphing calculators can be set so that the conversion is either with or without automatic simplification. The ability of fraction calculators to go back and forth between fractions and decimals makes them valuable tools as students begin to connect fraction and decimal symbols.

In Activity 9.3, the fraction is already written in base-ten format, but typically, the fractions we are converting to decimals are familiar fractions, such as $\frac{3}{4}$ and $\frac{2}{5}$. The following two activities are designed to focus on decimal equivalences for familiar fractions in a conceptual manner.

Activity 9.4

CCSS-M: 4.NF.C.7

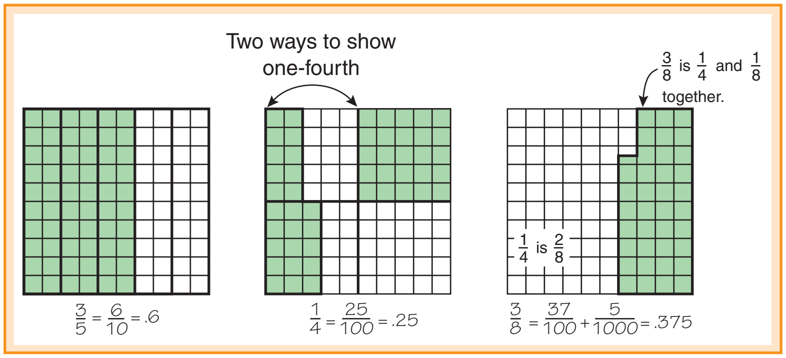
Familiar Fractions to Decimals

Distribute a 10 × 10 Grid to students and ask them to use the grid to determine equivalent names for each of the following fractions. Examples include $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{3}{8}$. Include fractions with the name "hundredths" (e.g., $\frac{45}{100}$). A good sequence is to start with halves and fifths, then fourths, and possibly eighths. Thirds are best done as a separate activity, as the decimal form is a repeating decimal. Adapt or add fractions, as appropriate, based on student needs. For a full lesson plan, see Expanded Lesson: Familiar Fractions to Decimals.

Figure 9.7 shows three examples of how fractions might be illustrated on a 10 × 10 grid. Notice that for fourths, two ways are shown. The question becomes how to translate this to decimals. In either case, students may see the 25 pieces representing 25 hundredths, but only one of the representations shows tenths and hundredths. Both ways are appropriate, but be sure that students see the connection between tenths and hundredths. The fraction $\frac{3}{8}$ represents an interesting challenge. A hint might be to find $\frac{1}{4}$ first, and then notice that $\frac{1}{8}$ is half of a fourth. Remember that the next smaller pieces are tenths of the "tinies". Therefore, half of a "tiny" is $\frac{5}{1000}$.

Figure 9.7

A 10 × 10 square can be used to convert familiar fractions to decimals.



The circle is also effective in making the fraction-to-decimal connections because it can be partitioned into any fractional amount.

Activity 9.5

CCSS-M: 4.NF.C.7

Estimate and Verify

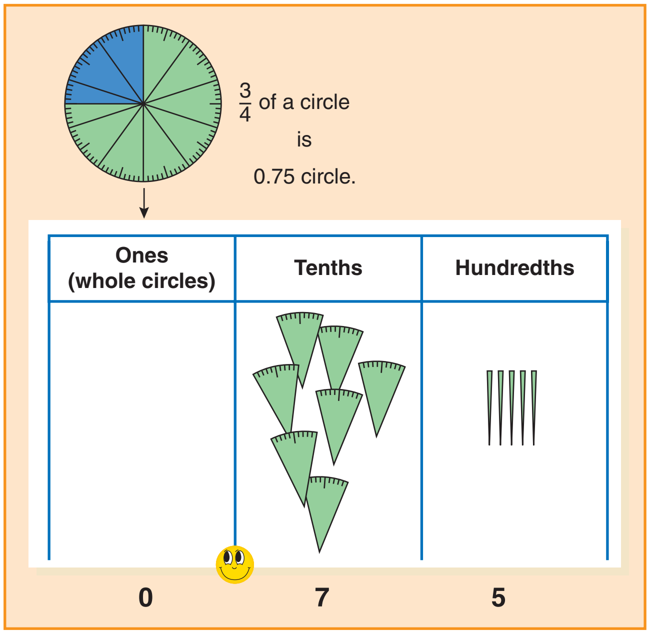
Use the **Rational Number Wheel** (see Figure 9.3). With the blank side of the wheel facing them, have students adjust the wheel to show a particular fraction—for example, $\frac{3}{4}$. Next, ask students to estimate how many hundredths they think are equivalent. Then, ask students to turn the wheel over and check their answer (note that the color reverses when the wheel is turned over). Ask students to justify how they decided what the decimal equivalent was. Repeat with other common fractions. For students with disabilities, you may need to use pieces from the **Rational Number Wheel** (on the hundredths side) that are cut into tenths; then, the students can use those pieces as comparison tools (see Figure 9.8).

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Figure 9.8

The rational number wheel can be cut apart to illustrate tenths and hundredths.



The exploration of modeling $\frac{1}{3}$ as a decimal is a good introduction to the concept of an infinitely repeating decimal, which is a standard for seventh grade (CCSSO, 2010). Try to partition the whole **10 × 10 Grid** into 3 parts by using strips and "tinies". Each part receives 3 strips, with 1 strip left over. To divide the leftover strip, each part gets 3 "tinies", with 1 left over. To divide the "tiny", each part gets 3 "tiny" strips, with 1 left over. (Recall that with base-ten pieces, each smaller piece must be $\frac{1}{10}$ of the preceding-size piece.) It becomes apparent that this process is never-ending. As a result, $\frac{1}{3}$ is the same as 0.333333 . . . , or $0.\overline{3}$. For practical purposes, $\frac{1}{3}$ is about 0.333. Similarly, $\frac{2}{3}$ is a repeating string of sixes, or about 0.667. Later, students will discover that many fractions cannot be represented by a finite decimal.

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

Students must also understand decimals on a number line, a more difficult representation than area or money based on data from the National Assessment of Educational Progress (NAEP). Over the years, NAEP results showed that most students could not find a decimal equivalent for a mixed number (Kouba et al., 1988) and struggled with finding decimals on a number line where the subdivisions on the number line were written as fractions (Kouba, Zawojewski, & Strutchens, 1997).

Teaching Tip

Length models need more emphasis in the teaching of decimals. Use paper strips, Cuisenaire rods, and number lines—all of which can show tenths and hundredths, and the relationship between them.

Even when partitions were written as decimals (increments were multiples of 0.2), only 56 percent of eighth graders correctly placed decimal numbers on a number line (Wearne & Kouba, 2000). This is a strong reminder that just telling students to convert to decimals by using division contributes nothing to their understanding of decimal equivalence. The following activity continues the development of fraction–decimal equivalences with a focus on the number line.

Activity 9.6

CCSS-M: 4.NF.C.6

Decimals and Fractions—Double Line Up

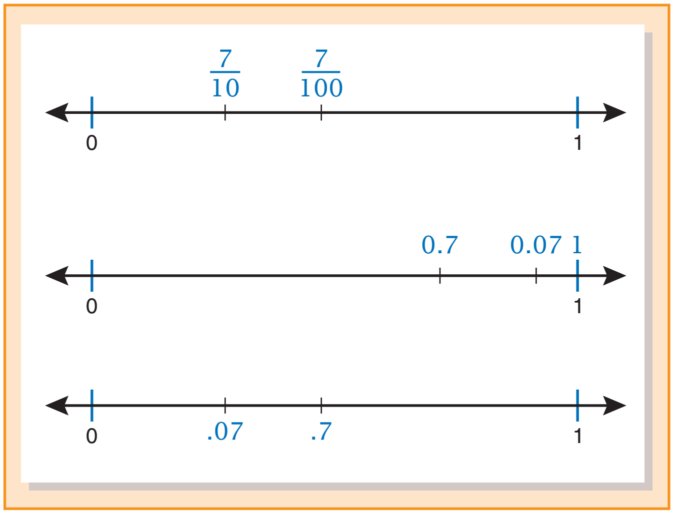
Give students decimal numbers that have familiar fraction equivalents. At first, keep the numbers between two consecutive whole numbers. For example, use 3.5, 3.125, 3.4, 3.75, and 3.66. Show two parallel number lines encompassing the same whole numbers. You can use an empty number line (no partitions), or you can have the line partitioned by fourths or tenths as benchmarks. The students' task is to locate each of the decimal numbers on one number line and to provide the fraction equivalent for each on the other number line, showing they are the same distance from the endpoints (in this case 3 and 4).

Formative Assessment Note

A simple yet powerful *performance assessment* to evaluate decimal understanding has students represent two related decimal numbers, such as 0.7 and 0.07, by using each of three or four different representations: a number line (not provided but student-drawn), a 10 × 10 grid, money, and base-ten materials (Martinie, 2014). Ask students to describe their representations. If students have significantly more difficulty with one model than with others, this may mean that they have learned how to use certain models but have not necessarily developed deep conceptual understanding of decimal fractions. Placement of decimals on an empty number line is perhaps the most challenging task and provides more revealing information (see Figure 9.9). The insights from this assessment can inform which models you need to emphasize.

Figure 9.9

The number lines here show the (inaccurate) attempts of 3 sixth graders..



Solving interesting problems that involve both decimals and fractions can strengthen student understanding of both fractions and decimals. The following are interesting investigations to be explored with a calculator:

- Which fractions have decimal equivalents that terminate? Can you discover a pattern? Is the answer based on the numerator, the denominator, or both?
- For a given fraction, how can you tell the maximum length of the repeating part of the decimal? Try dividing by denominators of 7, 11 and 13, and then make a conjecture.
- Explore all of the ninths—$\frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}$. Remember that $\frac{1}{3}$ is $\frac{3}{9}$ and that $\frac{2}{3}$ is $\frac{6}{9}$. Use only the pattern you discover to predict what $\frac{9}{9}$ should be. But doesn't $\frac{9}{9} = 1$?
- How can you find what fraction produces this repeating decimal: 3.454545 . . . ?

The last question can be generalized for any repeating decimal, illustrating that every repeating decimal is a rational number.

From Decimals to Fractions

In the real world, decimal numbers are rarely those with exact equivalents to familiar fractions. What fraction would you say approximates the decimal 0.52? NAEP found that only 51 percent of eighth graders selected $\frac{1}{2}$. The other choices were $\frac{1}{50}$ (29 percent), $\frac{1}{5}$ (11 percent), $\frac{1}{4}$ (6 percent), and $\frac{1}{3}$ (4 percent) (Kouba et al., 1997). This mediocre performance on such a basic problem is an indication that too many students are learning rules rather than developing an understanding of decimals. Through warm-up problems or full lessons, you can ask students to tell you what they know about 0.65 by using fraction statements (or vice versa). They may write or say such things as these:

Standards for Mathematical Practice

7 Look for and make use of structure. ▶

- It is greater than $\frac{1}{2}$ but less than $\frac{3}{4}$.
- It is about $\frac{2}{3}$.
- It is the fraction $\frac{65}{100}$.
- It is the same as $\frac{6}{10} + \frac{5}{100}$ *or* $0.6 + 0.05$ ("6 tenths and 5 hundredths").

The last example is very important to understanding decimal notation and building the connection back to what they know about whole numbers and fractions.

Middle-school students should have this conceptual foundation, but if they do not, doing approximation activities through warm-ups or learning centers can help them develop the foundation they will need for learning the operations. The first benchmarks that should be developed are 0, $\frac{1}{2}$, and 1. For example, is 7.3962 closer to 7 or to 8? Why? Expand these types of problems to other familiar fractions (thirds, fourths, fifths, and eighths). In this example, 7.3962 is close to 7.4, which is $7\frac{2}{5}$. A good number sense with decimals includes the ability to think of a familiar fraction that is a close equivalent to a decimal. This is what the *Common Core State Standards* mean by developing a "unified understanding of number" (CCSSO, 2010, p. 46). Activity 9.7 focuses on approximating the relative size of decimals.

Activity 9.7

CCSS-M: 4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a

Close to a Familiar Fraction

Use the Close to a Familiar Fraction Activity Page, or just project a list of about five decimals that are close to, but not exactly equal to, a familiar fraction equivalent. Include values less than and greater than 1. For example, use 0.49, 0.191, 6.59, 0.9003, and 7.7. Ask students to find a familiar fraction that is close to each of these decimals. Model for students how they might reason. For example, 6.59 is close to 6.6, which is $6\frac{3}{5}$. If different students select different equivalent fractions, discuss the reasonableness of the different choices. An interactive way to formatively assess is to have students write down their own familiar fraction. Next, ask them to write a nearby decimal on a sticky note, along with their name. Trade sticky notes. Have the recipient of each sticky note identify the familiar fraction and then check it with the originator of the decimal fraction.

Emphasizing Equivalence between Fractions and Decimals

The last two sections have shared conceptual ideas for converting fractions to decimals and then decimals to fractions, but the goal is *not* that students can convert, but that they can recognize equivalent representations and easily switch representations to the form that fits the situation. It is important in middle school to continue to reinforce this flexible approach to rational numbers, accepting responses in either form and asking students questions such as, "Who solved it using decimals? Fractions? What are the advantages or disadvantages of the two ways?" Activities like Activity 9.7 can be used for centers or even homework to continue to reinforce equivalence representations.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Games and activities can help students continue to practice moving between fractions and decimals. Activity 9.8 helps students estimate and move between representations.

Activity 9.8

CCSS-M: 4.NF.C.6

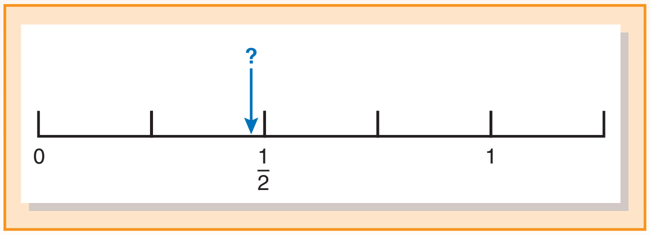
Best Match

Use Fraction-Decimal Cards or create a deck of cards of familiar fractions and decimals that are close to the selected fractions, but not exact. Students try to pair each fraction with the decimal that best matches it. Best Match can be played by laying all cards face up and students take turns finding a close match, or by having all cards face-down and flipping over two to find a match (memory). Alternatively, students can be dealt cards with some left in the center. When a card is turned up in the center, partners pull the Best Match from their hands and the person with the closest value keeps the cards (use a calculator to confirm). The difficulty is determined by how close the various fractions are to one another, so this game can be differentiated by providing different sets of cards to different students. Students with disabilities may benefit from prompts such as, "Is the decimal close to 0, close to $\frac{1}{2}$, or close to 1?" Or have a list of all the possible compatible fractions as a reference.

Formative Assessment Note

You can find out if your students have a unified understanding of fractions and decimals with a *diagnostic interview*. Here are a few examples:

- Write the fraction $\frac{5}{8}$ as a decimal. Use a drawing or a physical model (meter stick or 10 × 10 grid), and explain why your decimal equivalent is correct.
- What fraction is also represented by the decimal 0.004? Use words, a physical model, pictures, and numbers to explain your answer.
- Use both a fraction and a decimal to tell the name of this marked point on the number line. Explain your reasoning.



In the last example, it is especially interesting to see which representation students select first—fraction or decimal. Furthermore, do they then translate this number to the other representation or make a second independent estimate?

Comparing and Ordering Decimal Fractions

Comparing decimal fractions and putting them in order from least to greatest involves understanding the relative size of a number. But comparing decimal fractions has some important distinctions from comparing whole numbers, and because of that, students can make errors.

Stop and Reflect

What errors do you think students might make in ordering the following list from smallest to largest: 0.36, 0.058, 0.375, 0.97, 0, 2.0, and 0.4?

Common Errors and Misconceptions

Table 9.1 provides explanations and examples of six common errors and misconceptions that students exhibit when comparing and ordering decimals (Desmet, Gregoire, & Mussolin, 2010; Muir & Livy, 2012; Steinle & Stacey, 2004a, 2004b).

View videos of **Vanessa** and **Sean** as they compare decimals. Which of the common errors or misconceptions do you notice? How well do you think Sean understands addition of decimals?

Table 9-1. Common errors and misconceptions in comparing decimals and how to help.

Common Error or Misconception	What It Looks Like	How to Help
1. Longer is larger.	0.375 is greater than 0.97. 0.44 is less than 0.440.	• Students are using whole number reasoning and selecting the number with more digits as being larger. Have students use decimal models to show each number and compare. Two 10 × 10 grids with each number shaded will help students make the accurate comparison.
2. Shorter is larger.	0.4 is greater than 0.97 because "a tenth is larger than a hundredth."	• Have students create representations of these two decimals focusing on the quantities. Ask. for example, "Is any amount of tenths larger than any amount of hundredths?"
3. Internal zero.	0.58 is less than 0.078 thinking that "zero has no impact." Also, suggesting that 34.08 and 34.8 have the same value.	• When students are confused by a zero in the tenths position for example, have them build the number using Decimal Number Cards. Match this numerical value to a physical model if needed or match it to a number line.
4. Less than zero.	0.36 is less than 0 because zero is a whole number positioned in the ones column (to the left of the decimal point) and therefore is greater than a decimal fraction (to the right of the decimal point).	• Use contexts, for example, ask students whether they would rather have 0 or 0.50 of a dollar. • Use decimal representations on grid paper to visualize the size of each decimal as compared to zero.
5. Reciprocal thinking.	When students compare 0.4 and 0.6, they select 0.4 as larger because they connect 0.4 to $\frac{1}{4}$ and 0.6 to $\frac{1}{6}$ and erroneously decide 0.4 is greater.	• Use decimal materials such as shading 10 × 10 grids to visualize the size of each decimal.
6. Equality.	Students think that 0.4 is not close to 0.375 and/or that 0.3 is smaller than 0.30.	• Show the connections between these values through area models or placing these values on the number line to help students see if the amounts are close in size or as in the case of 0.3 and 0.30 are the same size.

Relative Size of Decimals

Because misconceptions stem from not understanding the size of values in the tenths, hundredths, and thousandths places and beyond, it is important to have students find values on the number line or other model. The following activity can help promote discussion about the relative sizes of decimal numbers.

Activity 9.9

CCSS-M: 4.NF.C.6; 4.NF.C.7; 5.NBT.A.1; 5.NBT.A.3a,b

Line 'Em Up

Use the Line 'Em Up Activity Page, or project a series of decimals that fall between two whole numbers but vary in the number of decimal places (e.g., 6.3, 6.03, 6.1123, 6.7, 6.07). A context such as the height of plants is a good connection to a vertical number line. As students reason about which plant (value) is taller, they will develop a deeper understanding of which digits contribute the most to the size of a decimal. After exploring, ask students to share how they made their decisions. In closing, ask, "Is 3.0917 closer to 3 or to 4? How do you know?"

Determining how close a particular decimal is to different place values can strengthen students' understanding of place value. After exploring decimals between two whole numbers, ask students to decide where a decimal fraction falls between two tenths. For example, "Is 3.0917 closer to 3.0 or to 3.1?" Repeat with hundredths and thousandths. Students may revert to thinking that tenths are only comparable to tenths and that there are no hundredths in between. When asked which decimal is closer to 0.19–0.2 or 0.21 students select 0.21 (ignoring the decimal point). They also are not sure that 0.513 is near 0.51 but just a little larger. They may also think that 0.3 is near 0.4 but far away from 0.31784. These misconceptions tell us that students need additional experiences focused on the relative size of decimals. These examples can be warm-ups or quick-shares with a partner.

Density of Rational Numbers

An important concept for middle-school students is that there is *always* another number between any two numbers. When students see only decimals rounded to two places, this may reinforce the notion that there are no numbers between 2.37 and 2.38 (Steinle & Stacey, 2004b). Finding the decimal located between any two decimals requires that students understand the density of decimals. Using a linear model helps to show that there is always another decimal to be found between any two decimals—an important concept that is emphasized in the following activities.

Activity 9.10

CCSS-M: 5.NBT.A.3b; 6.NS.C.6

Close Decimals

Have your students name a decimal between 0 and 1.0. Next, have them name another decimal that is even closer to 1.0 than the first. Continue for several more decimals in the same manner, each one closer to 1.0 than the previous decimal. Similarly, try close to 0 or close to 0.5. If students struggle, encourage them to use a number line to help them with their decision making.

Activity 9.11

CCSS-M: 5.NBT.A.3b; 6.NS.C.6

Zoom

Stretch a number line (e.g., clothesline or cash register tape) across the front of the room. Ask students to mark where 0.75 and 1.0 are on the line. Then ask students to "zoom in" to find and record three more values between those two values. Ask students to share their thinking strategies. Make sure ELLs are clear of the meaning of the word *between*, even demonstrating this relationship with students in the front of the room. For students with disabilities, you may need to give them a set of choices of decimals and ask them to select three that are between 0.75 and 1.0. See the **Expanded Lesson: Zoom: Finding Rational Numbers on the Number Line** for a full set of procedures, assessment ideas, and questions to pose.

Addition and Subtraction

There is much more to adding and subtracting decimals than knowing to "line up the decimal points." The *Common Core State Standards* (CCSSO, 2010, p. 33) say that fifth graders should "apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop fluency in these computations, and make reasonable estimates of their results."

Estimating Decimal Sums and Differences

Estimation is important. Often, an estimate is all that is needed. Even if an exact answer is required, estimating a reasonable answer helps students know if their answer makes sense. As with fractions, until students have a sound understanding of place value, equivalence, and relative size of decimals, they are not ready to develop understanding of the operations (Cramer & Whitney, 2010). An emphasis on estimation is very important, especially for students in the seventh and eighth grades who have learned the rules for decimal computation and may not be considering whether their answer is reasonable. Many students who rely on rules for decimals make mistakes without being aware of them because they are not using number sense.

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1. 45.907 + 123.01 + 56.1234
2. 459.8 − 12.345
3. 0.607 + 0.18
4. 89.1 − 0.998

Stop and Reflect

What strategies can be applied to these problems (front end, rounding, compatibles)? What strategies might students use?

Your estimates might be similar to the following:

1. Between 210 and 240
2. A little less than 450
3. Close to 0.8
4. About 88

In these examples, an understanding of decimal numeration and whole-number estimation skills (e.g., front-end, rounding, and compatibles) can produce reasonable estimates. When encouraging students to estimate, do not use rigid rules for rounding; instead, encourage a range of strategies.

Estimation activities can be used as warm-up or end-of-period challenges. Chapter 8 provides several estimation activities that are adaptable to decimals (including mixing decimals and fractions). See, for example, Activities 8.17 and 8.18.

Developing Algorithms

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them

Invented strategies receive significant attention with whole-number computation but less with fraction and decimal computation. Yet, invented strategies are grounded in place value, are efficient, and are often times more conceptual for students than standard algorithms. This is certainly true for adding and subtracting decimal fractions. Even after the standard algorithm is learned and understood, middle-grade students should be encouraged to pick the best method given the situation. This is what mathematically proficient students do. Consider this problem:

Sumiko and Tiana each timed their own quarter-mile run with a stopwatch. Jessica says she ran the quarter mile in 74.5 seconds. Tiana was more accurate in her timing, reporting that she ran the quarter mile in 81.34 seconds. Who ran it the fastest and how much faster was she?

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Students who understand decimal numeration should be able to tell approximately what the difference is—close to 7 seconds. Then, students can figure out the exact difference by using various methods. First, students might note that 74.5 and 7 equals 81.5, then figure out how much extra that is (0.16) and subtract the extra to get a total difference of 6.34. Second, students might use a counting-up strategy (e.g., on a number line), adding 0.5 to 74.5 to get 75, then 6 more seconds to 81, then add the remaining 0.34 to 6.5 for a total difference of 6.84. Third, students might change 74.5 to 74.50 and subtract with regrouping. Similar story problems for addition and subtraction, some involving different numbers of decimal places, will help develop students' understanding.

In addition to using many contexts, students should illustrate addition and subtraction with different tools. The number line is an important tool. Using an empty number line allows students to estimate different-size jumps for tenths, hundredths, and so on. The number line is a good connection to length contexts, like the race story described earlier. Because many uses of decimals are measurements, the number line should receive significant emphasis.

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

The base-ten pieces discussed earlier in this chapter are also an excellent tool for modeling addition and subtraction of decimals, in particular because students have often used them for modeling whole-number computation. This tool is a good match to the standard algorithm for adding and subtracting decimals, which the *Common Core State Standards* place in grade 6. As students solve various problems with base-ten pieces, they will notice the regrouping process and be able to generalize that place values must be added to (or subtracted from) the same place values (hundredths with hundredths and tenths with tenths, for example). This then leads them to the reason why the decimal points are lined up for adding and subtracting decimals.

technology note

The National Library of Virtual Manipulatives (NLVM) site offers a virtual activity, "Base Blocks—Decimals," that allows students to place base-ten blocks on a place-value chart to illustrate and solve addition and subtraction problems with decimals. The problems can be created or generated randomly. And the number of decimal places can be selected, so that any of the four blocks can be designated as the unit.

In middle school, students will have had experience adding and subtracting decimals. One way to assess how well they understand adding and subtracting decimals is to see if they can illustrate and come up with a context for a given problem, as in the next activity.

Activity 9.12

CCSS-M: 5.NBT.B.7

Represent and Review

Give students a copy of the Translation Task Activity Page or simply fold paper into four sections. In the first section, ask students to record a decimal fraction expression for example, 6.2+0.58. Next to it, ask them to write a situation that fits that problem. In the bottom two quadrants, students are to illustrate the operation by using (a) a number line and (b) base-ten pieces. Once they have finished, they can trade with someone else and review to see if their peer's representations accurately represent the problem.

Formative Assessment Note

As students complete Activity 9.12, use a *checklist* to record whether they are showing evidence of having an understanding of decimal concepts by determining a context, illustrating on the number line, and illustrating with base-ten pieces. Look to see whether students can do these representations with problems of varying difficulty.

As students become more proficient in adding and subtracting with the standard algorithm, continue to provide opportunities for them to estimate, illustrate by using one of the two models discussed here, use invented strategies, and explain a context to fit the situation. For example, the game "Circle 3" on the NLVM website is a great reasoning experience that challenges students to use logic as they combine decimals to add to 3 (it is not as easy as it sounds!). These types of continued experiences will ensure that students develop procedural proficiency for decimal addition and subtraction.

Multiplication

Multiplication of decimals tends to be poorly understood. Students (and adults) blindly count over how many decimal places they have to decide where the decimal goes in the answer. No attempt is made to assess if the answer is reasonable. Yet, being mathematically proficient means having a much deeper understanding of multiplication of decimals. Students need to be able to use concrete models or drawings, strategies based on place value (invented strategies), and properties of operations, and they must be able to explain the reasoning used (CCSSO, 2010). Estimation is essential in building that understanding.

Estimating Products

Estimate the problems listed below. Which ones were easy to estimate? Difficult?

1. 5.91 × 6.1
2. 145.33 × 0.109
3. 0.58 × 9.67023

A student's reasoning might be similar to the following:

1. This is about 6 times 6, so the answer is about 36.
2. This is like 145 dimes, so divide by 10 and it is about 14.50. *Or*, this is about a tenth of 145, so 14.5.
3. The first value is about one-half, so half of about 10 is about 5.

When problems involve two very small decimals, estimation is difficult, but it is still possible to look at the answer to see if it is relatively smaller than what the initial factor was (taking a small part of a small part results in an even smaller part).

Teaching Tip

As with any new operation, it is important to begin with the concrete. Situations (story problems) and visuals (manipulatives) provide that concrete foundation.

Developing Algorithms

Begin exploring multiplication of decimals by using problems in a context and by using physical models.

Using decimal values that include small whole-number values can help students estimate and begin to see the impact of multiplying decimals.

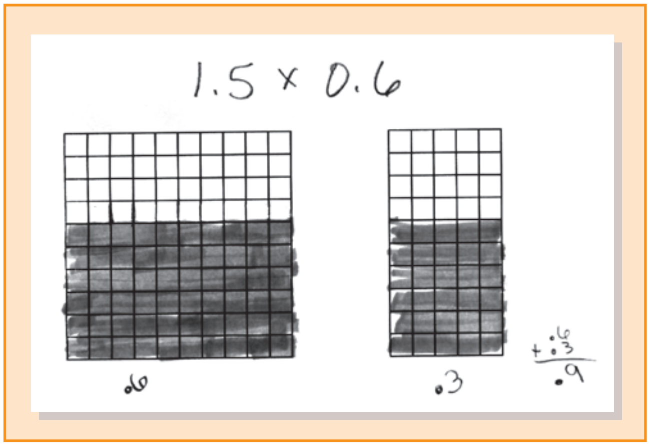
The farmer fills each jug with 3.7 liters of cider. If you buy 4 jugs, how many liters of cider is that?

Ask students, "Is it more than 12 liters? What is the most it could be?" Second, let students use their own methods for determining an exact answer (based on place value and properties). One strategy is repeated addition: 3.7+3.7+3.7+3.7. Another is to multiply 4 × 3, then count up 0.7 four times. Or, students may double 3.7 (getting 7.4) and double it again. Eventually, students will agree on the exact result of 14.8 liters. Connect these strategies to the number line, showing how jumps on the decimal number line match the invented strategies. Explore other problems involving whole-number multipliers. Multipliers, such as 3.5 and 8.25, that involve common fractional parts—here, one-half and one-fourth—are a good next step.

The area model is particularly useful in illustrating decimal multiplication because students know this model from whole-number multiplication (Rathouz, 2011). The rectangle model is especially useful for developing the standard algorithm for decimal multiplication, so creating stories that fit with a rectangle visual is a good idea.

Figure 9.10

A student's use of grids to reason about 1.5×0.6.



A gardener has a raised garden that is 1 meter by 1.5 meters. She decides to plant bluebells on 0.6 of the garden. On how many square meters will she plant bluebells?

In this problem, we are looking for 0.6 of 1.5 m^2, or 0.6×1.5. Begin with an estimate. An estimation might be 0.8 m^2, reasoning that it will be a little more than half of the available part of the garden. Figure 9.10 shows a student's solution on a grid diagram. Each large square represents 1 m^2, with each row of 10 small squares as 0.1 m^2 and each small square as 0.01 m^2. The shaded section shows $0.6\ m^2 + 0.3\ m^2 = 0.9\ m^2$. Notice that this is a proportional model, allowing students to "see" the values of the factors.

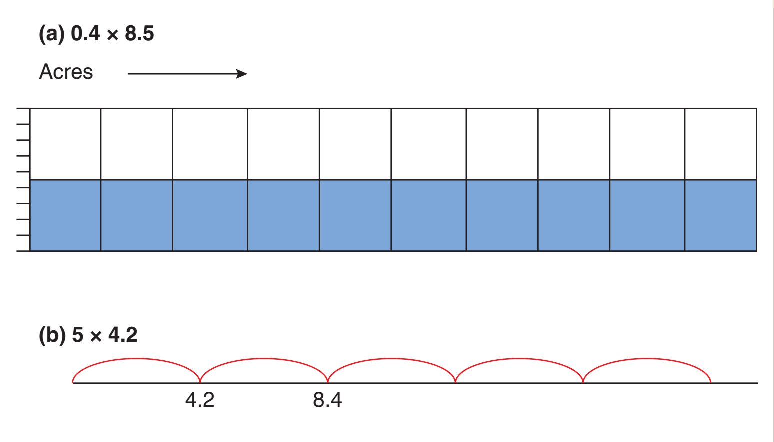
Use problems that can be illustrated with base-ten pieces or on the number line, such as these:

1. Four-tenths of an 8.5-acre farm is used for growing corn. How many acres of corn does the farm have?
2. A frog hops 4.2 inches at every hop. How far away is she from her starting point after 5 hops?

Figure 9.11 provides illustrations of each of these multiplication situations, a grid to illustrate the farm example (9.11a), and a line to illustrate frog leaps (9.11b). These illustrations of decimal multiplication should remind students of the strategies they used to learn whole-number multiplication, and this connection can be used in developing meaning for the standard algorithm for decimal multiplication.

Figure 9.11

Visuals (tools) used to illustrate multiplication of decimals.



Ask students to compare a decimal product with one involving the same digits but no decimal. For example, how are 23.4 × 6.5 and 234 × 65 alike? Interestingly, both products have exactly the same digits: 15210. (The zero may be missing from the decimal product.) Have students use a calculator to explore other products that are alike except for the location of the decimals involved. The digits in the answer are always the same. After seeing how the digits remain the same for these related products, do the following activity.

Activity 9.13

CCSS-M: 5.NBT.B.7; 6.NS.B.3

Where Does the Decimal Go? Multiplication

Have students compute the following product: 24 × 63. Using only the result of this whole number computation (1512) and estimation, have them give the exact answer to each of the following:

0.24 × 6.3   24 × 0.63   2.4 × 63   0.24 × 0.63

For each computation, have students write a rationale for how they placed the decimal point. For example, on the first one a student might explain that 0.24 is close to one-fourth and

(*continued*)

one-fourth of 6 is less than two, so the answer must be 1.512. They can check their results with a calculator. ELLs may apply a different mental strategy that is common in their country of origin. Even if they have trouble articulating their reasoning, it is important to consider alternative ways to reason through the problem. It is also important to have a class discussion about errors and how to avoid them.

Stop and Reflect

What is the value in having students explain how they placed the decimal, rather than just having students count over the number of places?

Another way to support full understanding of the algorithm is to rewrite the decimals in their fraction equivalents. So, if you are multiplying 3.4×1.7, that is the same as $\frac{34}{10} \times \frac{17}{10}$. With multiplication, you would get $\frac{578}{100}$. When this is rewritten as a decimal fraction, it is 5.78, which corresponds to moving the decimal two places to the left (Rathouz, 2011).

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

The method of placing the decimal point in a product by way of estimation is more difficult as the product gets smaller. For example, knowing that 37×83 is 3071 does not make it easy to place the decimal in the product of 0.037×0.83. But the standard algorithm can be developed from this problem, all the while helping students understand properties of numbers.

Here is the process:

$$0.037 \times 0.83 = \left(37 \times \frac{1}{1000}\right) \times \left(83 \times \frac{1}{100}\right)$$

$$\left(37 \times \frac{1}{1000}\right) \times \left(83 \times \frac{1}{100}\right) = 37 \times 83 \times \frac{1}{1000} \times \frac{1}{100}$$

$$37 \times 83 \times \frac{1}{1000} \times \frac{1}{100} = (37 \times 83) \times \left(\frac{1}{1000} \times \frac{1}{100}\right)$$

$$(37 \times 83) \times \left(\frac{1}{1000} \times \frac{1}{100}\right) = 3071 \times \frac{1}{100{,}000} = 0.03071$$

This may look too complicated, but if you just look at what is happening with the decimal fractions, you can see why you count the number of values to the right of each factor and then place the decimal in the product so that it has the same number of decimal places. The standard algorithm for multiplication is the following: Do the computation as if all numbers were whole numbers. When finished, place the decimal by reasoning or estimation, if possible. If not, count the decimal places, as illustrated above. Even if students have already learned the standard algorithm, they need to know the conceptual rationale centered on place value and the powers of ten for "counting" and shifting the decimal places. By focusing on rote applications of rules, students lose out on opportunities to understand the meaning and effects of operations and are more prone to misapply procedures (Martinie & Bay-Williams, 2003).

Questions such as the following keep the focus on number sense and provide useful information about your students' understanding.

1. Consider these two problems: $3\frac{1}{2} \times 2\frac{1}{4}$ and 2.276×3.18. Without doing the calculations, which product do you think is larger? Provide a reason for your answer that can be understood by someone else in this class.
2. How much larger is 0.76×5 than 0.75×5? How can you tell without doing the computation? (Kulm, 1994)

Student discussions and explanations as they work on these or similar questions can provide insights into their decimal and fraction number sense and the connections between the two representations.

Division

Like multiplication of decimals, division of decimals is often poorly understood, and estimation and concrete experiences are needed to build a strong understanding. Returning to the whole-number understanding of division can help build meaning for the division of decimals.

Estimating Quotients

Division can be approached in a manner parallel to that for multiplication. In fact, the best approach to a division estimate generally comes from thinking about multiplication rather than division. Consider the following problem:

The trip to Washington was 280 miles. It took exactly 4.5 hours to drive. What was the average rate in miles per hour?

To make an estimate of this quotient, think about what times 4 or 5 is close to 280. You might think $60 \times 4.5 = 240 + 30 = 270$, so maybe about 61 or 62 miles per hour.

Here is a second example without context.

Estimate $45.7 \div 1.83$. Decide if your estimate is too high or too low.

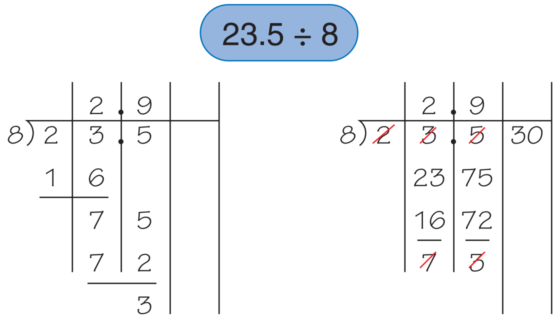
Because 1.83 is close to 2, the estimate is near 23. And since 1.83 is less than 2, the answer must be greater than 23—say 25 or 26. (The actual answer is 24.972677.)

Developing the Algorithm

Estimation can produce a reasonable result, but you may still require an exact answer. Figure 9.12 shows division by a whole number and how that can be carried out to as many places as you wish. (The explicit-trade method is shown on the right.)

Figure 9.12

Extending the whole-number division algorithm to decimal values.



Trade 2 tens for 20 ones, making 23 ones.
Put 2 ones in each group, or 16 in all.
That leaves 7 ones.

Trade 7 ones for 70 tenths, making 75 tenths.
Put 9 tenths in each group, or 72 in all.

Trade the 3 tenths for 30 hundredths.

(Continue trading for smaller pieces as long as you wish.)

An algorithm for division is parallel to that for multiplication: Ignore the decimal points, and do the computation as if all numbers were whole numbers. When finished, place the decimal by estimation. This is reasonable for divisors greater than 1 or close to a familiar value (e.g., 0.1, 0.5, 0.01). If students have a method for dividing by 45, they can divide by 0.45 and 4.5, thinking they will get ten times as many (as 4.5 is one-tenth of 45) and, therefore, multiply the answer by ten.

Activity 9.14

CCSS-M: 5.NBT.B.7; 6.NS.B.3

Where Does the Decimal Go? Division

Provide a quotient such as 146÷37 = 20857 correct to five digits but without the decimal point. The task is to use only this information and estimation to give a fairly precise answer to each of the following:

`146 ÷ 0.7    1.46 ÷ 7    14.6 ÷ 0.7    1460 ÷ 70`

For each computation, students should write a rationale for their answer and then check their results with a calculator. As noted in the section on multiplication, ELLs may apply a different mental strategy, and it is important to value alternative approaches. Again, engage students in explicit discussions of common errors or misconceptions and how to fix them.

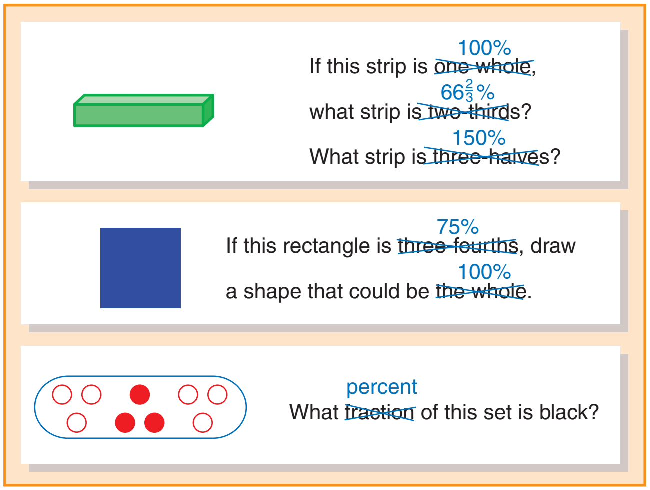
Percents

The term *percent* is simply another name for *hundredths* and as such is a standardized ratio with a denominator of 100. If students can express fractions and decimals as hundredths, the term *percent* can be substituted for the term *hundredth*. Consider the fraction $\frac{3}{4}$. As a fraction expressed in hundredths, it is $\frac{75}{100}$. As a decimal, it is 0.75. Both $\frac{75}{100}$ and 0.75 are read in exactly the same way: "seventy-five hundredths." When used as an operator, $\frac{3}{4}$ of something is the same as 0.75 or 75 percent of that same thing. Thus, percent is merely a new notation and terminology, not a new concept.

Chapter 8 explored fractions in which one part was unknown (see Figures 8.2 through 8.4). Figure 9.13 illustrates how these can be modified to focus on percents. Emphasize equivalence, but seek the equivalence for *hundredths*. Connect hundredths to percent, and replace fraction language with percent language.

Figure 9.13

Part–whole fraction problems can be adapted to percent challenges.

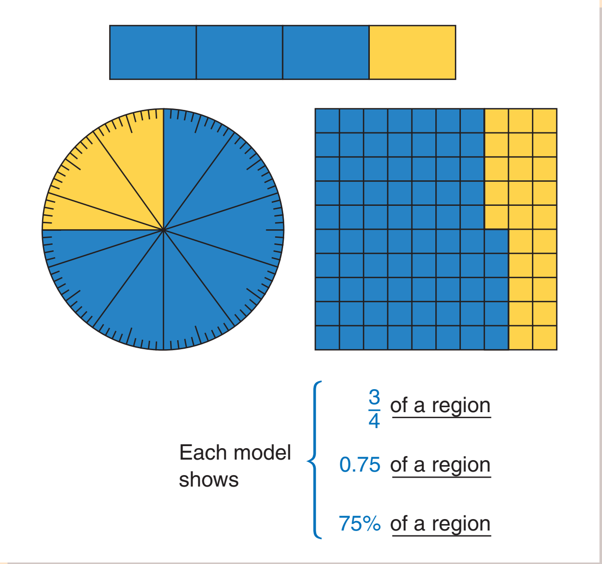


Tools and Terminology

Tools (manipulatives) illustrate connections among fractions, decimals, and percents, as shown in Figure 9.14 with a bar diagram, a **Rational Number Wheel** and a **10 × 10 Grid**. Students should develop an understanding of the percent equivalence of familiar fractions (halves, thirds, fourths, fifths, and eighths). Three-fifths, for example, is 60 percent as well as 0.6. For example, with the 10 × 10 grid, each little square inside is 1 percent of the grid and each row or strip of 10 squares is not only one-tenth but also 10 percent of the grid. Zambo (2008) recommends linking fractions to percents with a 10 × 10 grid. By marking one of every four squares on the grid, students can discover the link between $\frac{1}{4}$ and $\frac{25}{100}$ or 25 percent. Zambo goes on to suggest that even more complex representations, such as $\frac{1}{8}$, can lead to interesting discussions about the remaining squares left at the end, resulting in $12\frac{1}{2}$ of 100 squares or $12\frac{1}{2}$ percent.

Figure 9.14

Visuals illustrate three different representations.



Length representations are also important. A number line can be created that has 0 and 100 percent as end points, or 0 to 1, and percents can be posted on the number line in their appropriate places. In fact, an advantage of the number line is that it lends to showing percentages greater than 100 percent.

Teaching Tip

Use a classroom-long number line, and have students place fractions, decimals, and percents. Equivalent representations are placed on top of one another. This can be a lesson or a daily routine for five minutes at the start of class.

Another length representation is the percent necklace. Using fishing line or sturdy string, link 100 same-sized beads and knot them in a tight, circular necklace. Have students count the number of beads between any two lines that represent a wedge of the circle. For example, they might find that 24 beads are in the section of the circle graph that shows how many students' favorite music is country. That is an estimate that approximately 24 percent of the students favor country music. Counting the beads in a given category gives students an informal approach to estimating percent while investigating a meaningful physical model for thinking about the concept of per 100.

Activity 9.15

CCSS-M: 6.RP.A.3c

Percent Memory Match

Use the **Percent Cards** Activity Page to create a deck of cards of circle graphs with a percentage shaded in and cards that have the matching percents (e.g., a card with a circle that has $\frac{1}{2}$ shaded, and a matching card with the value 50% on it). A spreadsheet program like Excel can generate these graphs when you enter the percent of each part. Students pair each circle graph with the percent that best matches it in a memory game format. For students with disabilities, provide a **Rational Number Wheel** (see Figure 9.3) as a movable representation to help support their matching. For a virtual game that has the same goal, see the NCTM Illuminations virtual "Concentration," which uses representations of percents and fractions and a regional model. It is designed for one or two players.

technology note

The NCTM Illuminations activity "Fraction Models" explores equivalence of fractions, mixed numbers, decimals, and percents. You select the fraction and pick the type of model (length, area [rectangle or circle], or set), and it shows the corresponding visual and all the equivalences.

Percent Problems in Context

There are three possible missing values in percent stories. The percent, the part, and the whole. The sentence "________ is ________ percent of ________" has three spaces for values for example, "20 is 25 percent of 80." The classic story problems provide two of the values, and ask students to produce the third. Students tend to set up proportions but are not quite sure which numbers to put where. In other words, they are not connecting ratio reasoning or conceptual understanding with the procedure. This story structure needs to vary more to encourage students to make sense of percents. Commonly encountered percent situations, such as sales figures, taxes, food composition (percent fat), and economic trends are almost never in the straightforward "________ is ________ percent of ________" format.

To prepare students for these short statements, provide more elaborate and authentic examples. Begin with problems that involve familiar fractions and compatible numbers.

1. The baseball team won 80 percent of the 25 games it played this year. How many games were lost?
2. In Ms. Glasser's class, 20 students, or 66$\frac{2}{3}$ percent, were on the honor roll. How many students are in her class?
3. The PTA reported that 75 percent of the total number of families were represented at the meeting. If students from 320 families go to the school, how many families were represented at the meeting?
4. Erika bought her new computer at a 12$\frac{1}{2}$ percent discount. She paid $700. How many dollars did she save by buying it at a discount?
5. If Nicholas has read 60 of the 180 pages in his library book, what percent of the book has he read so far?
6. The hardware store bought widgets at 80 cents each and sold them for $1 each. What percent did the store mark up the price of each widget?

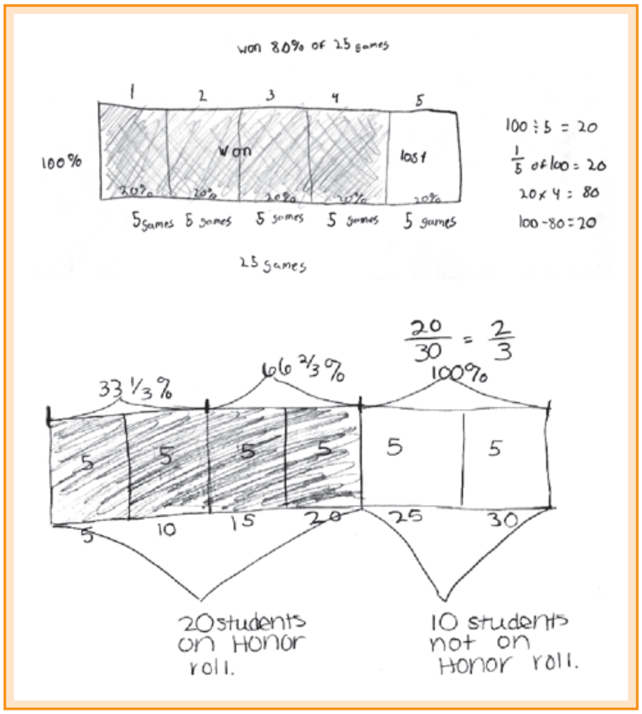
Stop and Reflect

Which types of problems might be more challenging to students? What visuals or physical models might be used to support their thinking?

Examples of students' reasoning with bar diagrams are illustrated in Figure 9.15.

Figure 9.15

Student-created bar diagrams for #1 and #2 of the percent problem examples.



The results of the 2005 NAEP exam revealed that only 37 percent of eighth graders could determine an amount following a given percent of increase. Many selected the answer obtained by adding the percent itself to the original amount. That is, for a 10 percent increase, they would select an answer that was 10 more than the original amount. In another NAEP item, only 30 percent of eighth graders could accurately calculate the percent of the tip when given the cost of the meal and the amount of the tip left by the diners. This weak performance is evidence of our failure to develop percent concepts meaningfully by using various representations and situations (a recurring theme related to rational numbers).

Formative Assessment Note

These context-based percent problems are an effective *performance assessment* to evaluate students' understanding. Assign one or two, and have students explain why they think their answers make sense. You might take a realistic percent problem and substitute a fraction for a percent (e.g., use $\frac{1}{8}$ instead of $12\frac{1}{2}$ percent) to compare how students handle the problems with fractions and with percents. If your focus is on reasoning and justification rather than number of problems correct, you will be able to collect all the assessment information you need to plan next instructional steps.

Estimating Percents

Many percent problems do not have simple (familiar) numbers. Frequently, in real life, an approximation or estimate in percent situations is enough to help one think through the situation. Even if a calculator will be used to get an exact answer, an estimate based on an understanding of the relationship confirms that a correct operation has been performed or that the decimal point is positioned correctly.

To help students with estimation in percent situations, two ideas that have already been discussed can be applied:

1. When the percent is not a simple one, substitute a close percent that lends to mental computation.
2. Select numbers that are compatible with the percent involved to make the calculation easy to do mentally.

In essence, convert the complex percent problem into one that is more familiar. Here are some examples – consider which value you might change in order to estimate the percent.

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1. The 83,000-seat stadium was 73 percent full. How many people were at the game?
2. The treasurer reported that 68.3 percent of the dues had been collected, for a total of $385. How much more money can the club expect to collect if all dues are paid?
3. Max McStrike had 217 hits in 842 at-bats. About what percent of the time is he getting a hit?

As with any estimation, there are multiple strategies. Here are some sample strategies for the three questions posed here:

1. Use $\frac{3}{4}$ and 80,000, so about 60,000 people at the stadium.
2. Use $\frac{2}{3}$ and $380; the club will collect $\frac{1}{3}$ more, which is half of $\frac{2}{3}$, so about $190.
3. The number of hits (217) is about $\frac{1}{4}$ of the number of at bats (842); $\frac{1}{4}$ is the same as 25 percent, so Max is getting a hit about 25% of the time.

There are several common uses for estimating percentages in real-world situations. As students gain full conceptual understanding and flexibility, there are ways to think about percents that are useful as you are shopping or in situations that bring thinking about percents to the forefront.

1. *Tips.* As mentioned previously, to figure a 15% tip, you can find 10 percent of the amount and then half of that again to make 15 percent.
2. *Taxes.* The same approach is used for adding on sales tax. For 6% sales tax, you can find 10 percent, take half of that, and then find 1 percent and add that amount: encourage other approaches as well. Students should realize that finding percents is a process of multiplication, and therefore commutative; therefore, finding 8 percent (tax) of $50 will generate the same result as finding 50 percent (half) of 8, or $4.
3. *Discounts.* A 30 percent decrease is the same as 70 percent of the original amount, and sometimes, depending on the original amount, one of those percents is easier to use in mental calculations than the other. If a $48 outfit is discounted 30 percent, for example, you are paying 70 percent. Round $48 to $50. And you have .70 × 50 (think 7 × 5), so your cost is about $35.

These are not rules to be memorized; they are reasoning activities to be explored. Such activities enhance a students' understanding of percent, as well as the commutative property.

Teaching Percents

In summary, here are suggestions for effective instruction with percents:

- Emphasize percents of familiar fractions (halves, thirds, fourths, fifths, eighths, and tenths), and use numbers compatible with these fractions. The focus of these exercises is on the relationships involved, not complex computation.
- Do not rush to develop rules or procedures for different types of problems—encourage students to notice patterns.
- Use the terms *part*, *whole*, and *percent* (or *fraction*). *Fraction* and *percent* are interchangeable. Help students see these percent exercises as the same types of exercises they did with fractions.
- Require students to use manipulatives, drawings, and contexts to explain their solutions. It is wiser to assign three problems requiring a drawing and an explanation than to give 15 problems requiring only computation and answers. Remember that the purpose is the exploration of relationships, not computational skill.
- Encourage mental computation and estimation.

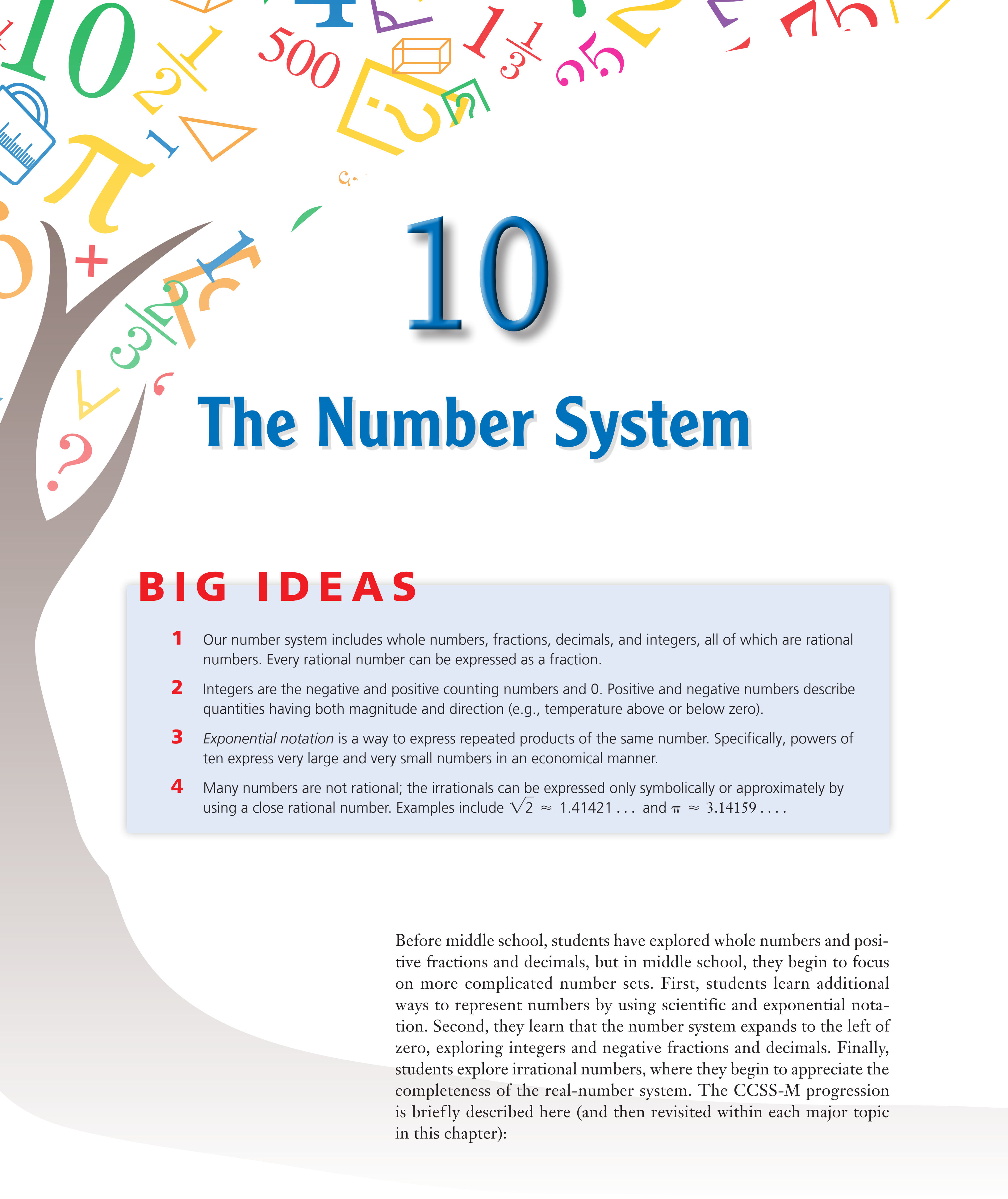
Literature Connections

The Phantom Tollbooth *Juster (1961)*

Mathematical ideas abound in this story about Milo's adventures in Digitopolis, where everything is number-oriented. There, Milo meets a half of a boy, appearing in the illustration as the left half of a boy cut from top to bottom. As it turns out, the boy is actually 0.58 because he is a member of the average family: mother, father, and 2.58 children. The boy is the 0.58. One advantage, he explains, is that he is the only one who can drive 0.3 of a car, as the average family owns 1.3 cars. This story can lead to a great discussion of averages that result in decimals. An extension is to explore averages that are interesting to the students (average number of siblings, etc.) and see where these odd decimal fractions come from. Illustrating an average number of pets can be very humorous!

Piece = Part = Portion: Fraction = Decimal = Percent
Gifford & Thaler (2008)

Illustrated with vivid photos, this book shows how fractions relate to decimals and percents. Written by a teacher, connections are made through common representations, such as one sneaker representing $\frac{1}{2}$ or 0.50 or 50 percent of a pair of shoes. Real-world links such as one-seventh of a week and one-eleventh of a soccer team will connect with students.



10

The Number System

BIG IDEAS

1. Our number system includes whole numbers, fractions, decimals, and integers, all of which are rational numbers. Every rational number can be expressed as a fraction.
2. Integers are the negative and positive counting numbers and 0. Positive and negative numbers describe quantities having both magnitude and direction (e.g., temperature above or below zero).
3. *Exponential notation* is a way to express repeated products of the same number. Specifically, powers of ten express very large and very small numbers in an economical manner.
4. Many numbers are not rational; the irrationals can be expressed only symbolically or approximately by using a close rational number. Examples include $\sqrt{2} \approx 1.41421\ldots$ and $\pi \approx 3.14159\ldots$.

Before middle school, students have explored whole numbers and positive fractions and decimals, but in middle school, they begin to focus on more complicated number sets. First, students learn additional ways to represent numbers by using scientific and exponential notation. Second, they learn that the number system expands to the left of zero, exploring integers and negative fractions and decimals. Finally, students explore irrational numbers, where they begin to appreciate the completeness of the real-number system. The CCSS-M progression is briefly described here (and then revisited within each major topic in this chapter):

Sixth grade: Students fluently compute whole numbers, fractions, and decimals, and begin to explore negative numbers by locating them on the number line and comparing them.

Seventh grade: A critical area is for students to develop a unified understanding of numbers, including developing a stronger understanding of the relationships between fractions, decimals, and percents, and performing operations involving negative numbers.

Eighth grade: Explore irrational numbers and explore radical expressions, including scientific notation.

Exponents

As numbers in our increasingly technological world get very small or very large, expressing them in standard form can become cumbersome. Exponential notation is more efficient for conveying numeric or quantitative information. In the *Common Core State Standards*, exponents are first introduced in fifth grade related to powers of ten and place-value. In sixth grade, students learn to write and evaluate numeric expressions involving whole-number exponents. And in eighth grade, students work with radicals and integer exponents.

Exponents in Expressions and Equations

The "rules" of exponents may be confusing for students. For example, with only a rule-based background, they may not remember whether you add or multiply the exponents when you raise a number to a given power. This is an indication that students lack a conceptual understanding of the operations and the notation. Students need to explore exponents with whole numbers before they use exponents with variables. By looking at whole-number exponents, they are able to notice patterns in solving problems and are able to generate (and understand) the "rules" of exponents themselves. A whole-number exponent is simply shorthand for repeated multiplication of a number times itself—for example, $3^4 = 3 \times 3 \times 3 \times 3$.

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

Symbols for exponents are abstract and unfamiliar and thereby require explicit attention. First, an exponent applies to its immediate base. For example, in the expression $2 + 5^3$, the exponent 3 applies only to the 5, so the expression is equal to $2 + (5 \times 5 \times 5)$. However, in the expression $(2 + 5)^3$, the 3 is an exponent of the quantity 2+5 and is evaluated as $(2 + 5) \times (2 + 5) \times (2 + 5)$, or $7 \times 7 \times 7$. Notice that the process follows the order of operations. As with any topic, start with what is familiar and concrete. With exponents, this means beginning by exploring powers of 2 and 3—operations that can be represented geometrically.

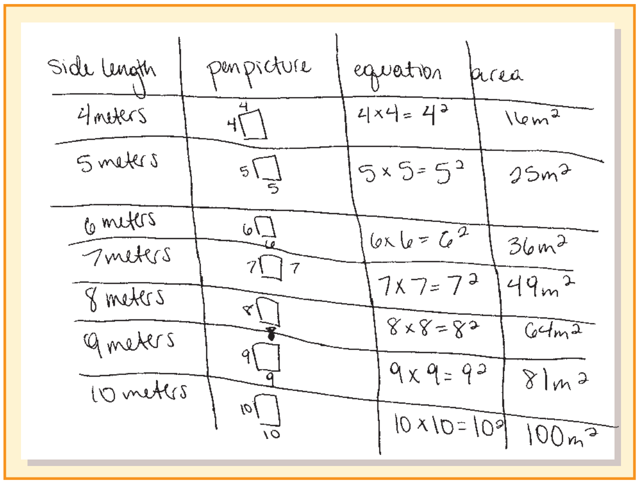
Minia knows that square animal pens are the most economical for the amount of area they provide (assuming straight sides). Can you provide a table for Minia that shows the areas of square pens that have between 4 m and 10 m of fence on each side?

Students may set up a table similar to the one in Figure 10.1, showing possible areas for square pens with different side lengths.

Another way to explore exponents is to explore algebraic growing patterns involving squares and/or cubes. The classic Painted Cube Problem, prepared on the **Painted Cube** Activity Page and illustrated in Figure 10.2, is an excellent way to explore squares and cubes.

Figure 10.1

A student records possibilities for making a square pen.

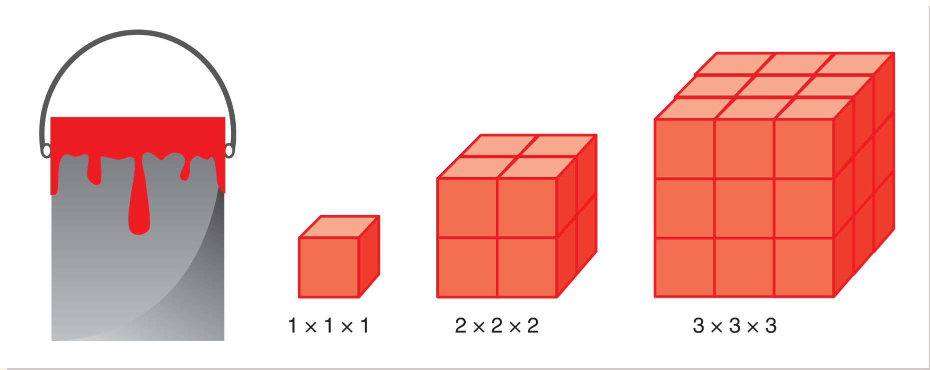


Side length	pen picture	equation	area
4 meters	4 □ 4	$4 \times 4 = 4^2$	16m²
5 meters	5 □ 5	$5 \times 5 = 5^2$	25m²
6 meters	6 □ 6	$6 \times 6 = 6^2$	36m²
7 meters	7 □ 7	$7 \times 7 = 7^2$	49m²
8 meters	8 □	$8 \times 8 = 8^2$	64m²
9 meters	9 □ 9	$9 \times 9 = 9^2$	81m²
10 meters	10 □ 10	$10 \times 10 = 10^2$	100m²

As the painted cube grows, so does the size of each square face (excluding the edges), as well as the number of cubes hidden inside the large painted cube. In a $2 \times 2 \times 2$ painted cube, the faces are 2×2; in a $3 \times 3 \times 3$ painted cube, the faces are 3×3. Note that although each face is 3×3, the outer cubes are corners or edges with more than one side painted, whereas the inner 1×1 square on each face is a centimeter cube with one side painted. Consider what is happening with the cubes in the *middle* of the painted cube. In a $2 \times 2 \times 2$ cube, there are no inside cubes; in a $3 \times 3 \times 3$ cube, there is one $1 \times 1 \times 1$ (1) “hidden” centimeter cube inside that will not be painted on any face. In a $4 \times 4 \times 4$ cube, there will be $2 \times 2 \times 2$ (8) hidden cubes. As you can see, the number of cubes with one side painted grows at a quadratic rate, and the number of “hidden” cubes grows at a cubic rate. In exploring the pattern, students get experience with algebraic rules that are linear, squared, and cubed.

Figure 10.2

The Painted Cube Problem asks students to figure out how many sides of the little cubes are painted when the larger cubes are dipped in paint.



technology note

Exponential growth is very interesting to explore in real-world contexts. The website “Otherwise” offers an applet that can engage students in experiments with population (exponential) growth. Another powerful exploration of exponential growth is to look at Powers of 10 on the Florida State University Molecular Expressions website. This video starts with photos of earth from way out in the Universe (10^{23}), then continues to zoom in by powers of 10 until it reaches earth, and keeps going until it zooms in on cells to a quark at 10^{-14}!

Order of Operations

Working with exponents extends the *order of operations*. As early as third grade, students need to know the order of operations for addition, subtraction, multiplication and division, and in sixth grade, exponents are added to the order of operations (CCSSM, 2010). An exponent indicates the number of times the base is used as a factor, so it indicates repeated multiplication and it precedes other multiplication and division, as well as addition and subtraction. In the expression $5 \times 4^2 - 6$, 4^2 is done first. If multiplication is done first, then the answer will be different. When we want to communicate that operations are to be computed in a different order, we have to use grouping symbols, such as parentheses.

Although the order of operations follows some conventions (e.g., working from left to right, using parentheses), the order of the computations can be developed conceptually (Bay-Williams & Martinie, 2015). A context can make this point clearer, which is the focus of Activity 10.1.

Activity 10.1

CCSS-M: 3.OA.D.8; 6.EE.A.2c

Stacks of Coins

Select a story situation that includes such things as stacks of coins, bricks, or notebooks. If you have the book *Two of Everything,* by Lily Hong, you can show that the Haktaks have stacks of coins from their magic pot and use that as the context. Tell stories and ask students to (1) write an expression and (2) tell you how many. For example: "Mrs. Haktak had one stack with seven coins and four stacks with ten coins. How many coins did she have? (Students should write $7 + 4 \times 10$ or $4 \times 10 + 7$ for the expression.) Ask, "Could we write it either way? Why or why not?" and "Could we add the seven to the four and then multiply by 10? Why or why not?" Then, write expressions with addition and multiplication and ask students to tell their own stories as they solve the problem.

Mnemonics should not replace an understanding of why the order of operations is what it is. The mnemonic "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally," or more simply PEMDAS, has led to many students thinking multiplication precedes division and addition precedes subtraction (Jeon, 2012). Rather than use this mnemonic, consider an improved version that involves writing the order in hierarchical levels (therefore avoiding the common errors students make):

P = parenthesis
E = exponents
MD = multiplication and division (whichever is first from left to right)
AS = addition and subtraction (whichever is first from left to right)

Teaching Tip

Different countries use different mnemonics for order of operations (e.g., in Canada, the United Kingdom, and other English-speaking countries). These include BODMAS, BEDMAS, and BIDMAS, with the word bracket replacing parenthesis, and order, exponents, or indices used for "exponents" (Bay-Williams & Martinie, 2015). Mnemonics should never replace understanding.

A visual can more effectively illustrate that multiplication and division are at the same level and addition and subtraction are at the same level (Ameis, 2011). Two such possible posters are pictured in Figure 10.3.

The order of operations is not as rigid as the list might imply. For example, consider the expression $14 \times 7 - 5 \times 7$. It doesn't matter which product is figured first, as long as both products are determined before adding. In fact, the CCSS-M describes mathematically proficient students as those who will look closely to notice the structure of a problem, for example, noticing that in this case it doesn't matter which product is determined first, and even recognize that they could apply the distributive property and factor out a 7, rewrite as $7(14 - 5)$, then subtract $14 - 5$ first.

Standards for Mathematical Practice

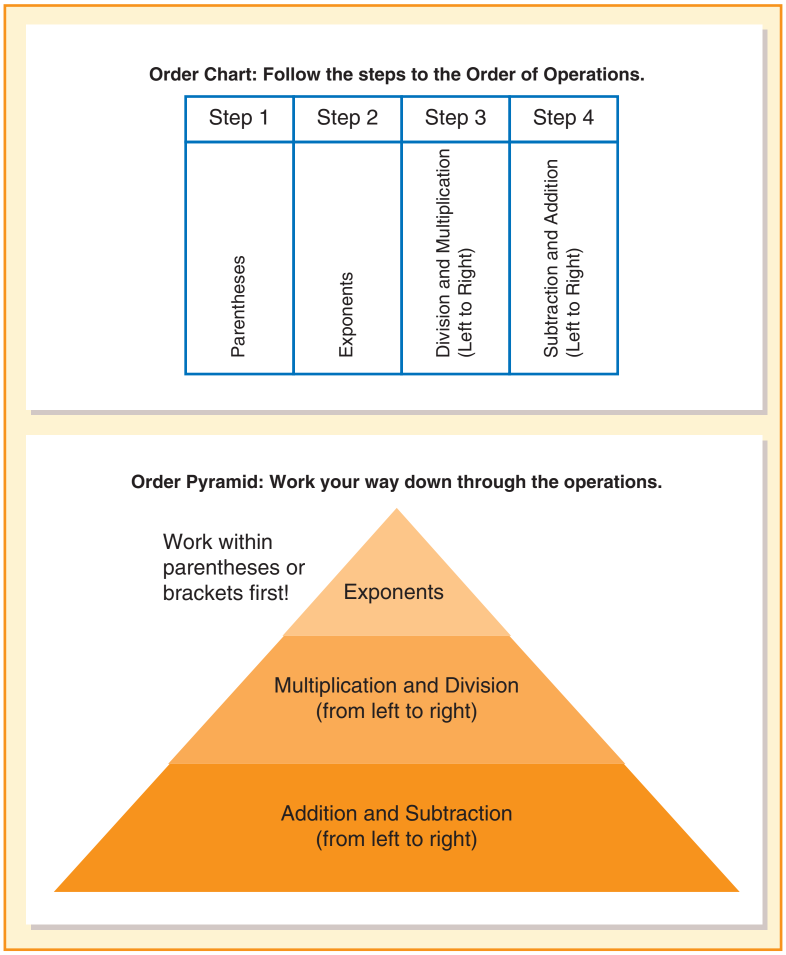
7 Look for and make use of structure.

Stop and Reflect

How would you calculate: 13(5 + 10)? Would you apply the distributive property, then add 65 + 130, or would you work inside the parenthesis first, and then multiply 13 × 15? Is one way more efficient?

Figure 10.3

Two possible posters for illustrating the Order of Operations.



Order Chart: Follow the steps to the Order of Operations.

Step 1	Step 2	Step 3	Step 4
Parentheses	Exponents	Division and Multiplication (Left to Right)	Subtraction and Addition (Left to Right)

Order Pyramid: Work your way down through the operations.

technology note

IXL has an applet, “Rags to Riches,” that engages students in practicing the Order of Operations using a game format similar to “Who Wants to Be a Millionaire.”

You can also strengthen students’ understanding of order of operations by having them use appropriate symbols to record expressions, as in the activity here.

Activity 10.2

CCSS-M: 6.EE.A.2a, b, c; 6.EE.B.6

Guess My Number

In this activity, you will give hints about a number, and students will think backward to find it (by using logical reasoning). For ELLs and students with disabilities, provide the statements verbally and in writing. Students create equations, using parentheses appropriately to reflect the clues you give, as in the following three examples:

- I am thinking of a number; I add 5, double it, and I get 22. [$(n + 5) \times 2 = 22$]
- I am thinking of a number; I subtract 2, square it, and I get 36. [$(n - 2)^2 = 36$]
- I am thinking of a number; I double it, add 2, then cube the quantity, and I get 1000. [$(2n + 2)^3 = 1000$]

For students with disabilities, you may want to start with a known number rather than an unknown number—for example, start with 5, square it, add 11, and divide by 6. They should write $(5^2 + 11) \div 6 = n$.

Standards for Mathematical Practice

4 Model with mathematics.

Formative Assessment Note

Give students an expression that includes all the operations and the use of parentheses—for example, $(4 + 2)^2 \times 2 \div 4$—and ask them to write a matching story, using a context of their choice, to fit the expression. Having students write these stories in *journals* provides an excellent assessment of their understanding of the order of operations.

As you review students' stories, see if the stories show students' understanding that multiplication and division (and addition and subtraction) are equal in the hierarchy of order and should therefore be solved from left to right.

Another way to engage students or assess their understanding of exponents and order of operations is to have them determine if given equations are true or false. This is the focus of Activity 10.3.

Teaching Tip

Students can be overzealous in using parentheses when writing expressions. Regularly ask students, "Are the parentheses required or optional in this equation?"

Activity 10.3

CCSS-M: 3.OA.D.8; 6.EE.A.3; 6.EE.A.4

Order of Operations: True or False Equations

Write an equation that addresses one or more aspects of the order of operations. For example: $24 \div (4 \times 2) = 24 \div 4 \times 2$. If students think that multiplication comes before division, they

(*continued*)

will write "true." Or, if they are not aware of the left-to-right prioritizing, they may also write "true." Use the **Order of Operations: True or False Equation** Activity Page. Examples from the Activity Page are listed here:

$17 \times 3 = 15 + 2 \times 3$

$2 + 5^3 = 7^3$

$3.2^2 + 3.2^2 = 3.2^4$

$4\ (2 + x) = 8 + 4x$

$4y - y = 4$

$3.2 - 1.2 + 0.04 = (3.2 - 1.2) + 0.04$

$(3.6 + 0.4)^2 = 4^2$

$6 \cdot 2^4 = 12^4$

$x + x^2 = x^3$

$3z + z = z + z + z + z$

There are several ways to incorporate true/false equations into instruction. First, one or two of these equations can be part of a daily warmup routine. Second, cards can be made using the Activity Page. Students work in partners and independently decide if the answer is true or false. Then, one student says "true" or "false," and the partner agrees or disagrees. If the partners disagree, they try to convince each other of the correct responses. They create a stack of their true equations and a stack of their false equations. When you check each group's progress, you can formatively assess by seeing if the cards in the true and false piles are correct. Create additional cards, or invite students to create cards, focusing on what they think might be a common error or misunderstanding related to the order of operations.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

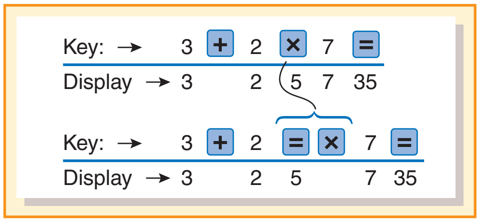
True/false tasks set up an excellent opportunity for students to debate, justify, and critique the justifications of their peers. The true/false statements can also uncover and clear up commonly misunderstood aspects of the order of operations.

Stop and Reflect

How does the strategy of true/false statements support the learning of students with disabilities and students who struggle? What aspects of the order of operations do you think are commonly misinterpreted by students (and why)?

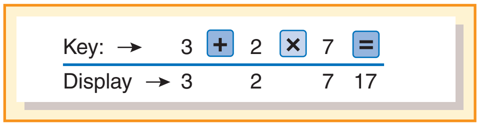
Exponent Notation on the Calculator

Most simple four-function calculators do not use algebraic logic, so operations are processed as they are entered. On calculators without algebraic logic, the following two keying sequences produce the same results:



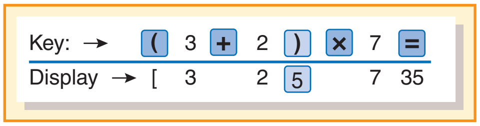
Whenever an operation sign is pressed, the effect is the same as pressing = and then the operation. Of course, neither result is correct for the expression $3 + 2 \times 7$, which should be evaluated as $3 + 14$, or 17.

Calculators designed for middle grades often use algebraic logic (follow the order of operations) and include parenthesis keys, so that both $3 + 2 \times 7$ and $(3 + 2) \times 7$ can be keyed in the order in which the symbols appear. See the difference in the following displays:



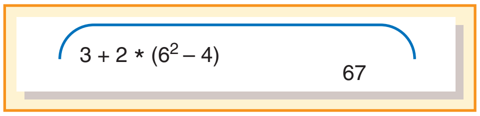
Key: →	3	$\boxed{+}$	2	$\boxed{\times}$	7	$\boxed{=}$
Display →	3		2		7	17

Notice that the following display does not change when × is pressed and a right parenthesis is not displayed. Instead, the expression that the right parenthesis encloses is calculated and that result displayed.



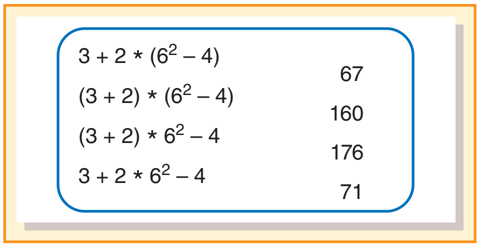
Key: →	$\boxed{(}$	3	$\boxed{+}$	2	$\boxed{)}$	$\boxed{\times}$	7	$\boxed{=}$
Display →	[	3		2	$\boxed{5}$		7	35

Some calculators show the expression $3 + 2 \times (6^2 - 4)$. Nothing is evaluated until you press Enter or EXE. Then, the result appears on the next line to the right of the screen:



3 + 2 * (6^2 – 4)
67

The last expression entered can be recalled and edited so that students can see how different expressions are evaluated.



3 + 2 * (6^2 – 4)
67
(3 + 2) * (6^2 – 4)
160
(3 + 2) * 6^2 – 4
176
3 + 2 * 6^2 – 4
71

Calculators are a powerful tool for exploring the impact of operations. For example, to evaluate 3^8, press 3 $\boxed{\times}$ 3 $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$. (The first press of $\boxed{=}$ will result in 9, or 3 $\boxed{\times}$ 3.) Students will be fascinated by how quickly numbers grow. Enter any number, press $\boxed{\times}$, and then repeatedly press $\boxed{=}$. Try two-digit numbers. Try 0.1.

Give students many opportunities to explore expressions involving exponents and the order of operations.

Activity 10.4

CCSS-M: 6.EE.A.3; 6.EE.A.4

Entering Expressions

Provide students with numeric expressions to evaluate with simple four-function calculators. Ask, "How will you have to enter these to apply the order of operations correctly?" Rewrite the expression the way it needs to be entered. Here are some examples of expressions to try:

$3 + 4 \times 8$
$4 \times 8 + 3$

$3^6 + 2^6$
$(3 + 2)^6$

$3^4 \times 7 - 5^2$
$(3 \times 7)^4 - 5 \times 2$

$3^4 \times 5^2$
$(3 \times 5)^6$

$\frac{5^3 \times 5^2}{5^6}$

$4 \times 3 - 2^3 \times 5 + 23 \times 9$

$\frac{4 \times 3^5}{2}$ $4 + \frac{3^5}{2}$

A common misconception with exponents is to think of the two values as factors, so 5^3 is thought of as 5×3 rather than as the correct equivalent expression of $5 \times 5 \times 5$. Exponent expressions become more problematic when students hear such things as, "It is five three times." Explicitly connect language to symbols. Analyze confusing language. In addition, develop meaning for the symbols by having students state and write the equivalent expressions. Students can re-write equivalent expressions without using exponents or include parentheses to show groupings. For example:

$$(7 \times 2^3 - 5)^3 = [7 \times (2 \times 2 \times 2) - 5] \times [7 \times (2 \times 2 \times 2) - 5] \times [7 \times (2 \times 2 \times 2) - 5]$$

$$= [(7 \times 8) - 5] \times [(7 \times 8) - 5] \times [(7 \times 8) - 5$$

$$= (56 - 5) \times (56 - 5) \times (56 - 5)$$

$$= 51 \times 51 \times 51$$

For many expressions, there is more than one way to proceed, and sharing different ways is important in helping students develop procedural fluency. Activity 10.3, "True or False Equations," can be adapted to focus on the equivalence of simplified and expanded forms of a variety of equations.

Even though calculators with algebraic logic will automatically produce correct results (i.e., follow the order of operations), students must know the order of operations, including when they have to do one operation before another and when it doesn't matter which goes first. This flexibility becomes the foundation for symbolic manipulation in algebra.

Standards for Mathematical Practice

8 Look for in repeated and express regularity reasoning.

Negative Exponents

What does 2^{-4} mean? This is a good question to ask students who have been working with positive exponents. The following two related options can help students explore the possibilities of negative exponents. First, looking for patterns as the power of 10 changes directly relates to place value and helps students see the regularity in the base 10 system.

Ask students to continue the pattern of 10^n as follows:

$$10^4 = 10{,}000$$

$$10^3 = 1000$$

$$10^2 = 100$$
$$10^1 = 10$$
$$10^0 = ?$$
$$10^{-1} = ?$$

To continue the pattern, 10^0 would be 1, which it is! (This is the definition of 10^0, in fact, any number to the zero power is 1). The next value would be one-tenth of 1. And, each successive number is one-tenth of the one that comes before it:

$$10^{-1} = 0.1 = \frac{1}{10}$$

$$10^{-2} = 0.01 = \frac{1}{100} = \frac{1}{10^2}$$

$$10^{-3} = 0.001 = \frac{1}{1000} = \frac{1}{10^3}$$

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

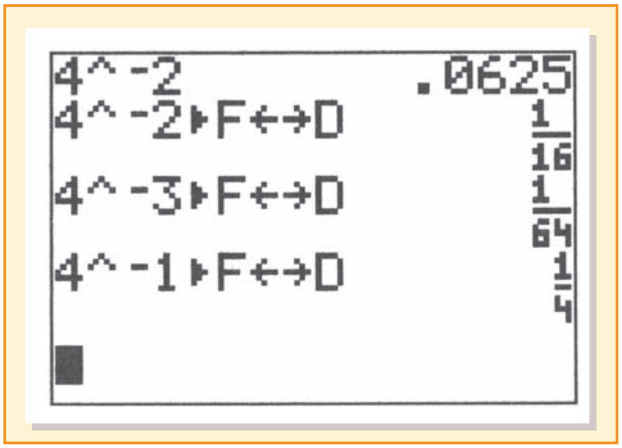
Students may notice that the negative exponent is the reciprocal of the value it would be without the negative sign.

Second, students can explore negative exponents on a calculator. For example, ask students to figure out what 4^{-3} or 2^{-5} equals. If the calculator has the decimal-to-fraction conversion function, suggest that students use that feature to help develop the meaning of negative exponents. Figure 10.4 gives an example of how this might look on a graphing calculator (courtesy of Texas Instruments Inc.). Ask students to notice patterns that are generalizable and test their conjectures.

Students often confuse exponent rules. Using worked examples and identifying a mistake in someone else's work is an effective way to help students think about the correct (and incorrect) order in a problem, as illustrated in Activity 10.5.

Figure 10.4

Graphing calculators evaluate expressions as decimals. This figure shows the screen of a TI-73 calculator. The F⟶⟵D key converts fractions to decimals (and decimals to fractions).



```
4^-2             .0625
4^-2►F↔D             1
                    ──
                    16
4^-3►F↔D             1
                    ──
                    64
4^-1►F↔D             1
                    ─
                     4
█
```

Activity 10.5

CCSS-M: 8.EE.A.1

Find the Error

Make a copy of **Find the Error** Activity Page, or create your own set of problems that are solved incorrectly and ask students to explain what was done wrong and how to do it correctly. A few examples are provided here:

Tamara: $3(4)^2 = 24$ Yoli: $3^3 \times 3^{-5} = 3^{-2} = -9$ Wilma: $\frac{20x^8}{5X^2} = 4X^4$

Examples can be increasingly more challenging and can be mixed with correct solutions. For additional examples, see Johnson and Thompson (2009).

Scientific Notation

The more common it becomes to find very large or very small numbers in our daily lives, the more important it is to have convenient ways to represent them. Numbers can be written in common form, but when this becomes cumbersome, a better option is scientific notation. In the *Common Core State Standards*, scientific notation is an eighth-grade expectation within the domain of Expressions and Equations (CCSSO, 2010). In scientific notation, a number is changed to a number greater than or equal to 1 and less than 10 multiplied by a power of ten. For example, 3,414,000,000 can also be written as 3.414×10^9.

Stop and Reflect

What strategies can you use to bring real-life contexts into exploring scientific notation so that students understand the relative sizes of the numbers, as well as how to write them symbolically?

Different notations have different purposes and values. For example, the population of the world at midyear of 2016 is estimated to be 7,412,778,971 (U.S. Census Bureau, n.d.). This can be expressed in various ways:

7412 million

7.4×10^9

Approaching seven and a half billion

Each way of stating the number has value and purpose in different contexts. Rather than spend time with exercises converting numbers from standard form to scientific notation, consider large numbers found in newspapers, magazines, and atlases. How are they written? How are they said aloud? When are they rounded and why? What forms of the numbers seem best for the purposes? What level of precision is appropriate for the situation? How do these numbers relate to other numbers? How does the population of the world relate to the population in your state or your continent?

Standards for Mathematical Practice

6 Attend to precision.

technology note

Websites like that of the U.S. Census Bureau (www.census.gov) make population data readily available. The NCTM Illuminations lesson "The Next Billion" is a high-quality lesson for exploring when the world population will reach 8 billion. Students discuss their predictions, past trends in population growth, and social factors—a good interdisciplinary opportunity.

Contexts for Very Large Numbers

The real world is full of very large quantities and measures. We see references to huge numbers in the media all the time. Unfortunately, most of us have not developed an appreciation for extremely large numbers, such as those in the following examples:

- A state lottery with 44 numbers from which to pick 6 has over 7 million possible combinations. There are $44 \times 43 \times 42 \times 41 \times 40 \times 39$ possible ways in which the balls can come out of the hopper (5,082,517,440). But generally, the order in which they are picked is not important. Because there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ different arrangements of six numbers, each collection appears 720 times. Therefore, there are *only* 5,082,517,440 ÷ 720 possible lottery numbers, or 1 in 7,059,052 chances to win.

- The estimated size of the universe is 40 billion light-years. One light-year is the number of miles light travels in *1 year*. The speed of light is 186,281.7 miles per *second*, or 16,094,738,880 miles in a single day.
- The human body has about 100 billion cells.
- The distance to the sun is about 150 million kilometers.

Connect large numbers to meaningful points of reference to help students get an idea of their true magnitude. For example, suppose students determine that the population in their city or town is about 500,000 people. They can then figure that it would take approximately 13,300 cities of the same population size to generate the population of the world. Or, suppose students determine that it is about 4,600 km between San Francisco, California, and Washington, D.C. This means that it would take more than 32,000 trips back and forth between these two cities to equal the distance between the earth and the sun.

The following activity uses real data to develop an understanding of scientific notation and the relative size of numbers and lends to an interdisciplinary teaching opportunity.

Activity 10.6

CCSS-M: 8.EE.A.4

How Far away from the Sun?

Use the **How Far Away?** Activity Page, or use the data provided here (alternatively, students can search for the data online).

Mercury	57,909,000 km	Jupiter	778,400,000 km
Venus	108,200,000 km	Saturn	1,423,600,000 km
Earth	149,600,000 km	Uranus	2,867,000,000 km
Mars	227,940,000 km	Neptune	4,488,400,000 km

Explain to students that they are going to compare planetary distances from the sun and create a scaled illustration of the distances. Encourage students to develop strategies to figure out the relative distances between two planets. You can give a long strip of cash register tape to each group and have the students mark the sun on one end and Neptune on the other. For ELLs, reinforce the names of thousands, millions, and billions. (Note that *billion* can mean one million millions in some countries, not one thousand millions, as it does in the United States.)

Contexts for Very Small Numbers

It is also important to use real examples of very small numbers. As with large numbers, connecting very small numbers to points of reference can help students conceptualize how tiny these numbers really are, as shown by the following real-world examples:

- The length of a DNA strand in a cell is about 10^{-7} m. This is also measured as 1000 *angstroms*. (Based on this information, how long is an angstrom?) For perspective, the diameter of a human hair is about 2.54×10^{-5} m.
- Human hair grows at the rate of 10^{-8} miles per hour.

Teaching Tip

Finding real data that are very, very small or very, very large can build the meaning of small and large numbers *and* provide insights into interesting and culturally relevant contexts.

- The mass of one atom of hydrogen is 0.000 000 000 000 000 000 000 001 675 g; for comparison, the mass of one paper clip is about 1 g.
- Sound takes 0.28 (2.8×10^{-1}) seconds to travel the length of a football field. In contrast, a TV signal travels a full mile in about 0.000005368 (5.3×10^{-6}) second. So, a TV viewer at home hears the football being kicked before the receiver on the field does.

Activity 10.7

CCSS-M: 8.EE.A.4

At a Snail's Pace

The speed of garden snails has been clocked at about 3×10^{-2}. Ask students to estimate how long it will take a snail to travel 1 mile. To explore, have them record the decimal equivalent of 3×10^{-2} (0.03). They can use the calculator's counting function (enter .03 + .03 =). On most calculators, when you hit = repeatedly, the calculator counts by the last value entered (.03). Each press of the = represents 1 hour. Ask students figure it out mathematically or by counting. Share strategies. When students have shared their results, ask what it would mean if the rate had been 3×10^{-3}. They can explore this problem, too, and should conclude that it would take 10 times longer.

Scientific Notation on the Calculator

Students may learn how to multiply by 10, by 100, and by 1000 by simply moving the decimal point. Help students understand *why* this works by examining powers of 10 on a calculator that handles exponents.

Activity 10.8

CCSS-M: 8.EE.A.3

Exploring Powers of Ten

Have students use any calculator that permits entering exponents to explore some of the following:

- Explore 10^n for various values of n. What patterns do you notice? What does 1E15 mean? (1E15 is the typical calculator form of 1×10^{15}.) What does 1E–09 mean?
- What does 4.5E10 mean? 4.5E–10?
- What does 5.689E6 mean? Can you enter this another way?
- Try sums like (4.5×10^n) + (27×10^k) for different values of n and k. What can you find out? Does it hold true when n and k are negative integers?
- What happens with products of numbers like those in the previous item?

Students need to become familiar with the power-of-ten expressions in written forms and the calculator form. For example, on a calculator, the product of 45,000,000 × 8,000,000 is displayed as 3.6E14, meaning 3.6×10^{14}, or 360,000,000,000,000 (360 trillion).

One misconception students can develop is that the exponent tells the number of zeros to add onto the number. Address this explicitly in class discussions. For example, ask students, "Why are there 13 zeros and not 14? Is there a relationship between the exponent and the number of zeros?" (No, it depends on how many nonzero digits are in the number.)

Scientific notation has advantages in terms of operating on large or small numbers, especially for multiplication.

Compute $(4.5 \times 10^7) \times (8 \times 10^6)$, or 4.5E7 × 8.0E6, mentally.

Notice that the significant digits can be multiplied mentally $(4.5 \times 8 = 36)$ and the exponents added to produce 36×10^{13} or 3.6×10^{14}.

Positive and Negative Numbers

Every day, students experience phenomena that involve negative numbers, as shown in the following list:

- Temperature
- Altitude (above and below sea level)
- Golf (above and below par)
- Money
- Time lines, including Before Common Era (BCE)
- Football yardage (gains/losses)

Generally, negative numbers are introduced with integers—the whole numbers and their negatives or opposites—instead of with fractions or decimals. However, it is a mistake to only focus on integer values because students must understand where numbers like $^{-}4.5$ and $-1\frac{1}{4}$ are positioned on the number line in relation to integer values. In fact, because noninteger negative numbers are not addressed adequately in middle school, many students have misconceptions about where noninteger negative numbers are located on the number line. For example, students will place $-1\frac{1}{4}$ between $^{-}1$ and 0 instead of between $^{-}2$ and $^{-}1$. In the *Common Core State Standards*, integers are introduced and developed in sixth grade, and in seventh grade, students "solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically" (CCSSO, 2000, p. 49).

Contexts for Exploring Positive and Negative Numbers

As with any new topic or type of number, it is important to start with familiar contexts so that students can use prior knowledge to build meaning. With integers, students often get confused about which number is bigger or in which direction they are moving when they compute, so having a context is particularly important.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

For many students, and for ELLs in particular, it is important to include visuals with the contexts to support language development (Swanson, 2010). As students learn to compare and compute, they can use the contexts to ground their thinking and justify their answers. The importance of using contexts is emphasized in the *Common Core State Standards* in sixth grade: "[Students must] understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation" (CCSSO, 2010, p. 43). These contexts and others are described in the next sections.

Quantity Contexts

Quantity contexts provide an opportunity for students to match opposites (4 and −4) to equal zero. Quantity contexts can be illustrated with two-color counters or other counting objects.

Golf Scores. In golf, scores are often written in relationship to a number considered par for the course. So, if par for the course is 70, a golfer who ends the day at 67 has a score of $^{-}3$, or 3 strokes under par. Consider a player in a four-day tournament with day-end scores of $^{+}5$, $^{-}2$, $^{-}3$, $^{+}1$. What would be his or her final score for the tournament? How did you think about it? You could match up the positive and the negatives (in this case, $^{+}5$ with $^{-}2$ and $^{-}3$ to get a net score of 0), then see what is left (in this case, $^{+}1$). The notion that opposites (5 and $^{-}5$) equal zero is an important concept in teaching positive and negative numbers. You can post a mixed-up set of golf scores and ask students to order the players from first through tenth place. Emphasize that first place is the *lowest* score and therefore the *smallest* number.

Money: Payments and Deposits. Suppose that you have a bank account. At any time, your records show how many dollars are in your account. The difference between the payments and deposit totals tells the amount of money in the account. If there is more money deposited than paid out, the account has a positive balance, or is "in the black." If there are more payments than deposits, the account is in debt, showing a negative cash value, or is "in the red." This is a good context for exploring addition and subtraction of integers, as in Figure 10.5. Net worth is a similar way to look at positive and negative numbers (assets and debts). Considering the net worth of famous people can engage students in making sense of positive and negative numbers (Stephan, 2009), and students can successfully draw on their experiences with assets, debts, and net worth values to create meaning for addition and subtraction involving positive and negative numbers (Stephan & Akyuz, 2012).

Figure 10.5

A checkbook used as a context for adding and subtracting positive and negative numbers.

Item	Payments or Deposits	Balance
Mowing lawn	+12.00	$34.00
Phone bill	–55.00	–21.00
iTunes downloads	–9.00	–30.00
Paycheck	+120.00	90.00

Activity 10.9

CCSS-M: 6.NS.C.5

What Is Her Net Worth?

On the Internet, look up the net worth of someone interesting to your students (e.g., a singer, athlete, or actor). List two to three assets and two to three debts, and ask your students to figure out her net worth. Then, with the students, look up the net worth of other people of their choice. Have them suggest possible assets and debts for that person. One clever way to do this is to have a net worth page filled out with two to three assets and two to three debts. Put a smudge on the paper so that the only thing students can see is the net worth (Stephan, 2009). This visual is particularly important for students with disabilities because they can see the missing value in a real situation.

Eventually, debts can be represented as negative values, and a connection is made to integer addition and subtraction.

Linear Contexts

Many of the real contexts for negative numbers are linear. The number line provides a good tool for understanding the ordering of negative numbers and can support reasoning in doing operations with positive and negative numbers (Bishop et al., 2014). CCSS-M emphasizes the need for sixth graders to be able to represent integers on a number line as well as a coordinate axis (CCSS, 2010). See Math Goodies website for a good introduction to integers on a number line.

Temperature. The "number line" measuring temperature is vertical. This context demonstrating negative numbers may be the most familiar to students because they either have experienced temperatures below zero or know about temperatures at the North or South Pole. A good starting activity for students is finding where various temperatures belong on a thermometer. For example, Figure 10.6 displays a thermometer marked in increments of five degrees, and students are asked to place on the number line the following temperatures from a week in North Dakota: 8°, $^{-}2$°, $^{-}12$°, 4°, $^{-}8$°, 0°, $^{-}3$°. Ask students to order the temperatures from the coldest to the warmest (least to greatest).

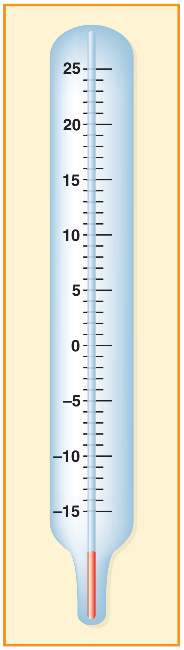
Teaching Tip

Temperature as a context for negative numbers has the advantage that you can also use fraction and decimal values.

Altitude. Another vertical number-line model, altitude, is also a good context for positive and negative numbers. The altitudes of sites below sea level are negative, such as that of the town of Dead Sea, Israel, with an altitude of $^{-}1{,}371$ feet, and Badwater, California (in Death Valley), which has an altitude of $^{-}282$ feet. Positive values for altitude include Mount Denali (the tallest mountain in North America), at 20,322 feet. Students can order the altitudes of various places around the world (data easily found on the Internet) or find the difference between the altitudes of two different places—a good context for subtraction of integers (interpreting subtraction as *difference* rather than as take-away).

Figure 10.6

Thermometers are excellent tools for exploring positive and negative numbers.



Time Lines. Asking students to place historical events on a time line is an excellent interdisciplinary opportunity. The time line is useful for examples with larger values (e.g., 1950) as well as negative values (e.g., $^{-}3000$, or 3000 BC). Or, students can explore their own personal time line (Weidemann, Mikovch, & Hunt, 2001), in which students find out key events that happened before they were born (e.g., the birth of an older sibling) and have happened since they were born (e.g., the move to a new house). Students place these events on a number line, with zero representing the day they were born. By partitioning a year into months, students can gain experience with rational numbers (halves, fourths, or twelfths) on the number line. Continue to reinforce the connection to the size of numbers, asking students, "Which number (year) is the smallest (earliest)?"

Football Statistics. A statistic reported on every play in a football game is yards gained and yards lost, which provides a good context for exploring integers, especially when it comes to comparing and adding integers. Students can be asked questions like these: "If the Steelers started their drive on the 20-yard line and the first three plays were recorded as $^{-}4$, $^{+}9$, $^{+}3$, did they get a first down?" "On the Ravens' first play, the yardage is $^{-}4$. Where are they in relation to the line of scrimmage?" (Use negatives if they are behind the line of scrimmage, in this case $^{-}4$.) "Where are they in relation to the first down marker?" ($^{-}14$).

Activity 10.10

CCSS-M: 6.NS.C.5; 6.NS.C.6a

(American) Football Statistics

Look up the average yards gained for some of the best running backs in the NFL or from college teams popular with your students. Ask students to use average yards gained per down to create a possible list of yardage gains and losses for each player. For example, if a player had an average of 4 yards per carry in a game, the following could have been his data:

$$10,\ {}^{-}3,\ {}^{-}2,\ 21,\ {}^{-}5,\ 3,\ {}^{-}1,\ 5,\ {}^{-}1,\ 13$$

You may want to do one like this together, then have students create their own. The football context provides an excellent way to *use* integers meaningfully, integrated with the important concept of averages. ELLs may not be familiar with American football because football in most countries is what is called *soccer* in the United States. Role playing the game with students is a fun way to be sure the game is understood by all. Also, a yard is a U.S. measurement that may not be familiar and could be confused with the other meaning of *yard.* Comparing a yard to a meter can provide a point of reference that will help build meaning for this activity.

Meaning of Negative Numbers

Negative numbers are defined in relation to their positive counterparts. For example, the definition of negative 3 is the solution to the equation $3 + ? = 0$. In general, the *opposite of n* is the solution to $n + ? = 0$. If n is a positive number, the *opposite of n* is a negative number. The set of integers, therefore, consists of the positive whole numbers, the opposites of the whole numbers (or negative integers), and zero, which is neither positive nor negative. Like many aspects of mathematics, abstract or symbolic definitions are best understood when conceptual connections link to the formal mathematics.

Absolute Value

Absolute value is introduced in sixth grade in the *Common Core State Standards*. The *absolute value of a number* is defined as the distance between that number and zero. Knowing the distance between two points, either on a number line or on a plane, is often needed in applications of mathematics. For example, we need to be able to determine how far a helicopter is from a hospital, regardless of its direction. Opposites, such as ${}^{-}12$ and 12, are the same distance from zero and therefore have the same absolute value.

When students' absolute value experiences are limited to simplifying expressions like $|{}^{-}8|$ or $|6 - 10|$, they do not connect the procedure with the meaning of absolute value or see any real purpose for doing this. Add in a context to make it meaningful. For example, $|6 - 10|$ can be the distance between the 10-mile marker and the 6-mile marker. In this example, you can see that both $10 - 6$ and $6 - 10$ can lead to the answer and that distance is positive (absolute value), so the answer is 4.

Stop and Reflect

What other contexts can be used to build meaning for absolute value?

Minus Sign Notation

Students often have a limited understanding of the minus sign, which can interfere with their ability to solve equations and to make sense of variables (Lamb et al., 2012). There are three meanings of the minus symbol (Bofferding, 2014; Lamb et al., 2012). Each is illustrated with examples here:

Subtraction: $25-12=$___ OR $9-$___$= 4.5$

Negative number: $25+{}^{-}12=$___ OR $-4.2+$___$= {}^{-}13.04$

The opposite of: ${}^{-}({}^{-}5) = 5$ OR $-x$

To further the challenge of the minus sign, it can switch meanings in the middle of simplifying an equation. For example, consider the equation $3.5 - x = -0.6$. A step in simplifying might be to add -3.5 to both sides, leaving $-x = -4.1$. Here, the equation reads, "The opposite of x is equal to negative 4.1 (or the opposite of 4.1). The different heights at which the negative sign may appear (e.g., ${}^{-}7$ and -7) may also be confusing. Another new notation is parenthesis used to separate the operation and the number for example, $8 - ({}^{-}5)$. Students have not seen parentheses used in this way and may think they should multiply. It is important to connect to their prior knowledge and explicitly build meaning for the new use of the minus sign and parentheses. The following Activity 10.11, based on Lamb et al. (2012), is designed to do this.

Activity 10.11

CCSS-M: 6.NS.C.5; 6.NS.C.6a

Greater, Less, Equal, or Don't Know?

Use Greater, Less, Equal or Don't Know Activity Page. Write an equation on the board that addresses different meanings of the minus symbol and/or parentheses. Have students "vote" on whether it is equal or not equal. Encourage students to work with a partner to prepare a rationale for whether they think it is equal or not equal. Facilitate a debate, as appropriate. Examples include:

$2 - ({}^{-}3)$____1 ${}^{-}x$____x ${}^{-}({}^{-}8)$_____8

In this case, you might ask students, "When do we use parentheses in mathematics?" Students might say they are used for grouping a series of computations to show what to do first and that it can also mean multiplication. Point out that parentheses are also used to make a number sentence more readable separating the negative number from the operation. ELLs may benefit from additional vocabulary support, such as hand gestures, writing the term "parentheses," and practicing it. Also, not all countries use the same notations, so you may see different ways of notating expressions involving the minus sign.

Tools for Illustrating Positive and Negative Numbers

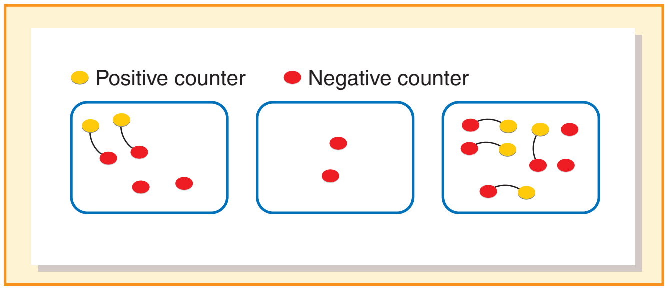
Two tools, one denoted by quantity and the other by linear operations, are popular for helping students understand comparisons and the four operations (+, −, ×, and ÷) with positive and negative numbers. Counters can be used for integer values (counting numbers, including 0), and number lines are needed to illustrate positive and negative noncounting numbers.

Counters

Two-color counters are a great fit for showing integers because one side (color) can represent positive counts and the other side (color) can represent negative counts. One counter of each type results in zero (${}^{+}1 + {}^{-}1 = 0$), illustrating that they are opposites. Consider money: If yellow counters are credits and red counters are debits, having 5 yellows and 7 reds is the same as having 2 reds or 2 debits and is represented as $^{-}2$ (Figure 10.7). It is important for students to understand that it is always possible to add to or remove from a collection of any number of pairs consisting of one positive and one negative counter without changing the value (i.e., it is like adding equal quantities of debits and credits).

Figure 10.7

Each collection is a model of negative 2.



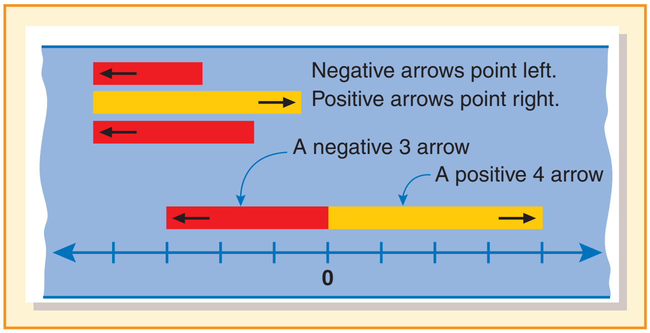
Number Lines

The number line is the second model or tool for computing with negative numbers, and it has a number of advantages. A number line:

1. Is familiar to students from their computations with whole numbers, fractions, and decimals (see Chapters 8 and 9).
2. Is an excellent tool for representing the operations conceptually in terms of increasing or decreasing amounts. To add a context, consider using small cutouts of grasshoppers that jump up and down the line (Swanson, 2010). Or, use your school's mascot! Students can see that as the animal moves to the left, it goes to smaller numbers, and as it moves to the right, it goes to larger numbers.
3. Shows the distance from zero (or the absolute value of the number).
4. Allows students to explore noninteger negative and positives values (e.g., $^{-}4\frac{1}{4} + 3\frac{3}{3}$ or $^{-}9.2 - 4.5$), which is difficult to do with counters.
5. Connects to the coordinate axis, which involves both a vertical and a horizontal number line.

Figure 10.8

Using the number-line to model negative and positive numbers.



Arrows can be used in integer computation to show distance and direction. For example, 4 can be modeled with an arrow four units long starting at any location on the number line and pointing to the right, and $^{-}3$ can be modeled with an arrow three units long starting at any location on the number line and pointing to the left (Figure 10.8). The arrows help students think of integer quantities as directed distances. A positive arrow never points left, and a negative arrow never points right. Furthermore, each arrow is a quantity with both length (magnitude or absolute value) and direction (sign). These properties are constant for each arrow regardless of its position on the number line.

Which Model to Use

Although the two models of counters and the number line appear quite different, they are alike mathematically. Positive and negative numbers involve two concepts—magnitude and direction. Magnitude is represented by the number of counters or the length of the arrows. Direction is represented by colors or directions. Seeing how positive and negative numbers are represented across these two tools and making connections between them can help students extract the intended concepts. The context might also influence the representation: if the context is height, the number line is a better fit.

Operations with Positive and Negative Numbers

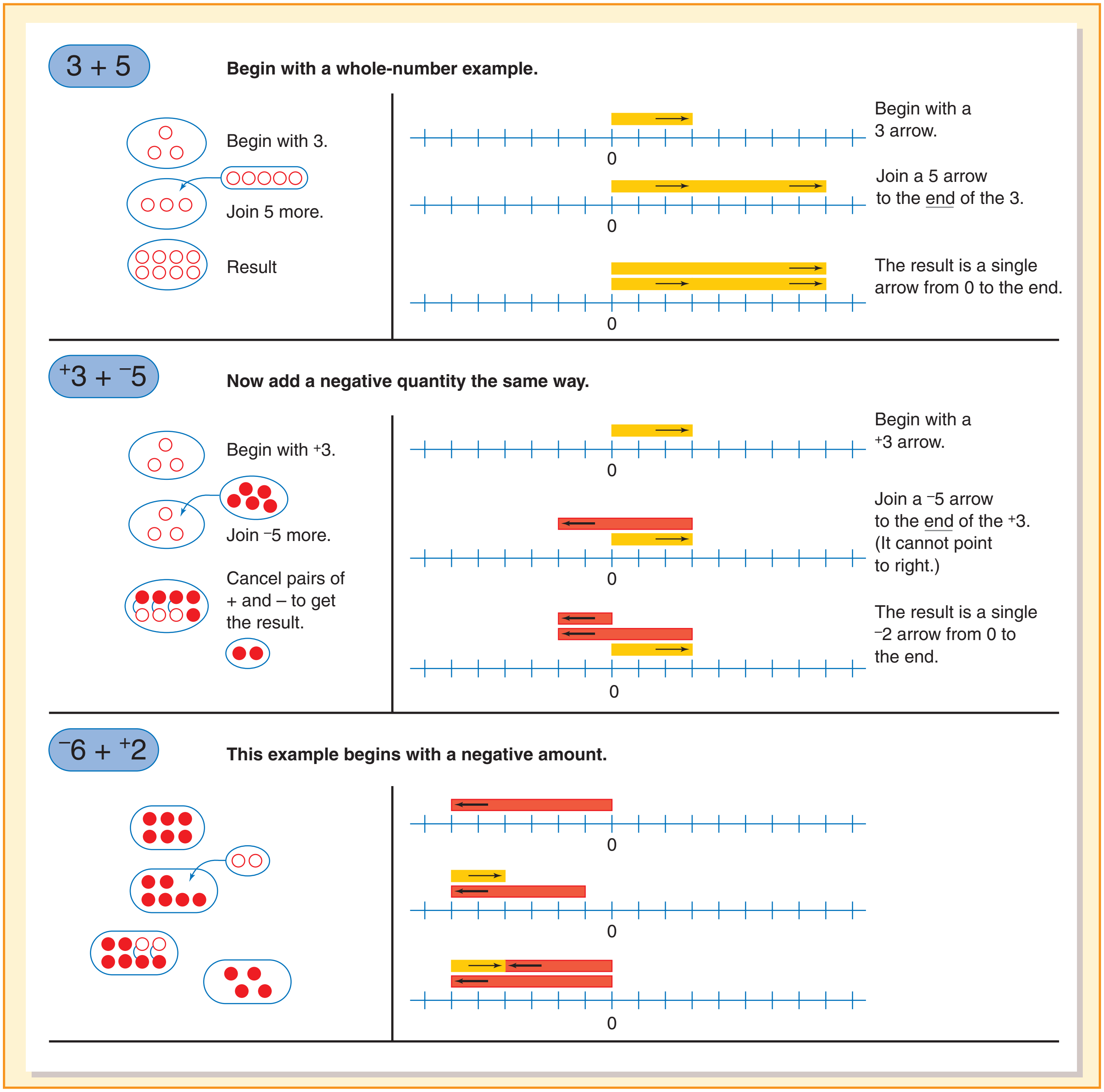
Once your students understand how integers are represented, in particular as a value on a number line, they are ready to solve problems involving operations with negative numbers.

Addition and Subtraction

Introduce negative values in one of the contexts discussed earlier, such as golf scores. Personalize the story by telling students that each weekend, you golf a round on Saturday and on Sunday. The first weekend, your results were $^{+}3$ and $^{+}5$, the next weekend you scored $^{+}3$ and $^{-}5$, and on the last weekend you scored $^{-}6$ and $^{+}2$. How successful was your game each weekend? Overall? Because this is a quantity model, counters are a good choice for illustrating (although number lines can also be used). A linear context could be used with football yards gained and lost on two plays. See Figure 10.9 for illustrations of how to use both models for addition.

Figure 10.9

Addition with counters and number lines.



Conversely, ask students to create their own stories for integer operations. One way to scaffold this is to ask students three prompts (Swanson, 2010):

1. Where did you start?
2. How far did you go?
3. Where are you now?

So for $^{-}3 - 5 = ?$, a student might say or write, “I was 3 feet under water, then dove down 5 feet. Where am I now?” (-8).

Standards for Mathematical Practice

2 Reason abstractly and quantitatively. ▶

Contexts are an important place to begin, but students also need to be able to reason about the numbers themselves. It is important to focus on the meaning of opposites. Activity 10.12 provides an excellent way to explore integers quantitatively, with a focus on having students use opposites as they think about integer addition (Friedmann, 2008).

Activity 10.12

CCSS-M: 7.NS.1a, b, c

Find the Zero

Before beginning the activity, ask students to tell you the sums of several opposites, such as $4 + {}^{-}4$. Distribute Finding the Zero Activity Page. Ask students to look at a sum of integers that are not opposites, such as $7 + {}^{-}4$, and see if they can “find a zero” by decomposing one of the numbers—for example, $(3 + 4) + {}^{-}4$ and solve. Students, particularly students with disabilities, may benefit from creating a “zero box” below each problem as they solve it, as illustrated. (For details, see Expanded Lesson: Finding the Zero.)

$$12 + {}^{-}5 =$$

$$\text{Zero box } \boxed{5 + {}^{-}5}$$

$$(7 + 5) + {}^{-}5 = 7 + (5 + {}^{-}5) = 7 + 0 = 7$$

Students must continue to illustrate what is happening when they are adding and subtracting negative numbers. If they do not, they will get lost in the symbols and will not know in which direction to head on the number line. Figure 10.9 provides examples of addition problems that are illustrated by using different approaches with the two models: with positive and negative counters and with the number line and arrows.

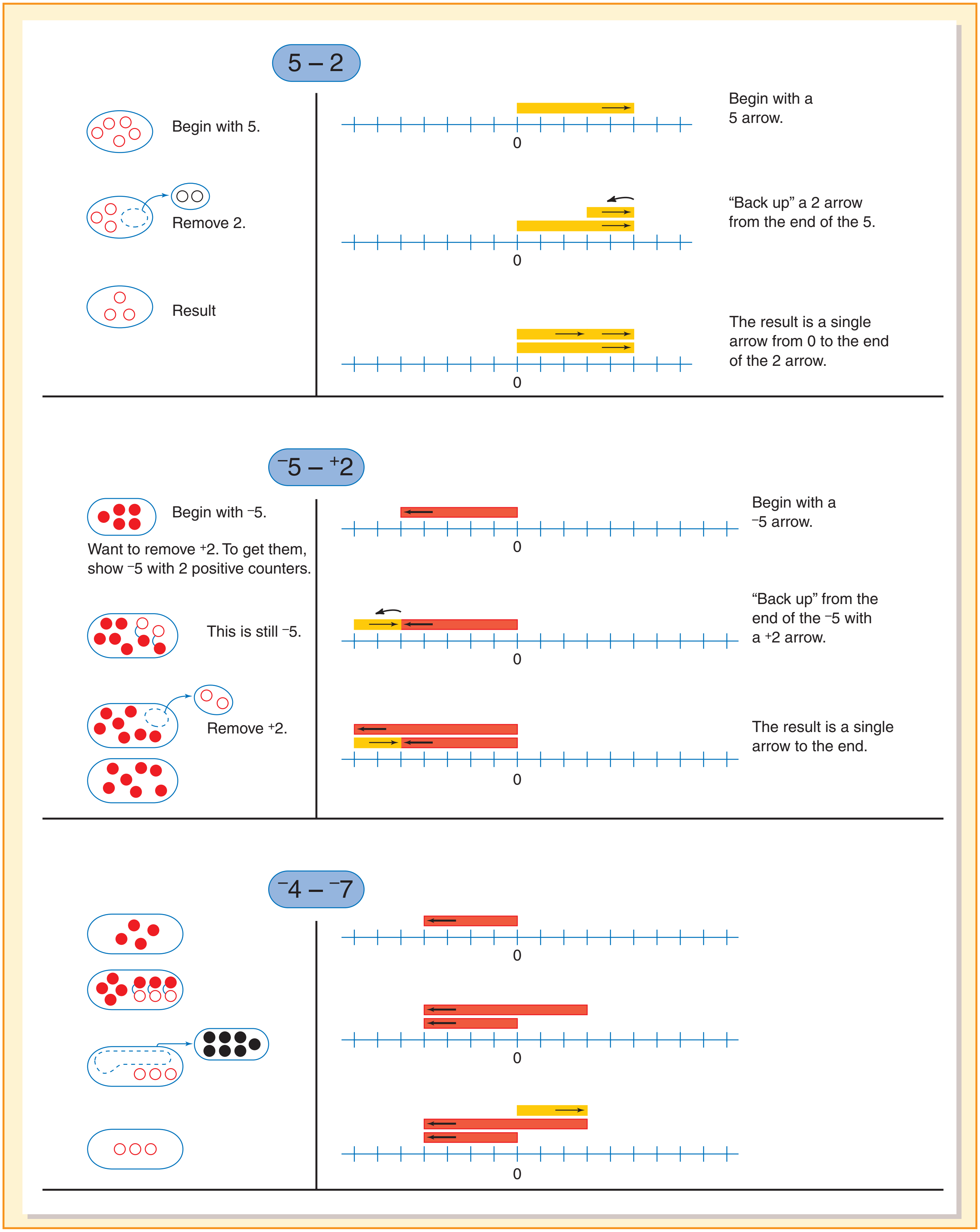
To add with the number line, each added arrow begins at the head of the previous arrow. Subtraction can be used for separate (take-away) situations (e.g., start with 7 and take away 10) or for comparison situations (e.g., what is the difference or distance between 7 and $^{-}3$?). An advantage to the number-line model is that it can be used for both separate and comparison situations.

Consider the problem $^{-}5 - {}^{+}2$, the second example illustrated in Figure 10.10. If a quantity model is used, the context could be money, such as, “I start with a debit of \$5 and then withdraw (take out) \$2 more from my account. What balance will my bank account show (if no fees have been charged yet for my overdrawn account)?”

To show this problem, you start with the five red counters. To remove two positive counters from a set that has none, a different representation of $^{-}5$ must first be made. Because any number of neutral pairs (one positive, one negative) can be added without changing the value of the set, two pairs are added so that two positive counters can be removed. The net effect is to have more negative counters.

Figure 10.10

Subtraction illustrated with counters and on the number line.



With the number line, subtraction can be illustrated by using arrows for separate and comparison situations. Consider a separate (take-away) situation as a way to think about the second example in Figure 10.10. Using temperature as a context, the explanation could be, “The day begins at 5° below zero. Then the temperature drops 2, which means it just got colder and is now $^{-}7$°.” The difficulty in the take-away thinking comes when you try to provide an authentic explanation of subtracting a negative value.

Consider, for example, $^{-}4 - {}^{-}7$ (see the third example in Figure 10.10). You start with taking away, but because it is negative temperature (or coldness) that is being taken away, you are in fact doing the opposite—warming up by 7°. With the number line, you start at $^{-}4$, then reverse the arrow going left to one going right for 7 moves.

Number lines can also be used for comparison or distance situations. For subtraction, this can make a lot more sense to students (Tillema, 2012). In this example, the comparison question is, “What is the difference from $^{-}7$ to $^{-}4$? In other words, how do you get *from* $^{-}7$ *to* $^{-}4$? You count up 3. Notice that if this were written in reverse ($^{-}7 - {}^{-}4$), it would be the difference *from* $^{-}4$ *to* $^{-}7$, which is still 3, but the direction is to the left, so $^{-}3$.

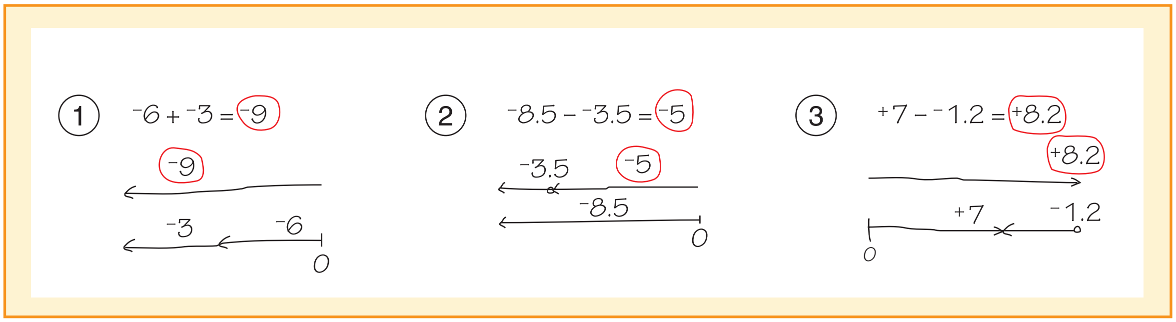
Stop and Reflect

Try to explain the problems in Figures 10.9 and 10.10 by using both a quantity and a linear context. For subtraction, explain each in a separate situation and a comparison situation. What kinds of stories or contexts might fit with these different ways to think about addition and subtraction?

An important and challenging task for students is to connect visual representations to their symbolic work. One way to help students is to ask them to make basic number line illustrations with their equations. Figure 10.11 illustrates how a student might draw arrows to represent addition and subtraction exercises. You can also have students write story situations as a third representation. This helps students to build a stronger understanding while helping you better identify any misconceptions or error patterns.

Figure 10.11

Students use arrow sketches to represent addition and subtraction.

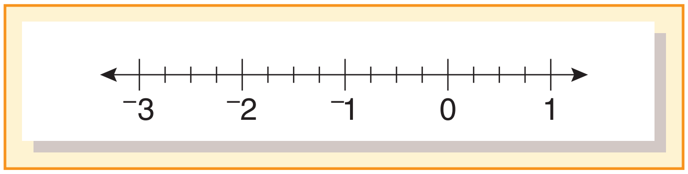


Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

It is important for students to see that $^{+}3 + {}^{-}5$ is the same as $^{+}3 - {}^{+}5$, and that $^{+}2 - {}^{-}6$ is the same as $^{+}2 + {}^{+}6$. Illustrating addition and subtraction problems on the number line and explicitly discussing how the two expressions are related will help students see the connections between these expressions.

In seventh grade, students must learn to “solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically” (CCSSO, 2010, p. 49). The examples in this section have been limited to integers, but with the use of a ruler or any number line partitioned into fractional amounts, the same arrow illustrations can help students reason about rational number addition and subtraction.



technology note

Two good websites have applets for exploring addition and subtraction of integers: (1) The National Library of Virtual Manipulatives has “Color Chips—Addition” and “Color Chips—Subtraction,” and (2) NCTM Illuminations has “Volt Meter.” There are audio podcasts for adding and subtracting integers available on iTunes; for example, there is a videocast of a teacher sharing how she uses Inspiration software.

Multiplication

Multiplication of integers is an extension of multiplication of whole numbers, fractions, and decimals. One way to think of whole-number multiplication is equal groups, or repeated addition. The first factor tells how many sets there are or how many are added in all, beginning with zero. This translates to integer multiplication quite readily when the first factor is positive, regardless of the sign of the second factor. The first example in Figure 10.12 illustrates a positive first factor and a negative second factor ($3 \times {}^{-}4$), which translates into the question, “If I have three groups of $^{-}4$, how much do I have?” With a context, this could be any of the following: I lost four dollars three days in a row; how much have I lost? Three days in a row Hans scored $^{-}4$ in his golf tournament; what is his score? Three times today the temperature dropped four degrees; how much has the temperature changed? Connecting to contexts and looking at repeated examples can then lead to students creating conjectures about the ‘rules’ for multiplying (and dividing) with negative numbers (Choppin, Clancy, & Koch, 2012). That is the focus of the next activity.

Activity 10.13

CCSS-M: 7.NS.A.2a, b

Creating Stories and Conjectures for Operations with Negative Numbers

This activity is described here for multiplication but could be replicated for any of the operations with negative numbers. Ask students to make a conjecture about what they think the sign will be on a product of a positive factor and a negative factor. Discuss briefly, or ask students to submit a vote electronically. Next, give students a set of multiplication problems, such as:

$3 \times {}^{-}10 =$ $\quad$ $5 \times \frac{^{-}1}{4} =$ $\quad$ $8 \times {}^{-}0.5 =$

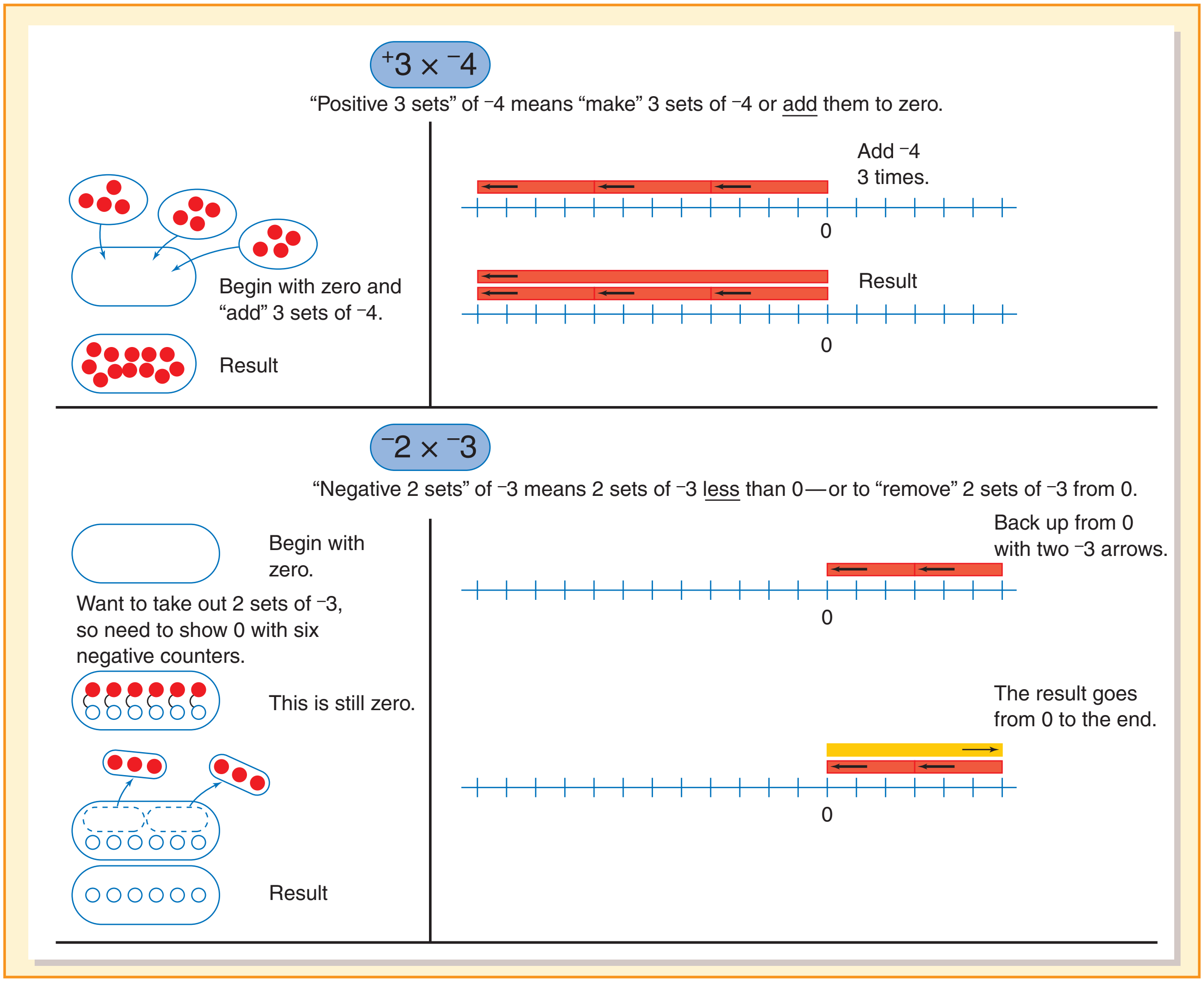
Invite students to create the stories to match each of the equations. You can display a menu of contexts (see the list of contexts provided in the section above). After telling stories for each, ask students to repeat the process for situations involving a negative times a positive number:

$^{-}3 \times 4 =$ $\quad$ $^{-}5 \times \frac{1}{2} =$ $\quad$ $^{-}10 \times 0.25 =$

Then ask students to revisit and revise (as needed) their conjectures regarding the product of a negative factor times a positive factor. Be sure to connect the conjectures to the examples students have created.

Figure 10.12

Multiplication by a positive first factor is repeated addition. Multiplication by a negative first factor is repeated subtraction.



What could the meaning be when the first factor is negative, as in $^{-}2 \times {}^{-}3$? If a positive first factor means repeated addition (how many times added to zero), a negative first factor should mean repeated subtraction (how many times subtracted from zero). The second example in Figure 10.12 illustrates how multiplication with the first factor negative can be represented.

Teaching Tip

Be sure that *students* are creating stories to match the situations. Create a menu of contexts and display them for reference.

technology note

The National Library of Virtual Manipulatives has two virtual explorations that support student understanding of multiplication with negative numbers: "Rectangle Multiplication of Integers" and "Integer Arithmetic."

Division

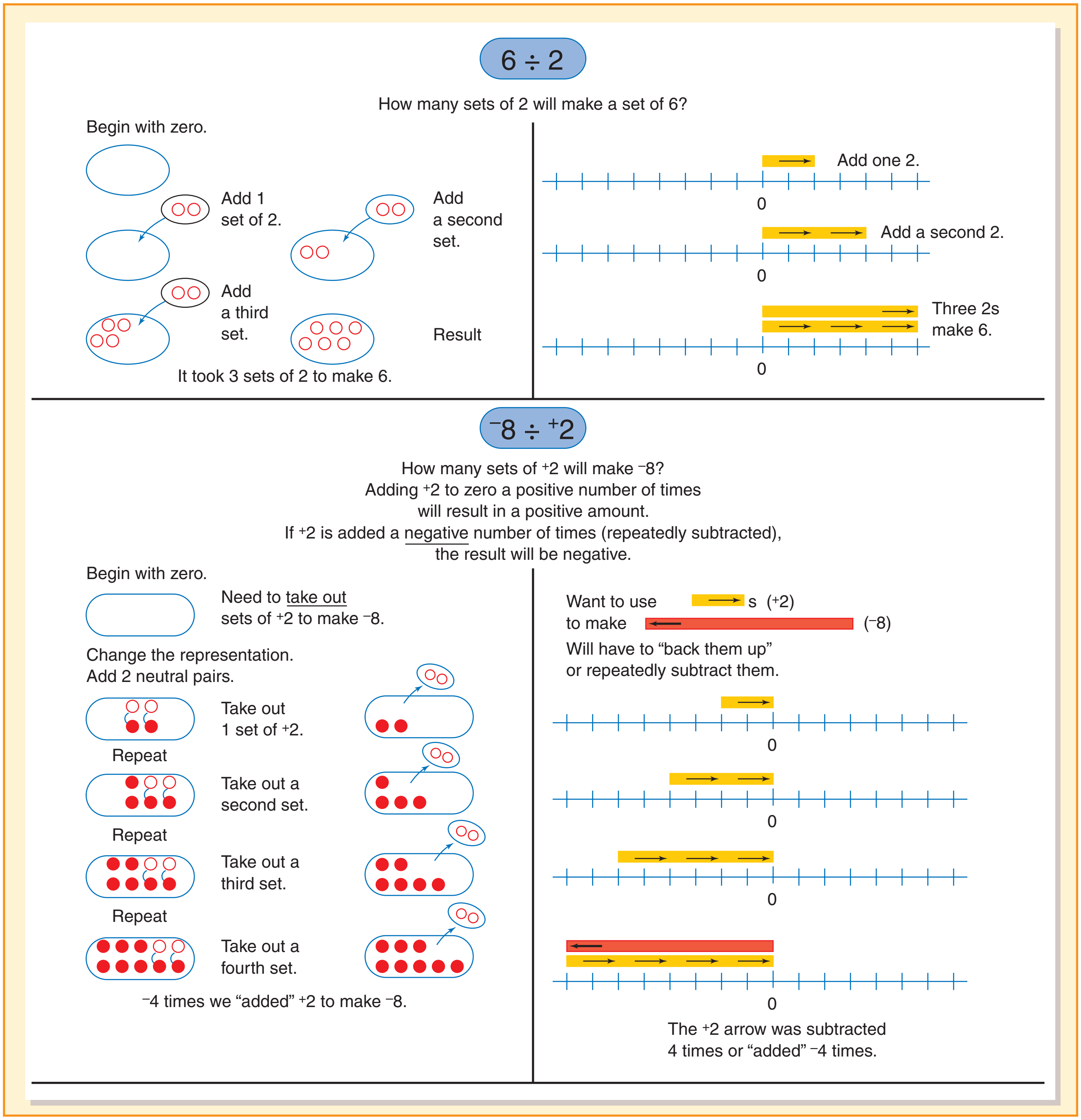
Connect division of negative numbers with what students know about positive numbers. Recall that 8 ÷ 4 with whole numbers has two possible meanings corresponding to two missing-factor expressions. The equation ? × 6 = 24 asks, "How many groups of six equal twenty-four?"

(number of groups unknown). This fits well with thinking about negative values because it lends to skip counting to 24. The first example in Figure 10.13 illustrates how the two visual models (two-color counters and number line) work for whole numbers.

Understanding integer division rests on a good conceptual understanding of multiplication problems in which the first factor is negative (e.g., $^{-}3 \times 5$) and knowledge of the relationship between multiplication and division. Encourage students to think first about how to visualize the whole-number situation and then connect that understanding to negative numbers. Extend these explorations to decimal and fraction values.

Figure 10.13

Division of integers following a measurement approach.



Real Numbers

Whole numbers, fractions and decimals, and integers are all rational numbers because they can all be written as a fraction with an integer over a nonzero integer. An interesting fact is that there are infinitely many rational numbers between any two numbers. Exploring this idea can deepen students' understanding of rational numbers.

Activity 10.14

CCSS-M: 6.NS.C.5; 7.NS.A.2d

How Many in Between?

Post two fractions for students, such as $\frac{1}{2}$ and $\frac{9}{10}$. Ask students to locate those fractions on a number line, and then to insert four additional fractions *between* the two given fractions. Encourage students to use different strategies to reason about what fractions come in between. Students may convert one-half to five-tenths and readily find $\frac{6}{10}$, $\frac{7}{10}$, and $\frac{8}{10}$, , but be challenged to find the fourth equivalence (there are many, e.g., $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{8}{9}$). Or, they may convert to decimal equivalencies to find decimals and then convert the decimals back to fractions. See Expanded Lesson: How Many In Between? for a full lesson exploring rational numbers between two rational numbers.

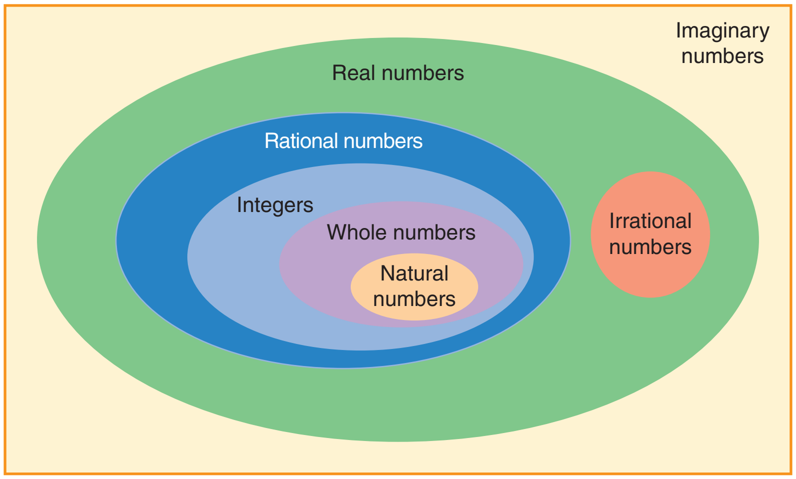
Irrational numbers are numbers such as $\sqrt{2}$ numbers whose value cannot be written as a fraction and whose exact value can only be estimated. Eighth graders begin to explore irrational numbers and find their rational approximation (CCSSO, 2010). All these numbers are a part of the *real numbers*, which are the only types of numbers students explore until high school, where they consider the square roots of negative numbers, called *imaginary numbers.* These sets of numbers are interrelated, and some are subsets of other sets. Figure 10.14 provides an illustration of the types of numbers and how they are interrelated.

Standards for Mathematical Practice

7 Look for and make use of structure.

Figure 10.14

An illustration of the organization of the real numbers.



Rational Numbers

Rational numbers comprise the set of all numbers that can be represented as a fraction—or a ratio of an integer to an integer. Even when numbers are written as whole numbers or as terminating decimals, they can also be written as fractions and thus are rational numbers. In fact, in school mathematics the term *rational numbers* is often used to refer to fractions, decimals (terminating and repeating), and percents. These are rational numbers, but so are integers, including whole numbers.

Moving among Representations

In sixth grade, students should be able to recognize a rational number as a point on a number line (CCSSO, 2010). In seventh grade, "students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers" (CCSSO, 2010, p. 46). This means that given any value, students are able to think about it and operate on it across representations, moving flexibly among different representations. For example, students should be able to explain equivalence, as noted here.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

- $4\frac{3}{5}$ is equivalent to 4.6 because $\frac{3}{5}$ is six-tenths of a whole, so 4 wholes and six-tenths is 4.6.
- $4\frac{3}{5}$ is equivalent to $\frac{23}{5}$, and that is the same as $23 \div 5$, or 4.6 if I use decimals.
- 4.6 is read "four and six-tenths," so I can write that as $4\frac{6}{10} = 4\frac{3}{5}$.

Similarly, compare these three expressions:

$$\frac{1}{4} \text{ of } 24 \qquad\qquad \frac{24}{4} \qquad\qquad 24 \div 4$$

This discussion can lead to a general development of the idea that a fraction can be thought of as division of the numerator by the denominator, or that $\frac{a}{b}$ is the same as $a \div b$.

When a fraction is converted to a decimal, the decimal either terminates (e.g., 3.415) or repeats (e.g., 2.5141414 . . .). Is there a way to tell in advance whether a given fraction is a terminating decimal or a repeating decimal? The following activity can be used to discover if that prediction is possible.

Activity 10.15

CCSS-M: 7.NS.A.2d

Repeater or Terminator?

Have students generate a table listing in one column the first 20 unit fractions $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{21}\right)$. The second column is for listing the prime factorization of the denominators, and the third column is for listing the decimal equivalents of the fractions (use calculators to get the precise decimal form).

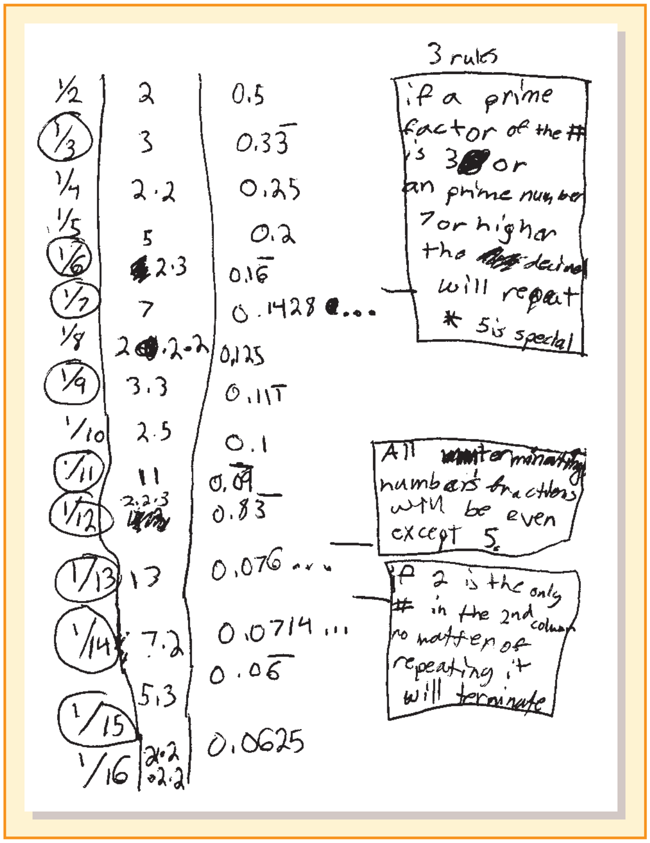
After they have completed the table, ask students if they can determine a rule that will tell in advance whether the decimal will repeat or terminate. They can test the rule with fractions that have denominators beyond 21. Challenge students to confirm that their rule applies even if the numerator changes.

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

Figure 10.15

A student notes patterns while exploring the "Repeater or Terminator" activity.



1/2	2	0.5
1/3	3	$0.3\overline{3}$
1/4	2.2	0.25
1/5	5	0.2
1/6	2.3	$0.1\overline{6}$
1/7	7	0.1428...
1/8	2.2.2	0.125
1/9	3.3	$0.1\overline{1}$
1/10	2.5	0.1
1/11	11	$0.\overline{09}$
1/12	2.2.3	$0.8\overline{3}$
1/13	13	0.076...
1/14	7.2	0.0714...
1/15	5.3	$0.0\overline{6}$
1/16	2.2.2.2	0.0625

3 rules

if a prime factor of the # is 3 or an prime number 7 or higher the decimal will repeat
* 5 is special

All terminating numbers fractions will be even except 5.

If 2 is the only # in the 2nd column no matter of repeating it will terminate

As students work on this task, they will notice various patterns, as can be seen in the student work provided in Figure 10.15. As this student has discovered, the only fractions with terminating decimals have denominators that factor only with combinations of twos and fives. Why is this so?

Middle-school students must understand that any rational number, positive or negative, whole or not whole, can be written as a fraction and as a decimal. So, $^{-}8$ can be written as the fraction $\frac{^{-}8}{1}$ or $\frac{^{-}16}{2}$, $\frac{^{-}800}{100}$ or as the decimal $^{-}8$ or $^{-}8.0$. In fact, there are infinite ways to write equivalences for $^{-}8$. Fluency with equivalent representations is critical and requires much more than teaching an algorithm for moving from one representation to another.

Irrational Numbers

Students encounter *irrational numbers* in seventh grade when they learn about π and continue exploration of irrational numbers more explicitly in eighth grade (CCSSO, 2010). As noted earlier, *irrational* numbers are not rational, meaning they cannot be written in fraction form. The irrational numbers together with the rational numbers make up the *real numbers*. The real numbers fill in all the holes on the number line, even when the holes are infinitesimally small.

technology note

YouTube has some interesting videos. First, there is a very old but great middle-school video, "The Weird Number." This is a scary (funny) story of a village of whole numbers in the mountains that "hear" there are other numbers beyond the hills. Second, see the Discovery Education video about rational, irrational, and real numbers.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them

Students' first experiences with irrational numbers typically occur when they explore square roots of whole numbers. The following activity provides a good introduction to square roots and cube roots.

Activity 10.16

CCSS-M: 8.NS.A.1; 8.EE.A.2

Edges of Squares and Cubes

Show students pictures of three squares (or three cubes), as in Figure 10.16. The sides (squares) and edges (cubes) of the first and last figure are consecutive whole numbers. The areas or volumes of all three figures are provided. Ask students to use a calculator to find the length of the sides (squares) or edges (cubes) of the figure in the center. Explain to students that they are not to use the square root or cube root key; they are to estimate what they think the length of the side would be and test it by squaring or cubing it. Ask students to continue to estimate until they have found a value to the hundredths place that gets as close to 45 as possible (or 30 in the case of the cube). Solutions will satisfy these equations:

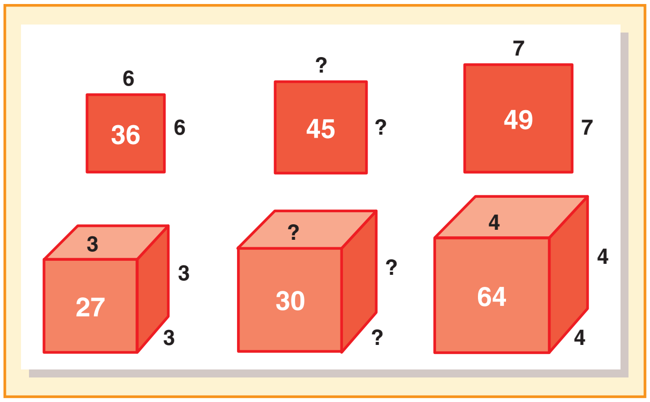
$\square \times \square = 45$, or $\square^2 = 45$

and

$\square \times \square \times \square = 30$, or $\square^3 = 30$

Figure 10.16

A geometric interpretation of square roots and cube roots.



To solve the cube problem, students might start with 3.5 and find that 3.5^3 is 42.875, which is much too large. Through trial and error, they will find that the solution is between 3.1 and 3.2. Continued use of strategic trial and error will lead to a close approximation. Although a calculator can find these square or cube roots quickly, the estimation activity strengthens students' understanding of squares and square roots and the relative sizes of numbers.

From this introduction, students can be challenged to find solutions to equations such as $\square^2 = 8$. At this point, students are ready to understand the general definition of the nth root of a number, x, as the number that when raised to the nth power equals x. The square and cube roots are simply other names for the second and third roots. It is important to point out that $\sqrt{6}$ is a *number*, not a computation (since it looks so much like a division problem). The cube root of eight is the same as $\sqrt[3]{8}$, which is equivalent to 2.

Teaching Tip

Point out that $\sqrt{6}$ is a number, *not* a computation. (This is a significant point of confusion for students because the symbol looks like a division problem.) $\sqrt[3]{8}$ is equivalent to 2. Both are numbers.

Pythagorean Theorem

In middle school, students encounter irrational numbers primarily when working with the Pythagorean Theorem ($a^2 + b^2 = c^2$), which is used to find the distance between two points (the distance being the diagonal, or c). If $a = 3$ and $b = 4$, then $c = 5$. All sides are rational numbers. But this result is only in special cases. More often, sides will be something like 4 and 7 units, in which case $c = \sqrt{16 + 49} = \sqrt{69}$. Although there is sometimes a perfect square that can be simplified, in this case there isn't one, and the distance is $\sqrt{69}$, an irrational number.

An engaging middle-school project applying the Pythagorean Theorem and irrational numbers is a Wheel of Theodorus, described in Activity 10.14. Theodorus was one of the early believers in the existence of irrational numbers. (This was quite a contentious issue for the Pythagoreans, who were against the idea of irrationals!)

Activity 10.17

CCSS-M: 8.NS.A.2; 8.EE.A.2

Wheel of Theodorus

Ask students to construct a right triangle that measures 1 cm on each side adjacent to the right angle and then draw the hypotenuse and record its measure. They then use the hypotenuse as side *a* of a new right triangle, drawing side *b* 1 cm long. Connect the end points of side *a* and side *b*. Draw and record the new hypotenuse ($\sqrt{3}$). Create the next triangle, which will have sides of $\sqrt{3}$ and 1 and a hypotenuse of $\sqrt{4}$ or 2, and so on. Doing this about 30 times will form a wheel. (See Bay-Williams & Martinie, 2009, for a complete lesson, or search online for instructions and diagrams of a Wheel of Theodorus.)

technology note

The Math Page has a link called "The Evolution of the Real Numbers" this is an interesting description of many topics related to the real-number system. Although mostly text, the pages are filled with interactive questions.

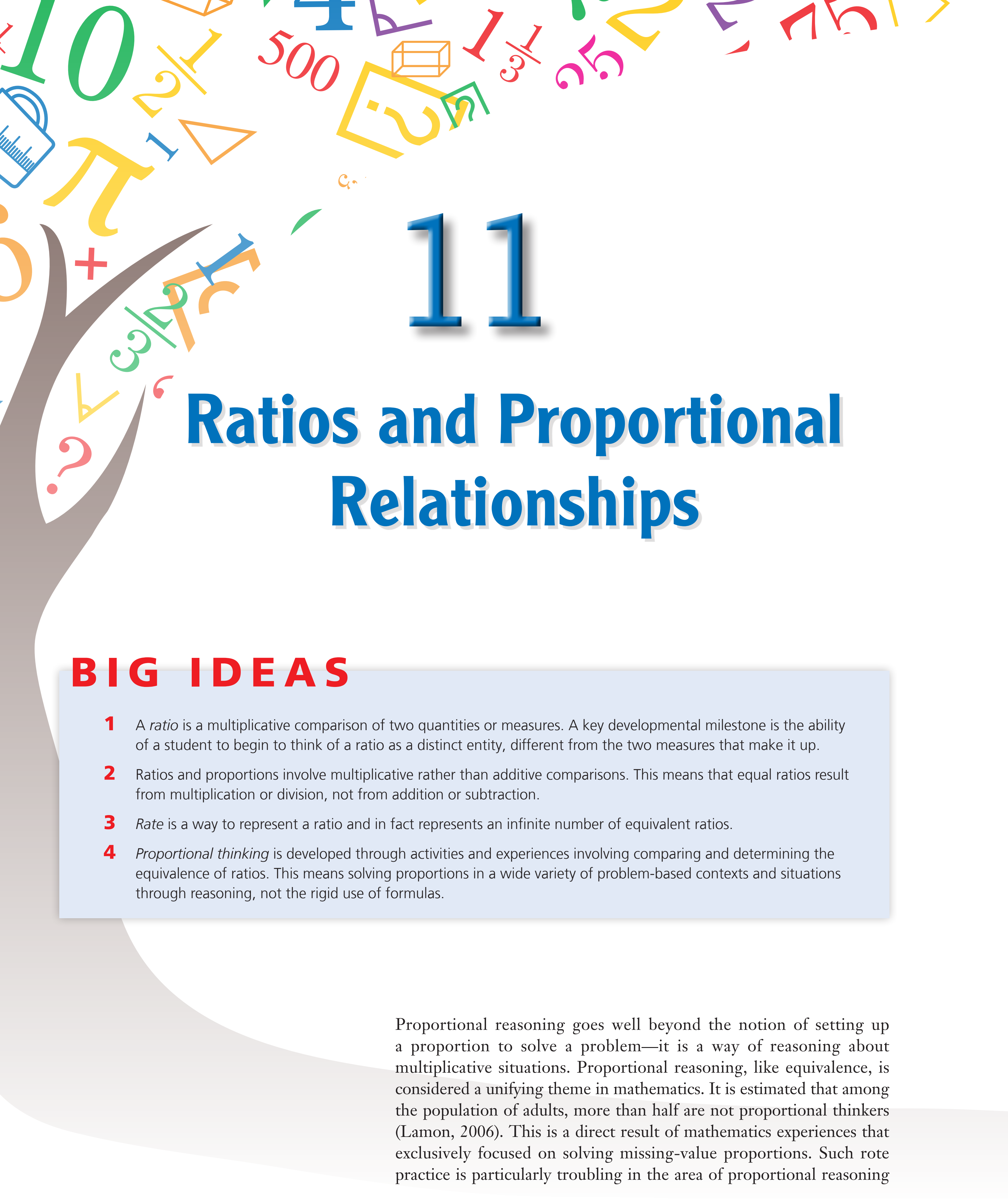
Literature Connections

The Number Devil *Enzensberger, 1997*

Full of humor and wit, *The Number Devil* presents a collection of interesting ideas about numbers. Robert, a boy who hates mathematics, meets up with a crafty number devil in each of 12 dreams (one per chapter). On the fourth night's dream, Robert learns about infinitely repeating decimals and the "Rutabaga of two" (the square root of two), providing a connection to rational and irrational numbers.

Oh, Yikes! History's Grossest, Wackiest Moments *Masoff and Sirrell, 2006*

In this picture-rich reference book, the authors describe important historical events and people with facts that are interesting to middle schoolers (e.g., "Aztec Antics," "Cruel Constructions," "Humongous Hoaxes," "Pirates"). Selecting a topic, such as brushing teeth, the author describes how this was handled across all of history—an opportunity for time lines that include dates such as 2500 BC. Students can create a time line that is proportionally accurate to tell the events related to their topic. This lesson involves integers, measuring, proportional reasoning, and fractions.



11

Ratios and Proportional Relationships

BIG IDEAS

1. A *ratio* is a multiplicative comparison of two quantities or measures. A key developmental milestone is the ability of a student to begin to think of a ratio as a distinct entity, different from the two measures that make it up.
2. Ratios and proportions involve multiplicative rather than additive comparisons. This means that equal ratios result from multiplication or division, not from addition or subtraction.
3. *Rate* is a way to represent a ratio and in fact represents an infinite number of equivalent ratios.
4. *Proportional thinking* is developed through activities and experiences involving comparing and determining the equivalence of ratios. This means solving proportions in a wide variety of problem-based contexts and situations through reasoning, not the rigid use of formulas.

Proportional reasoning goes well beyond the notion of setting up a proportion to solve a problem—it is a way of reasoning about multiplicative situations. Proportional reasoning, like equivalence, is considered a unifying theme in mathematics. It is estimated that among the population of adults, more than half are not proportional thinkers (Lamon, 2006). This is a direct result of mathematics experiences that exclusively focused on solving missing-value proportions. Such rote practice is particularly troubling in the area of proportional reasoning because proportional reasoning is at the core of so many important concepts, including "similarity, relative growth and size, dilations, scaling, pi, constant rate of change, slope, speed, rates, percent, trigonometric ratios, probability, relative frequency, density, and direct and inverse variations" (Heinz & Sterba-Boatwright, 2008, p. 528). Wow, that's a lot of concepts! Study of proportional relationships begins in elementary school with work in measurement and in multiplication and division. In the CCSS-M, the progression looks like this:

> *Grade 4:* Solve multiplication problems involving a multiplicative comparison.
>
> *Grade 5:* Identify relationships between two varying quantities.
>
> *Grade 6:* Critical area that focuses on connecting ratio and rate to multiplication and division; students solve problems involving ratios and rates.
>
> *Grade 7:* Another critical area, this year focuses on extending knowledge of ratios to proportionality, solving multistep problems. Proportions are written as equations and graphed, as well as used to solve problems involving interest, tax, gratuities and so on.

Ratios

A *ratio* is a number that relates two quantities or measures within a given situation in a multiplicative relationship (in contrast to a difference or additive relationship). Reasoning with ratios involves paying attention to two quantities that covary.

Types of Ratios

Part-to-Part Ratios

A ratio can relate one part of a whole (9 girls) to another part of the same whole (7 boys). This can be represented as $\frac{9}{7}$, meaning "a ratio of nine to seven," not "nine-sevenths" (the fraction). In other words, a part-to-part ratio is not a fraction, although it can be written with the fraction bar; the context is what tells you it is a part-to-part ratio.

Part-to-part ratios occur across the curriculum. In geometry, corresponding parts of similar geometric figures are part-to-part ratios. The ratio of the diagonal of a square to its side is $\sqrt{2}$. In algebra, the slope of a line is a ratio of *rise* for each unit of horizontal distance (called the *run*). The probability of an event is a part-to-whole ratio, but the odds of an event occurring are a part-to-part ratio.

Part-to-Whole Ratios

Ratios can express comparisons of a part to a whole—for example, the ratio of the number of girls in a class (9) to the number of students in the class (16). This can be written as the ratio $\frac{9}{16}$ or can be thought of as nine-sixteenths of the class (a fraction). Percentages and probabilities are examples of part–whole ratios.

Ratios as Quotients

Ratios can be thought of as quotients. For example, if you can buy 4 kiwis for $1.00, the ratio of money for kiwis is $1.00 to 4 kiwis. The cost per kiwi ($0.25) is the unit rate.

Ratios as Rates

Rates are a subset of ratios that involve two different units and how they relate to each other. Miles per gallon, square yards of wall coverage per gallon of paint, passengers per busload, and roses per bouquet are all rates. For example, 36 roses in 6 bouquets, 600 roses in 100 bouquets, and 120 roses in 20 bouquets are all equivalent ratios. Relationships between two units of measure are also rates—for example, inches per foot, milliliters per liter, and centimeters per inch. A rate represents an infinite set of equivalent ratios (Lobato, Ellis, Charles, & Zbiek, 2010).

Ratios Compared to Fractions

Ratios are closely related to fractions, but ratios and fractions should be thought of as overlapping concepts with important distinctions (Lobato et al., 2010). Because they are both represented symbolically with a fraction bar, it is important to help students see that fractions and ratios are related but not the same thing. In the three examples given here, ask, "Is this a fraction, ratio, or both?"

---

1. The ratio of cats to dogs at the pet store is $\frac{3}{5}$.

   This ratio is not a fraction because fractions are not part-to-part ratios.

2. The ratio of cats to pets at the pet store is $\frac{3}{8}$.

   This can be adapted to say that three-eighths of the pets are cats. Since this is a part-to-whole ratio, it is both a ratio and a fraction.

3. Mario walked three-eighths of a mile ($\frac{3}{8}$ mile).

   This is a fraction of a length and is not a ratio because there is not a multiplicative comparison.

Stop and Reflect

The distinctions between ratios and fractions are subtle. How might you help students notice and be able to articulate these distinctions?

Two Ways to Think about Ratios

Unfortunately, ratios are often addressed in a superficial manner, with students recording the symbols (3:5) to tell the ratio of girls to boys. Instead, ratios should be taught as relations that involve multiplicative reasoning. Lobato et al. (2010, p. 22) point out that "forming a ratio is a cognitive task—not a writing task." What they mean is that ratio is a relationship, and that the relationship can be thought of in different ways, regardless of whether its notation is $\frac{2}{5}$ or 2:5 or 2 ÷ 5. It is important to understand two ways to think about ratios: as multiplicative comparisons and as composed units.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Multiplicative Comparison

A ratio represents a multiplicative comparison, and that comparison can go either way. Consider the following relationship: Wand A is 8 inches long, and wand B is 10 inches long.

The ratio of the two wands is 8 to 10. But this statement does not necessarily communicate the *relationship* between the measures. There are two ways to compare the relationship multiplicatively:

> The short wand is eight-tenths as long as the long wand (or four-fifths the length).
>
> The long wand is ten-eighths as long as the short wand (or five-fourths or $1\frac{1}{4}$ the length).

The questions related to multiplicative comparisons can be worded two ways: "How many times greater is one thing than another?" Or "What fractional part is one thing of another?" (Lobato et al., 2010, p. 18). Notice that the ratio of the wands (a:b) can also be described as a unit rate ($\frac{a}{b}$). Understanding this relationship is a Grade 6 expectation in the CCSS-M. Also in Grade 6 in the CCSS-M is the expectation to solve real world problems involving ratios. These problems should include multiplicative comparisons, and those problems should increase in their difficulty (Cohen, 2013).

Activity 11.1

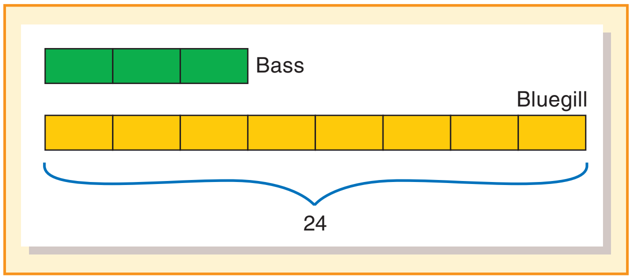
CCSS-M: 6.RP.A.1; 6.RP.A.2; 6.RP.A.3

Stocking the Pond

Use Stocking the Pond Activity Page or simply pose each of the three problems, one at a time, for students to solve. Ask students to use tape diagrams (also referred to as strip diagrams) as a tool to represent each problem. The first task is included here:

> **For every 3 bass that are put in a pond, Environmental Edwin puts in 8 blue gill. If Edwin puts 24 blue gill in the pond, how many bass does he need to put in the pond to preserve this ratio?**

Tape or Strip Diagram to show the ratio of bass to blue gill:



After students have solved the problem, ask questions, such as "What fractional part is the bass compared to the blue gill?" (three-eighths) and "What fractional part is the blue gill compared to the bass?" (eight-thirds). Continue to more questions that vary what information is unknown in the story. For ELLs, it is helpful to keep the same context to reduce the linguistic load and to provide visuals for the fish names. Also, be sure that the word "if" is understood in the situation to mean "assume."

Composed Unit

The term *composed unit* refers to thinking of the ratio as one unit. For example, if kiwis are 4 for $1.00, you can think of this as a unit and then think about other multiples that would also be true, like 8 for $2.00, 16 for $4.00, and so on. (Each of these would be a unit composed of the original ratio.) This is *iterating* (also discussed in Chapter 8). There can also be other *partitioning* of the composed unit: 2 for $0.50 and 1 for $0.25. Any number of kiwis can be priced by using these composed units.

It is important that students be able to apply both types of ratios. Activity 11.2 provides a context for thinking of a composed unit and then a multiplicative comparison.

Activity 11.2

CCSS-M: 6.RP.A.1; 6.RP.A.2; 6.RP.A.3

Birthday Cupcakes

Explain to students that they are going to be icing cupcakes and selling them at school. In a recipe for icing, the instructions say that to ice 1 batch of cupcakes with aqua icing, you will need 2 green drops of food coloring and 5 blue drops of food coloring. Ask students to figure out how many drops of food coloring will be needed for 1 batch of cupcakes, 2 batches, 5 batches, and so on (composed-unit thinking). Students may want to record their data in a table.

Next, ask students to figure out how many blue drops for 1 drop of green, and how much green for 1 drop of blue (multiplicative comparison). Ask students to think about how this information helps them in determining the number of drops of each color for the various numbers of batches. Students—particularly students with disabilities—may benefit from visualizing this comparison, which can be done by lining up green and blue tiles or counters (or use drawings).

Standards for Mathematical Practice

2 Reason abstractly and quantitatively. ▶

Proportional Reasoning

Contexts such as interest, taxes, and tips, as well as geometric contexts such as similar figures, graphing, and slope, involve comparing ratios, which requires proportional reasoning. Reasoning proportionally begins in kindergarten with one-to-one correspondence and continues through the grades with place value, fraction concepts, and multiplicative reasoning (Seeley & Schielack, 2007). And, in middle school it is perhaps the most significant topic, providing the foundation for algebra and geometry.

Proportional reasoning is difficult to define in a simple sentence or two. It is not something that you either can or cannot do. According to Lamon (2012), proportional thinkers do the following:

- Understand a *ratio as a distinct entity* representing a relationship different from the quantities it compares (see earlier discussion about the composed units)
- Recognize proportional relationships as distinct from nonproportional relationships in real-world contexts
- Have a sense of *covariation* (they understand relationships in which two quantities vary together and are able to see how the variation in one coincides with the variation in the other)
- *Develop a wide variety of strategies* for solving proportions or comparing ratios, most of which are based on informal strategies rather than prescribed algorithms

In other words, this list describes what fluency looks like related to ratios and proportions. These last three areas are addressed in the sections that follow.

Proportional and Nonproportional Situations

Students need opportunities to compare situations and discuss whether the comparison is due to an *additive*, *multiplicative*, or *constant* relationship (Van Dooren, De Bock, Vleugels, & Verschaffel, 2010). Importantly, a ratio is a number that expresses a multiplicative relationship (part-to-part or part-to-whole) that can be applied to a second situation in which the relative quantities or measures are the same as in the first situation. For example, in the kiwi problem, the first situation was 4 kiwis for $1.00, and this relative quantity (4 for $1.00 or 1 for $0.25) is true regardless of how many kiwis you buy (the second situation).

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

1. Janet and Jeanette are walking to school, each at the same rate. Jeanette started first. When Jeanette has walked 6 blocks, Janet has walked 2 blocks. How far will Janet have walked when Jeanette is at 12 blocks?
2. Lisa and Linda are planting corn on the same farm. Linda plants 4 rows, and Lisa plants 6 rows. If Linda's corn is ready to pick in 8 weeks, how many weeks will it take for Lisa's corn to be ready?
3. Kendra and Kevin are using the same recipe to bake cookies. Kendra will make 6 dozen cookies, and Kevin will make 3 dozen cookies. If Kevin uses 6 ounces of chocolate chips, how many ounces will Kendra need?

Stop and Reflect

Can you figure out which of the three problems above is an *additive, multiplicative,* or *constant* relationship? What are the differences in the wording of these situations that make each one an *additive, multiplicative*, or *constant* relationship? How can you help students distinguish between these types?

Think about the way quantities compare in each situation. The first situation is additive. Janet will still be 4 blocks behind, so 8 blocks. If the problem is incorrectly solved through multiplicative reasoning, however, you will get 4 blocks. The second situation is constant. It will take 8 weeks for the corn to grow, regardless of how many rows are planted. If the problem is solved through multiplicative reasoning, the incorrect answer is 12 weeks. The final situation is multiplicative, and the answer is 12 ounces. How did you do?

The way to get students to distinguish between these types of reasoning is to provide opportunities for them to discuss and compare the distinctions between these problem types. Consider the following sample problem, suggested by Cai and Sun (2002, p. 196) in their discussion of how teachers in Chinese classrooms introduce the concept of ratio:

Mr. Miller's 25 students are asked if they are basketball fans (yes or no). Of these students, 20 say yes and 5 say no. Describe as many relationships as you can about those who are basketball fans and those who are not.

Students might describe several different relationships:

- There are 15 more fans than nonfans.
- There are 4 times as many fans as nonfans.
- For every 4 students who like basketball, there is 1 student who does not.

Of these, the first is an additive relationship—focusing on the difference between the two numbers. The other two are variations of the multiplicative relationship, each expressing the 4-to-1 ratio of fans to nonfans in a slightly different way.

The following problem, adapted from the book *Adding It Up* (National Research Council, 2001), involves a comparison. As you read it, decide if you think it is additive or multiplicative.

Two weeks ago, two flowers were measured at 8 inches and 12 inches. Today they are 11 inches and 15 inches tall. Did the 8-inch or the 12-inch flower grow more?

Additive reasoning would lead to the response that they both grew the same amount—3 inches. Reasoning multiplicatively leads to a different conclusion: The first flower grew $\frac{3}{8}$ of its height, while the second grew $\frac{3}{12}$ of its height. So the first flower grew more. This is a proportional view of the change situation. Here, both additive reasoning and multiplicative reasoning produce valid, albeit different, answers. As students critique these different approaches, they are better able to understand the difference between additive and multiplicative comparisons.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

The following activities provide more opportunities for students to make the distinction between additive reasoning and multiplicative reasoning.

Activity 11.3

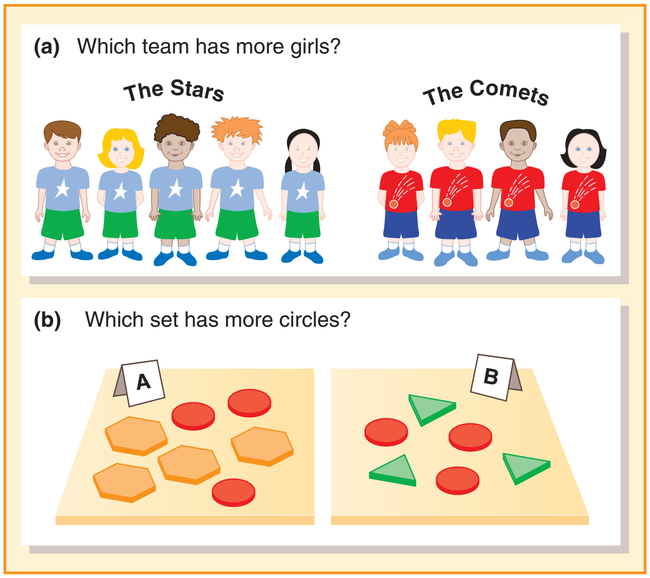
CCSS-M: 6.RP.A.1

Which Has More?

Provide students with situations similar to those in Figure 11.1. Ask students to decide which has more and share a rationale for their thinking. As students share their reasoning, help them see the difference between looking at the difference (additive reasoning) and looking at the ratio (multiplicative reasoning). For English language learners (ELLs), take time to build meaning for these terms—connect *additive* with the word *add,* and *multiplicative* with *multiple* and *multiply.* Reasoning can be modeled through illustrations or explanations. If no one suggests one of the options, introduce it. For example, say, "Amy says it is the second group. Can you explain why she made that choice?" or "Which class team has a larger proportion of girls?"

Figure 11.1

Two pictorial situations that can be interpreted with either additive or multiplicative comparisons.



Activity 11.4

CCSS-M: 6.RP.A.1

Weight Loss

Show students the data in the following chart:

Week	Max	Moe	Minnie
0	210	158	113
2	202	154	107
4	200	150	104

Max, Moe, and Minnie are all on a diet and have recorded their weights at the start of their diet and at two-week intervals. After four weeks, which person is the most successful dieter?

Ask students to make three different arguments—each favoring a different dieter. (The argument for Moe is that he is the steadiest in his loss.)

Additive and Multiplicative Comparisons in Story Problems

When comparisons are embedded in a story situation, they may be additive *or* multiplicative. Using additive reasoning in a situation that calls for multiplicative reasoning, or vice versa, leads to incorrect answers.

Stop and Reflect

Solve the five-item assessment (see Figure 11.2) devised to examine students' appropriate use of additive or multiplicative reasoning (Bright, Joyner, & Wallis, 2003). Which ones are multiplicative situations? Additive? What is the difference between items 2 and 4?

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Figure 11.2 Five items that can be used to assess proportional reasoning.

For each problem, circle the correct answer.

1. Mrs. Allen took a 3- by 5-inch photo of the Cape Hatteras Lighthouse and made an enlargement on a photocopier. She used the 200 percent option. Which is "more square," the original photo or the enlargement?
   a. The original photo is "more square."
   b. The enlargement is "more square."
   c. The photo and the enlargement are equally square.
   d. There is not enough information to determine which is "more square."
2. The Science Club has four separate rectangular plots for experiments with plants:

   1 foot by 4 feet   7 by 10 feet

   17 by 20 feet   27 by 30 feet

   Which rectangular plot is most square?
   a. 1 foot by 4 feet
   b. 7 by 10 feet
   c. 17 by 20 feet
   d. 27 by 30 feet
3. Sue and Julie were running equally fast around a track. Sue started first. When Sue had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?
   a. 45 laps
   b. 24 laps
   c. 21 laps
   d. 6 laps
4. At the midway point of the basketball season, you must recommend the best free-throw shooter for the all-star game. Here are the statistics for four players:

   Novak: 8 of 11 shots   Peterson: 22 of 29 shots

   Williams: 15 of 19 shots   Reynolds: 33 of 41 shots

   Which player is the best free-throw shooter?
   a. Novak
   b. Peterson
   c. Williams
   d. Reynolds
5. Write your answer to this problem:

   A farmer has three fields. One is 185 feet by 245 feet, one is 75 feet by 114 feet, and one is 455 feet by 508 feet. If you were flying over these fields, which one would seem most square? Which one would seem least square? Explain your answers.

Source: Reprinted with permission from Bright, G. W., Joyner, J. J., & Wallis, C. (2003). Assessing Proportional Thinking. *Mathematics Teaching in the Middle School*, *9*(3), p. 167. Copyright 2003 by the National Council of Teachers of Mathematics. All rights reserved.

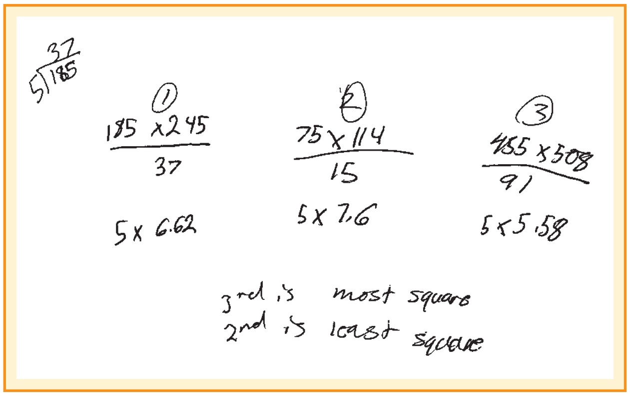
Notice that the items involving rectangular representations (1, 2, and 5) cannot be answered correctly by using additive reasoning. Students are often challenged to determine which type of reasoning to use. When 132 eighth- and ninth-grade students were asked these questions, scores on items 1 through 4 ranged from 45 percent to 67 percent correct. Item 5 proved very difficult (37 percent correct for most square, 28 percent correct for least square).

Formative Assessment Note

All five of these items could be used as *performance assessments*, or a few of them (at least one additive) could be used as a *diagnostic interview*. For example, item 5 was given to an eighth grader, who first solved it incorrectly by using an additive strategy (subtracting the sides). When asked if a very large rectangle, 1,000,000 feet by 1,000,050 feet, would look less square, he replied, "No—*oh*, this is a proportional situation." He then solved it with a novel strategy (Figure 11.3).

Figure 11.3

A solution to #5 in Figure 11.2 (Which field is most square? Least square?)



37
5)185

①
185 x245
37
5x 6.62

②
75x114
15
5x 7.6

③
455 x508
91
5x5.58

3rd is most square
2nd is least square

"Jacob noticed that each length was divisible by 5; therefore, he simplified each ratio to have a side of 5 and then compared the widths."

Teaching Tip

Acting out a proportional situation can help students make sense of it. For example, have students act out item 3 after they have attempted it.

Return for a moment to item 3 in Figure 11.2. This item has been used in other studies showing that students try to solve this as a proportion problem, though it is an additive situation. (The two runners will end up six laps apart, which is how they began.) Because there is a common style to word problems involving proportions, students begin to recognize the style and automatically arrange four quantities (three known and one unknown) into a proportion, without paying attention to whether the numbers have a multiplicative relationship (Watson and Shaughnessy, 2004). They are focusing on the structure of the proportion, not the concept of the proportion (Heinz & Sterba-Boatwright, 2008). Activity 11.5 can help students consider both additive and multiplicative reasoning.

Activity 11.5

CCSS-M: 6.RP.A.3a; 7.RP.2b

Pencil to Pencil

Hold up a cut-out of a very large pencil (e.g., 30 inches in length). Explain to students that this is the exact size of a pencil used by a giant. Ask, "If this is her pencil, what else can you tell me about her?" For students (particularly those with disabilities) who need more structure or guidance, ask specific questions: "How tall is the giant? How long would her hand be?" After students have determined measurements of the giant, have them list their findings on posters and illustrate or explain how they found the measures. ELLs are likely to be more familiar with centimeters, so have students choose what measurement system they use, or have all students use centimeters. As an alternative, and to use fractional measurements, begin with a tiny pencil and consider the size of the tiny person.

Source: Based on Che, S. M. (2009). Giant pencils: Developing proportional reasoning. *Mathematics Teaching in the Middle School, 14*(7), 404–408.

When students first engage in this activity, they may focus on the (additive) difference of the pencil they are shown and a real pencil they have. If they reason about this difference, then they will find that the giant is only 24 inches taller than they are. Thinking about this should raise some doubt about this line of reasoning because there are real people who are 2 feet taller, and a 30-inch pencil would still be too big for them to manage. They might then start thinking of how many of their pencils would equal the extra large pencil. By counting (iterating), they might notice it takes about 5 lengths of their pencils. In debriefing this activity with students, it is important to discuss the thought processes and why the situation is a multiplicative comparison, not an additive comparison.

Teaching Tip

Ask students to write examples of additive differences and multiplicative differences. This can help them distinguish between these ideas, which they must be able to do to reason with ratios (which is multiplicative).

Contrasting two very similar problems can also support students' emerging skills at reasoning proportionally (Lim, 2009). Consider these two tasks and how they are the same and different:

1. A red car and a silver car are traveling at the same constant rate. When the red car has traveled 20 miles, the silver car has traveled 12 miles. How far will the red car be when the silver car has traveled 32 miles?
2. A red car and a silver car are traveling at different but constant rates. They pass Exit 95 at the same time. When the red car has traveled 20 miles past Exit 95, the silver car has traveled 16 miles. How far will the red car have traveled when the silver car has traveled 32 miles?

Analyze the situation using equations. In the first case, the relationship is red $=$ silver $+$ 8 because the red car is 8 miles in front of the silver car. In the latter case, the relationship is red $= \frac{5}{4}$ silver because for every 5 miles the red car travels, the silver car travels 4 miles (Lim, 2009).

Standards for Mathematical Practice

4 Model with mathematics.

Teaching Tip

Asking students to use variables to write the relationships can help them see the difference between additive and multiplicative situations.

Covariation

Covariation can sound like a high school or college concept, but it simply means that two different quantities vary together. For example, 5 mangos cost $2.00 (two quantities in a multiplicative relationship); as the number of mangos varies (for example, to 10 mangos), so does the cost. And, as the cost changes, so does the number of mangos you will get. Once you know either a new price or a new number of mangos, you can determine the missing variable.

Within Ratios and Between Ratios

A ratio of two measures in the same setting is a *within* ratio. For example, in the case of the mangos, the ratio of mangos to money is a within ratio—that is, it is "within" the context of that example.

Teaching Tip

Have students create a drawing similar to the one in Figure 11.4. This representation will be very helpful in setting up and understanding proportions, especially for students who struggle with setting up proportions.

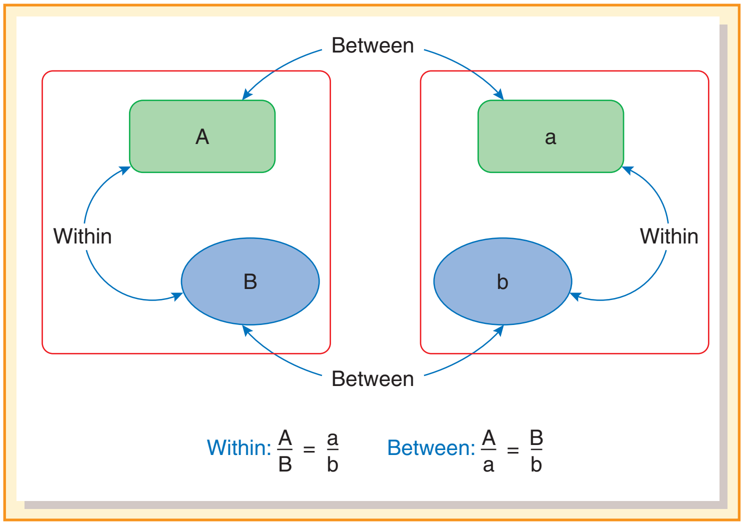
A *between* ratio is a ratio of two corresponding measures in different situations. In the case of the mangos, the ratio of the original number of mangos (5) to the number of mangos in a second situation (10) is a between ratio—that is, it is "between" the two situations. The drawing in Figure 11.4 is an effective way of looking at two ratios and determining whether a ratio is between or within.

Standards for Mathematical Practice

5 Use appropriate tools strategically. ▶

Figure 11.4

In a proportional situation, the two between ratios will be equivalent, as will the two within ratios.



Activity 11.6

CCSS-M: 6.RP.A.1; 6.RP.A.2

Different Objects, Same Ratios

Download **Ratio Cards** Activity Page and cut out the cards (see Figure 11.5) (or create your own using clipart of objects that are relevant to your students). Given the set of cards, students are to select the cards on which the ratio of the two types of objects is the same. This task

moves students toward a numeric approach rather than a visual one and introduces the notion of ratios as rates. In this context, it makes the most sense to find the boxes per truck as the rate (rather than trucks per box). Finding the unit rate (amount for one unit) for pairs of quantities facilitates comparisons (just as the unit prices provided in grocery stores allow you to compare different brands).

Covariation in Measurement and Geometry

Within and between ratios apply to measurement conversions. Consider that the capital A and B in Figure 11.4 is the conversion of inches to feet. How might you use within and between ratios to determine the number of feet in 60 inches?

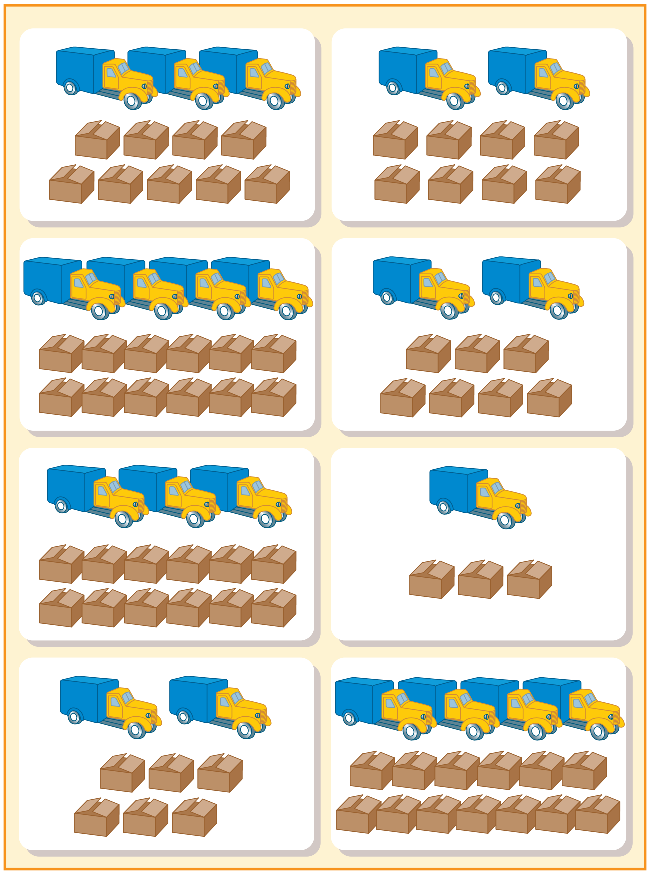
$$\frac{\text{12 inches}}{\text{1 foot}} = \frac{\text{60 inches}}{\text{? feet}}$$

You might notice that the between ratio is × 5 (left to right), or you might notice that the within ratio is ÷ 12 (inches to feet). Measurement conversions are difficult even for adults, and they are a goal in grade 6 in the CCSS-M. Setting up a between and within equation such as this one and analyzing the relationships can help students see the options they have for finding the conversion.

Within and between ratios are particularly relevant in exploring similarity with geometric shapes (a grade 8 topic in the CCSS-M). Students often struggle to determine which features to compare. The following activity can help students begin to analyze which features to compare.

Figure 11.5

Sample cards for comparing ratios and rates.



Activity 11.7

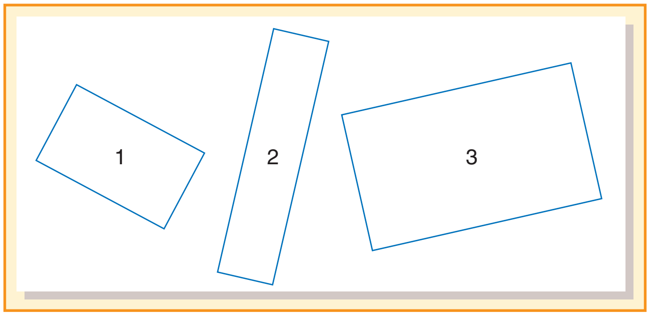
CCSS-M: 6.RP.A.1; 6.RP.A.2a; 6.RP.A.3a

Look-Alike Rectangles

Provide students with cut-outs of the rectangles provided in Look-Alike Rectangles Activity Page (or have students cut out a set for themselves (see Figure 11.6)). [Rectangles A, I, and D have sides in the ratio of 3 to 4. Rectangles C, F, and H have sides in the ratio of 5 to 8. Rectangles J, E, and G have sides in the ratio of 1 to 3. Rectangle B is a square, so its sides are in the ratio of 1 to 1.]

(*continued*)

Ask students to group the rectangles into three sets that “look alike.” If your students know the word *similar*, use that term To explain what “look alike” means, draw three rectangles on the board with two that are similar and one that is clearly dissimilar, as in the following example. Have students use ratio language to explain why rectangles 1 and 3 are alike.



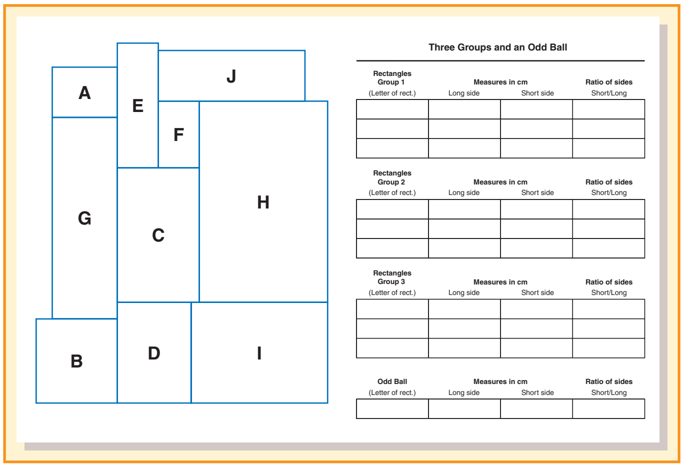
When students have decided on their groupings, stop and discuss the reasons why they classified the rectangles the ways they did. Be prepared for some students to try to match sides or look for rectangles that have the same amount of difference between the sides. Encourage students to critique the explanations given. Next, have the students measure and record the sides of each rectangle to the nearest half centimeter. Use the **Look Alike Rectangles Recording Sheet** (see Figure 11.6). Discuss the results and ask students to offer explanations of how the ratios and groupings are related. If the groups are formed of proportional (similar) rectangles, the ratios within each group will be equivalent. Students with disabilities may need to have examples of one rectangle from each grouping as a starting point.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Figure 11.6

Activity pages for look-alike rectangles.



Three Groups and an Odd Ball

Rectangles Group 1 (Letter of rect.)	Measures in cm Long side	Short side	Ratio of sides Short/Long

Rectangles Group 2 (Letter of rect.)	Measures in cm Long side	Short side	Ratio of sides Short/Long

Rectangles Group 3 (Letter of rect.)	Measures in cm Long side	Short side	Ratio of sides Short/Long

Odd Ball (Letter of rect.)	Measures in cm Long side	Short side	Ratio of sides Short/Long

The connection between proportional reasoning and the geometric concept of similarity is very important. Similar figures provide a visual representation of proportions, and proportional thinking enhances the understanding of similarity. Discussion of the similar figures should focus on the ratios between and within the figures.

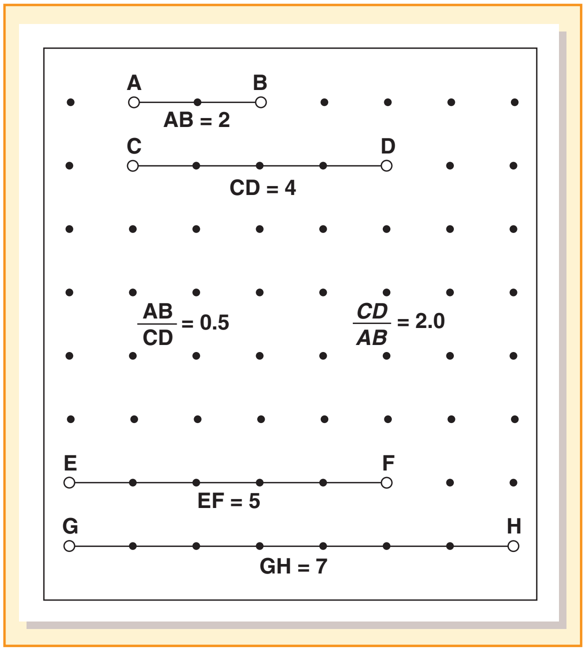
technology note

Dynamic software such as *GeoGebra* (a free download) offers a very effective method of exploring the idea of ratio. In Figure 11.7, two lengths are drawn on a grid with the software's "snap-to-grid" option. The lengths are measured, and two ratios are computed. As the length of either line is changed, the measures and ratios are updated instantly. You can also use dynamic software to explore dilations (i.e., similar figures) and corresponding measures. Figures can be drawn and then dilated (reduced or enlarged proportionally) according to any scale factor. The ratios of beginning and ending measures (lengths and areas) can then be compared to the scale factor.

Creating *scale drawings* is an application of similarity. Multiplication *is* scaling (e.g., making something three times bigger or one-half the size). Scale drawings, then, are an important way for students to see the connection between multiplicative reasoning and proportional reasoning. Students are asked to resize a sketch that is similar to one they are given. The next activity is about scaling.

Figure 11.7

Dynamic software can be used to draw line segments or geometric shapes to see if a proportional relationship exists. Dynamic software can also be used explore congruent or similar shapes.



Activity 11.8

CCSS-M: 6.RP.A.1; 7.RP.A.3a

Scale Drawings

On **1-Centimeter Grid Paper**, **0.5-Centimeter Grid Paper**, or **Dot Paper**, have students use straight lines with vertices on the dots to draw a simple shape. After one shape is complete, have them draw a "scaled" drawing—a larger or smaller shape that looks similar to the first. With ELLs, be sure the term *scale* is understood so that they don't confuse this use of the word with a machine that weighs things or what fish have. This can be done on a grid of the same size or a different size, as shown in Figure 11.8. First, compare ratios within (see the first problem in Figure 11.8). Then compare ratios between the figures (see the second problem in Figure 11.8).

Corresponding sides from one figure to the next should all be in the same ratio. The ratio of two sides within one figure should be the same as the ratio of the corresponding two sides in another figure.

Comparisons of corresponding lengths, areas, and volumes in proportional figures lead to some interesting patterns. If we know the length of the side of a figure, we can create the ratio of 1 to k, for example, to represent the relationship to a proportional figure. (The variable k is often used with proportions, whereas m is used with equations to describe slope—both refer to the rate or ratio between two values, which is called the *scale factor*.)

Standards for Mathematical Practice

4 Model with mathematics.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

If two figures are proportional (similar), then any corresponding linear dimensions will have the same scale factor.

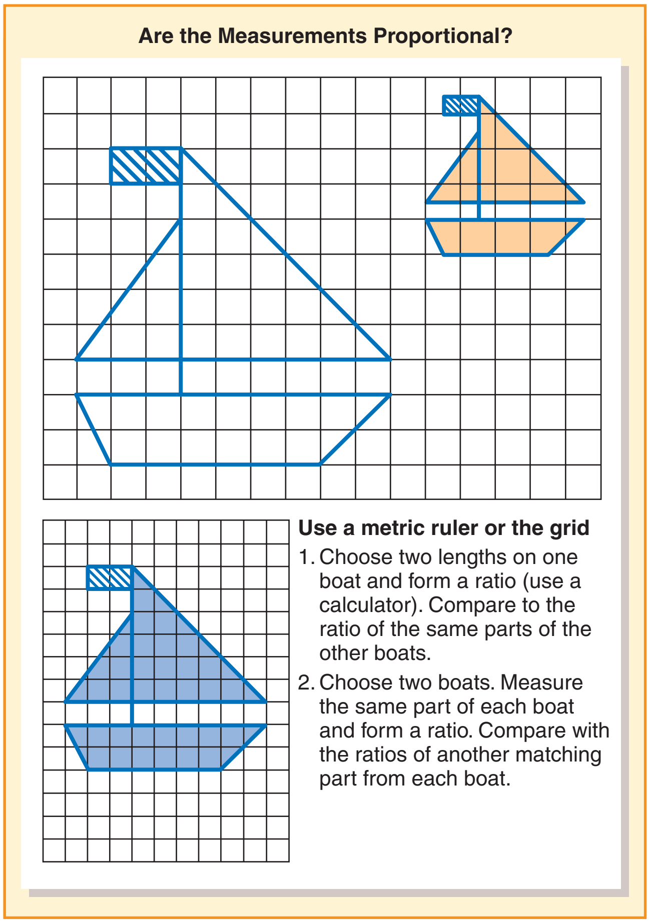
Imagine you have a square that is 3 by 3 and you create a new square that is 6 by 6. The ratio between the side lengths is 1:2. What is the ratio between the two areas? Why is it 1:4? Try the same idea with the volume of a cube—what is the relationship of the original volume to the new volume when you double the length of the edges? Why? Returning to the sailboat in Figure 11.8, what would you conjecture is the ratio between the areas of the two sailboats? Measure and test your hypothesis.

Here are some interesting situations to consider for scale drawings:

- If you want to make a scale model of the solar system and use a Ping-Pong ball for the earth, how far away should the sun be? How large a ball do you need to represent the sun?
- What scale should be used to draw a scale map of your city (or some region of interest) so that it will fit onto a standard piece of poster board?
- Use the scale on a map to estimate the distance and travel time between two points of interest.
- Roll a toy car down a ramp, timing the trip with a stopwatch. How fast was the car traveling in miles per hour? If the speed is proportional to the size of the car, how fast would this have been for a real car?
- Your little sister wants a table and chair for her doll. Her doll is 14 inches tall. How tall should you make the table? How tall is the chair?

Figure 11.8

Comparing similar figures drawn on grids.



Are the Measurements Proportional?

Use a metric ruler or the grid

1. Choose two lengths on one boat and form a ratio (use a calculator). Compare to the ratio of the same parts of the other boats.
2. Choose two boats. Measure the same part of each boat and form a ratio. Compare with the ratios of another matching part from each boat.

technology note

Google Earth is a great resource for doing authentic scaling activities (Roberge & Cooper, 2010). Find a Google Earth diagram that includes something for which the measure is known, students can figure out other measures. For example, you know that a standard football field is 100 yards from end zone to end zone (120 yards if you include the end zones), so zoom in on your school football field. By zooming to different levels, students can build an understanding of scale factor in an interesting context.

Scale City, available for free through Kentucky Education Television and aligned to the CCSS-M, features fun and engaging videos and interactive simulations for exploring scale drawings.

Covariation in Algebra

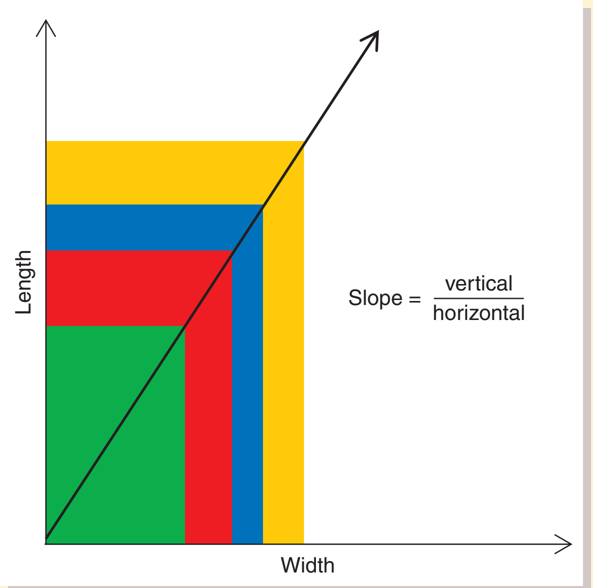
A great geometry-algebra connection is to actually stack the "Look-Alike Rectangles" (Activity 11.7) so that that they are aligned at one corner, as in Figure 11.9. Place a straight-edge across the diagonals and you will see that opposite corners also line up. If the rectangles are placed on a coordinate axis with the common corner at the origin, the *slope* of the line joining the corners *is the ratio of the sides*.

Proportional situations are linear situations. Graphing equivalent ratios is a powerful way to connect ratios to linear equations. The *Common Core State Standards* emphasizes the need to represent ratios and proportions graphically. The ratio or rate is the slope of the graph, and the line goes through the origin, so it is represented symbolically as $y = mx$, or sometimes as $y = kx$, where k takes on the specialized meaning of the scale factor.

Graphing ratios can be challenging for students. For example, students can struggle to decide what points to graph, deciding which axes to use for the two measures, and making sense of what the graph means (Kastberg, D'Ambrosio, Lynch-Davis, Mintos, & Krawczyk, 2014). With ratios, the choice of which variable is on the x-axis and which variable goes on the y-axis is arbitrary.

Figure 11.9

The slope of a line through a stack of proportional rectangles is equal to the ratio of the two sides.



Activity 11.9

CCSS-M: 7.RP.A.2a, b, c, d

Rectangle Ratios—Graph It!

This activity is connected to Activity 11.7. Using the **Look-Alike Rectangles: Graph It!** Activity Page, have students select one set of alike rectangles and record the measures in the ratio table, and then have them create three of their own examples of alike rectangles using their knowledge of equivalent ratios. Also, see if students can find a rectangle that has a noninteger side (e.g., $4\frac{1}{2}$cm).

(continued)

Standards for Mathematical Practice

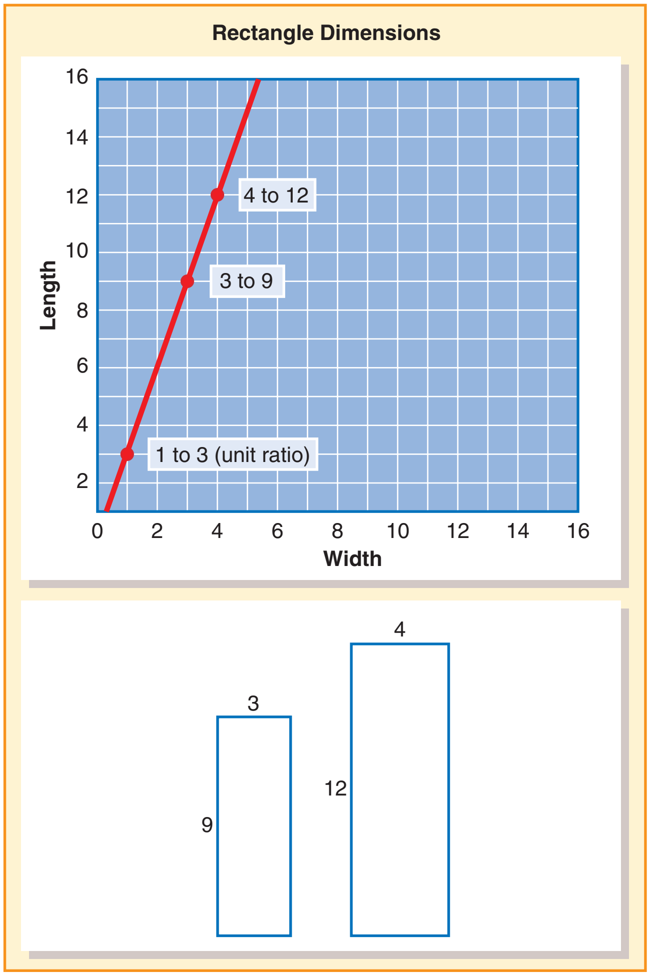
4 Model with mathematics.

Next, ask students to graph the data for their 6 rectangles. The graph in Figure 11.10 is based on the ratios of two sides of similar rectangles. After the ratios have been graphed, challenge students to use their graph to determine a look-alike rectangle. Also, ask questions to help students understand the graph, such as: What does a point on the graph mean (even if it is not one of the rectangles)? What is the length for a look-alike rectangle with a width of 1 cm? Challenge students to find the unit rate each way (also finding the width if the length is 1). Then ask students how, if they know the short side, they could find the long side (and vice versa).

In addition to using geometric illustrations, it is important to use relevant contexts in which students analyze proportional situations and represent them in various ways. Dripping faucets is one such context (Williams, Forrest, & Schnabel, 2006).

Figure 11.10

The sides of the rectangles can be graphed as coordinates (x, y) representing width (*x*) and length (*y*).



Activity 11.10

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.2b, c

Dripping Faucets

Pose the following problem to students:

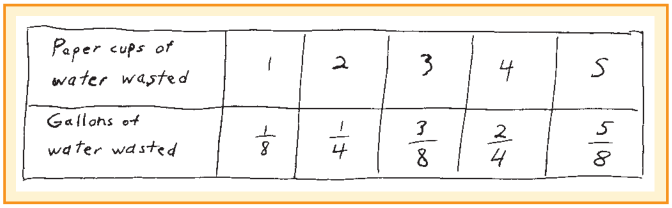
> **If you brush your teeth twice a day and leave the water running when you brush, how many gallons of water will you waste in one day? In two days? In a week? A month? Any number of days?**

Invite students to gather real data, if possible. Or, use the recommended time to brush teeth (e.g., 2 minutes). Ask students to explore the ratio of number of days to water wasted, and to illustrate their data using tables, graphs and equations (using graphing calculator or computer applications to create the displays). Challenge students to consider other such environmental issues, determining two variables, determining what ratio exists between them, and representing that ratio in multiple ways.

This environmental investigation involves real measurement and authentic data. Stemn (2008) implemented this investigation with students and found that it aided their understanding of the difference between multiplicative and additive situations. Students in her class were challenged to figure out how many of the paper cups that they used for measuring would fill a gallon. The class figured out that two paper cups were equal to one-quarter gallon. Each student reasoned how many gallons he or she wasted in a day. For example, a student who wasted five paper cups of water reasoned that this would be two-fourths, plus a half of another fourth (or one-eighth). So, in total, five-eighths of a gallon of water is wasted per day. The class recorded what they knew in a table:

Teaching Tip

There are many great books and poems about very large or very small people, and literature is an excellent way to bring meaning and relevance to mathematics! See Table 11.1.



Paper cups of water wasted	1	2	3	4	5
Gallons of water wasted	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{2}{4}$	$\frac{5}{8}$

The teacher encouraged students to rewrite the table with common denominators. Students recognized that the ratio of water in paper cups to water in gallon containers was $1\colon\frac{1}{8}$. Notice the connection to algebraic reasoning and to measurement. The formula $y = \frac{1}{8}x$ describes the relationship between volume of paper cups (x) to gallons (y). This investigation also showed the students how to reason through measurement conversions with a nonstandard measure (the paper cup).

More great contexts can be found in literature, and middle-school students love exploring contexts from interesting books, such as the ones in Table 11.1.

technology note

For a good selection of challenging ratio and proportion problems that involve work and everyday life tasks, see NCTM's "Figure This! Math Index: Ratio and Proportions."

Table 11.1. Great middle-school literature for ratio and proportions.

Literature with Proportional Situations **Many books include multiplicative comparisons and therefore can be used for proportional reasoning.**	
If You Hopped Like a Frog Schwartz, 1999	David Schwartz compares features of various creatures with those of humans. For example, in the title comparison, Schwartz deduces that if a person had a frog's jumping ability, he or she could jump from home plate to first base in one hop. This short picture book contains 12 more fascinating comparisons. Schwartz also provides the factual data on which the proportions are based. Students can figure out how strong or tall they would be if they were one of the featured animals.
Holes Sachar, 2000	A popular book and movie, this novel tells the story of boys in a "camp" who are digging holes every day, which provides an opportunity to look at daily rates of dirt removal. Pugalee, Arbaugh, Bay-Williams, Farrell, Mathews, and Royster (2008) describe an excellent activity with this book that involves not only proportional reasoning but also measurement and algebra.
Literature with Large and/or Small People **There is a plethora of literature involving very little or very big people (or animals). With any of these books, lengths of arms, legs, or noses can be compared as a way to explore within and between ratios.**	
Alice's Adventures in Wonderland Carroll, 1865/1982	In this classic, Alice becomes very small and very tall, opening doors to many ratio and proportion investigations. See Taber (2005) for ideas on using this book to teach proportions.
The Borrowers Norton, 1953	A classic tale of little folk living in the walls of a house. Furnishings are created from odds and ends of the full-size human world.

(*continued*)

Table 11.1. Great middle-school literature for ratio and proportions. (*continued*)

Literature with Large and/or Small People **There is a plethora of literature involving very little or very big people (or animals). With any of these books, lengths of arms, legs, or noses can be compared as a way to explore within and between ratios.**	
Gulliver's Travels Swift, 1726; amended 1735/1999	Yet another classic story. In this case, Gulliver first visits Lilliput, where he is 12 times the size of the inhabitants, and then goes to Brobdingnag, where he is one-tenth the size of the inhabitants.
Harry Potter and the Sorcerer's Stone Rowling, 1997	In the book, Hagrid is described as twice as tall and nearly five times as wide as the usual man. Students can measure their own shoulder width and height and create a scatter plot. Placing Hagrid's measurements on the graph illustrates that he is not made in proportion to the usual person. In the movie, he is more proportional to the other characters, although still nearly twice their height. Students can explore the size of his house, furniture, and motorcycle based on an estimated scale of how much bigger he is than the usual person.
Jim and the Beanstalk Briggs, 1970	What happened to the giant after Jack? Jim comes along. Jim wants to help the poor, pessimistic giant. This heartwarming story is great for multiplicative or proportional reasoning.
Kate and the Beanstalk Osborne, 2000	This version of the traditional *Jack and the Beanstalk* tale includes a giantess. The giantess falls to earth, and Kate finds out that the castle belongs to her family.
The Lord of the Rings Tolkien, 1965	Hobbits are described as approximately 3 feet tall (which can be estimated at 100 cm). This height can be used to set up a ratio with the height of the typical sixth grader and the ratio then used to determine the size of various objects. To connect to the movie, ask students to be in the role of producer and figure out the size of objects used in the movie. See Beckman, Thompson, and Austin (2004) for elaboration on this activity.
"One Inch Tall" in *Where the Sidewalk Ends* Silverstein, 1973	Shel Silverstein is a hit with all ages. This poem asks what it would be like if you were one inch tall.
Swamp Angel Isaacs, 1999	A swamp angel named Angelica grows into a giant. Students can compare birth height with current height or compare Angelica's measurements with their own.

Strategies for Solving Proportional Situations

Procedural fluency includes *flexibility*, *efficiency*, *accuracy*, and *selecting appropriate strategies*. Certainly that should be the goal in teaching students to solve proportional situations. Sometimes the numbers in the situation lend to simple reasoning strategies, and sometimes they require a more sophisticated strategy. The most commonly known strategy is cross products, but as the most abstract and least intuitive strategy, it should only be introduced after students have solved proportional problems through reasoning and by using drawings such as a tape diagram or double number line and using ratio tables. Posing problems that have multiple solution strategies can help students (Berk, Taber, Gorowara, & Poetzl, 2009; Ercole, Frantz, & Ashline, 2011). Strategies for solving missing value proportions include the following:

- Rate
- Scaling up or down
- Scale factors (within or between measures)
- Ratio tables
- Double number line
- Graphs
- Cross products

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them. ▶

It is worth repeating that all of these strategies are useful in particular situations, and all should be taught such that as students add on new strategies they are encouraged to select from the strategies they know (rather than just use the one that they just learned). The first three are the most intuitive; therefore, they are the ones students might invent, and a good place to begin.

Rates and Scaling Strategies

Unit rate and scale factor can be used to solve many proportional situations mentally. The key is to know both strategies and pick the one that best fits the particular numbers in the problem, as the next examples illustrate.

Tammy bought 3 widgets for $2.40. At the same price, how much would 10 widgets cost?

Tammy bought 4 widgets for $3.75. How much would a dozen widgets cost?

Stop and Reflect

Consider how students might use an approach other than the cross-product algorithm to solve these two problems. If they know the cross product, how might you encourage them to use a mental strategy?

In the first situation, it is perhaps easiest to determine the cost of one widget—the *unit rate* or unit price. This can be found by dividing the price of 3 widgets by 3. Multiplying this unit rate of $0.80 per widget by 10 will produce the answer. This approach is referred to as a *unit-rate* method. Notice that the unit rate is a within ratio. This approach applies the ratio as a multiplicative comparison.

In the second problem, a unit-rate approach could be used, but the division does not appear to be easy. Because 12 is a multiple of 4, it is easier to notice that the cost of a dozen is 3 times the cost of 4, or that the scale factor between the ratios is 4. This is called a *buildup strategy*. (This strategy could have been used for the first problem but would have been more difficult because the scale factor between 3 and 10 is $3\frac{1}{3}$.) Notice that the buildup strategy applies the ratio as a composed unit. Although the buildup strategy is a useful way to think about proportions, they are most frequently used when the numbers are compatible (i.e., the scale factor is a whole number).

Teaching Tip

Give students problems in which the numbers lend themselves to both the unit-rate approach and the buildup approach (but let them choose their approach). This way, they learn not only that two different approaches are possible, but also that sometimes one approach works better than another.

At the office superstore, you can buy 4 pencils for $0.59, or you can buy the same pencils in a large box of 5 dozen for $7.79. How much will you save per pencil if you buy the large box?

The price of a box of 2 dozen gumballs is $4.80. Bridget wants to buy 5 gumballs. What will she have to pay?

To solve the pencil problem, you might notice that the between ratio of pencils to pencils is 4 to 60 (5 dozen), or 1 to 15. If you multiply the $0.59 by 15, the factor of change, you will get the price of the box of 60 if the pencils are sold at the same price. In the gumball problem, the within ratio of 24 to $4.80 lends to finding the unit rate of $0.20 per gumball, which can then be multiplied by 5. See **Expanded Lesson: It's a Matter of Rates** and **It's a Matter of Rates** Activity Page for a full lesson using stories to develop strategies for reasoning about ratios and rates.

Follow these tasks with problems that have more difficult numbers, asking students to apply the same strategies to reason to an answer. For example, try to apply both strategies to the next problem.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Brian can run 5 km in 11.4 minutes. If he keeps running at the same speed, how far can he run in 23 minutes?

Selecting problems that can be solved many ways is important. The following is a classic proportional reasoning activity because it can be approached in so many ways.

Activity 11.11

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2a

Comparing Lemonade Recipes

Show students a picture of two pitchers of lemonade, as in Figure 11.11. The little squares indicate the amounts of water and lemonade concentrate used in each pitcher. In this case the recipes are:

3 cups water	4 cups water
2 cups concentrate	3 cups concentrate

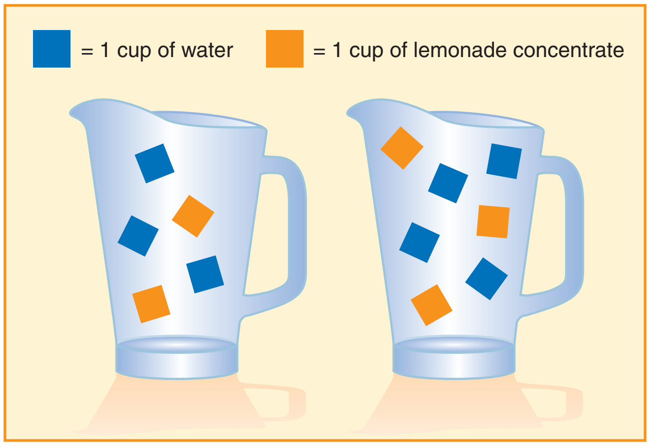
A yellow square is a cup of lemonade concentrate, and the blue is a cup of water. Ask whether the lemonade in one of the pitchers will have a stronger lemon flavor or whether the lemonade in the two pitchers will taste the same. Ask students to justify their answers. Recipes can be adapted to include fractional values. For example, ask students to reason about which of these has a stronger lemon flavor:

$\frac{3}{4}$ cup water	$\frac{3}{2}$ cup water
$\frac{1}{8}$ cup of concentrate	$\frac{1}{4}$ cup of concentrate

Look at the between and within ratios in this second example. What do you notice? The within ratio for both recipes is × 6 (or ÷ 6). The between ratio for both water and concentrate is × 2 (or ÷ 2). These recipes will taste the same! Watch a video of John Van de Walle and others discussing this problem.

Figure 11.11

A comparing-ratios problem: In which pitcher will the lemonade have the stronger lemon flavor, or will the flavor be the same in both pitchers?



Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Stop and Reflect

There are many ways to solve the lemonade problem through reasoning (see if you can think of three). How might you have students share their ways so that other students can understand and critique their approach? How will you help students see connections between the strategies?

Comparing Lemonade Recipes is a rich task because of the number of ways in which the comparison can be made. Here are a few ideas:

1. Figure out how much water goes with each cup of concentrate. As we will see later, this approach uses a unit rate: cups of water per cup of lemonade concentrate ($1\frac{1}{2}$ vs. $1\frac{1}{3}$).
2. Use part-part fractions (concentrate to water) and compare the fractions ($\frac{2}{3}$ vs. $\frac{3}{4}$ or the reverse)
3. Use part-whole fractions (lemonade concentrate to whole mixture) ($\frac{2}{5}$ vs. $\frac{3}{7}$).
4. Converting to percentages.
5. Use multiples of one or both of the pitchers until either the amounts of water or the amounts of lemonade concentrate are equal.

The lemonade task can be differentiated to different levels of challenge. As given, there are no simple relationships between the two pitchers. If the solutions are 3 to 6 and 4 to 8 (equal flavors), the task is much simpler. For a 2-to-5 recipe versus a 4-to-9 recipe, it is easy to double the first and compare it with the second. When a 3-to-6 recipe is compared with a 2-to-5 recipe, the unit rates are perhaps more obvious (1 to 2 vs. 1 to $2\frac{1}{2}$).

An excellent addition or alternative to mixing juice is to mix paint shades, creating a paint swatch.

Activity 11.12

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2a

Creating Paint Swatches

Ask students to write five different shades using two primary colors (e.g., use red and yellow and make recipes for shades of orange) or you can create a set of recipes. Ask students to order them (e.g., from the reddest to the yellowest), and try to include some that are equivalent ratios. Use real paint drops and have students test their ratios and create their own paint chart! See Beswick, 2011, for details

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

The last two activities were mixture type problems. The next activity is not a mixture but compares food portions, and it also can be approached in a variety of ways.

Activity 11.13

CCSS-M: 6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2a

Which Camp Gets More Pizza?

Pose the following story (or adapt it to be about your own students). Before solving it ask students to select the camp they think gets more pizza. Record the information for later. Ask students to use a rate or scaling strategy (reasoning strategy) to prepare a convincing argument for which camp gets more pizza.

> **Two camps of Scouts are having pizza parties. The leader of the Bear Camp ordered enough so that every 3 campers will have 2 pizzas. The leader of the Raccoon Camp ordered enough so that there will be 3 pizzas for every 5 campers. Did the Bear campers or the Raccoon campers have more pizza?**

After students have each found their own way to decide, pair them with another student to compare strategies and see if they agree on which camp gets more pizza.

Figure 11.12

Rate and scaling methods for comparing pizzas per camper in two camps.

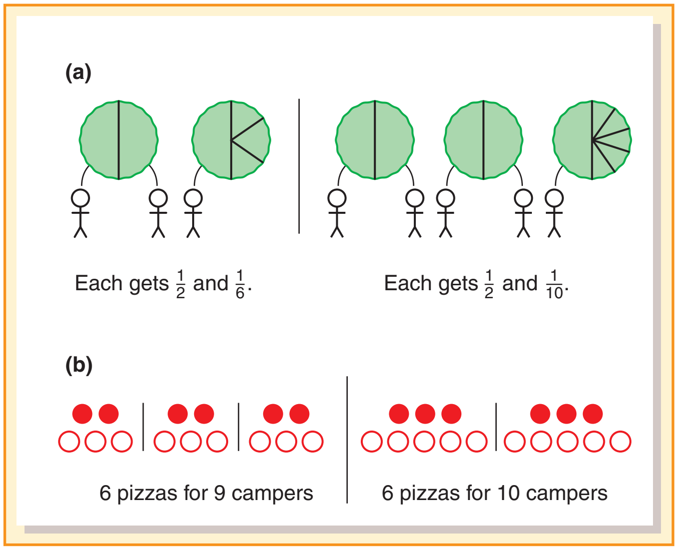


Figure 11.12 shows two different reasoning strategies. When the pizzas are sliced into fractional parts, as in Figure 11.12(a), the approach is to look for a unit rate—pizzas per camper. A partitioning approach has been used for each ratio (as in division). But, notice that this problem does not say that the Bear Camp and the Raccoon Camp have only 3 and 5 campers, respectively. Any multiples of 2 to 3 and 3 to 5 can be used to make the appropriate comparison, the same as in making multiple pitchers of lemonade. The iterative approach is illustrated in Figure 11.12(b). Three "clones" of the 2-to-3 ratio and two clones of the 3-to-5 ratio are made so that the number of campers getting a like number of pizzas can be compared. From the vantage of fractions, this is like getting common numerators. Because there are more campers in the Raccoon Camp ratio (larger denominator), there is less pizza for each camper. These strategies, and the strategies in the lemonade activity, emerge as students reason multiplicatively about the situations.

Perhaps you have noticed that some of the problems shared in this section include all the values and ask students to compare, whereas others ask for a missing value. Some lend themselves to a unit-rate method, some to a buildup strategy, and some to other strategies. The more experiences students have in comparing and solving situations that are proportional in nature, the better they will be able to reason proportionally.

Ratio Tables

Ratio tables, or charts that show how two variable quantities are related, are good ways to organize information. They serve as tools for applying buildup strategy but can also be used to determine unit rate. Consider the following table:

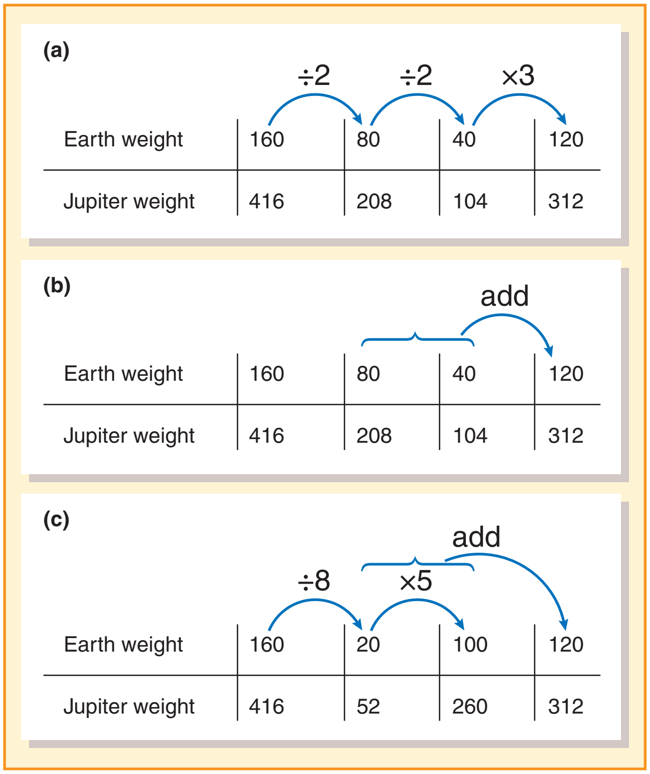
Acres	5	10	20	30	60
Pine trees	75	150	300		

In this situation, the problem posed might have asked, "How many trees for 60 acres of land?" Or it might be, "How many acres are needed for 750 pine trees?" Students can generate more columns in the table in an additive way (e.g., adding 10 to the top row and 150 to the bottom row). Or, they could look at the relationship between acres and pine trees (× 15). This is the rate (15 pine trees per acre). The equation for this situation is $y = 15x$, where x is number of acres and y is the number of pine trees. Once this is discovered, students can figure out that 15×65 acres $= 975$ pine trees. But, an advantage of the ratio table is that neither variables nor equations are needed, so it is less abstract than using proportions.

Ratio tables can be used to find a specific equivalent ratio. For example, **Factory Ratios** Activity Page provides an interesting factory context and connects ratio tables to graphs to explore production ratios. Because different ratios are represented in the ratio table, those ratios can be compared, or a missing value found. Therefore, they are a strategy for solving missing value proportions. The following activity provides examples, and Figure 11.13 gives illustrations of this use of a ratio table (ideas based on Dole, 2008 and Lamon, 2012).

Figure 11.13

Something weighing 160 pounds on Earth weighs 416 pounds on Jupiter. If something weighs 120 pounds on Earth, how many pounds will it weigh on Jupiter? Three solutions obtained with ratio tables.



(a)

Earth weight	160	80	40	120
Jupiter weight	416	208	104	312

(b)

Earth weight	160	80	40	120
Jupiter weight	416	208	104	312

(c)

Earth weight	160	20	100	120
Jupiter weight	416	52	260	312

Activity 11.14

CCSS-M: 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2b, c; 7.RP.A.3

Solving Proportional Problems Using Ratio Tables

Use the Proportional Problems Task Cards to create task cards. The Activity Page includes tasks such as:

If you run at the constant rate of $\frac{1}{2}$ mile in $\frac{1}{10}$ of an hour, how long will it take you to run a mini-marathon (13 miles)?

You can give a set of cards to each pair of students, or select different cards for different groups to differentiate the activity. For each task card, ask students to (1) make a ratio table, (2) solve the problem, and (3) represent the proportional relationship as an equation ($y = kx$). Note: These contexts may have language that needs reinforced with ELLs. Posting visuals or translations can support their reading comprehension.

You likely recognize these tasks as typical "solve-the-proportion" tasks. One ratio and part of a second are given, with the task being to find the fourth number. Figure 11.13 shows three different ways to use ratio tables to solve the Jupiter weight task. As this example illustrates, the ratio table has several advantages over the missing-value proportion. Students label each row, are more successful at placing the values appropriately, and therefore are able to compare. And working within the ratio table is more directly

Standards for Mathematical Practice

4 Model with mathematics.

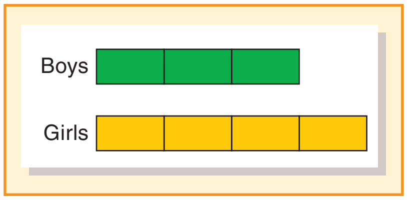
connected to the concept of seeking an equivalent ratio than to the concept of solving a missing-value proportion. Therefore, the ratio table should be introduced prior to using the cross-product approach. In connecting the ratio table to an equation, students are able to better understand the meaning of ratios and rates.

Formative Assessment Note

Because there are many ways to reason about proportional situations, it is important to capture *how* students are reasoning. *Writing* is an effective way to do this. Ask students to tell how they solved a problem or to explain how they used the ratio table. You can provide more structure by using specific writing prompts or sentence starters, such as, "In the ratio table, I used the values _____ to . . . " or by asking students to describe two different ways they can use the ratio table to arrive at the solution or ask students to describe two different ways they could arrive at the solution.

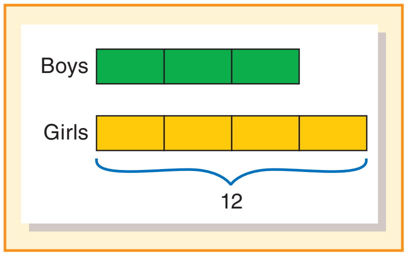
Tape or Strip Diagram

Tape or strip diagrams help students visualize the multiplicative relationships. CCSS-M defines a tape diagram as "a drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model" (CCSSO, 2010, p. 87). Consider this statement: The ratio of boys to girls in the class is 3 to 4. This can be set up in strips as shown here, or a similar diagram that looks more like a partitioned line segment. These can be drawn on **1-Centimeter Grid Paper**, **0.5-Centimeter Grid Paper**, or **Dot Paper**. They can also be created by folding and cutting paper strips:



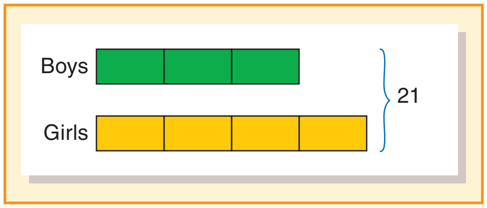
Once this basic ratio is provided, students use this sketch to solve problems. Let's look at three different ways these might be asked and sketched:

1. If there are 12 girls, how many are boys? [one part is given, the other part requested]



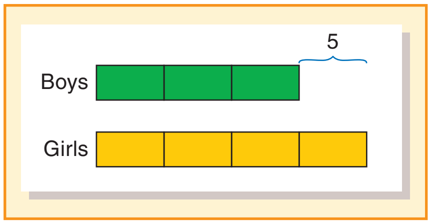
In observing the girls tape, students ‘see’ the need to divide 12 by 4, placing 3 in each partition. Therefore, there will be 9 boys.

2. If there are 21 children, how many are boys? [the whole is given, a part is requested]



This second situation is harder for students to solve (Cohen, 2013). In particular it can be difficult for students to set up proportions, but the tape shows 7 sections to equally distribute 21 students, so 9 boys.

3. There are 5 more girls than boys, how many are girls? [the difference between the parts is given, a part is requested]



We can see that the 5 represents one-fourth of the girls, so there are 20 girls.

Notice how the tape diagram makes solving each situation more accessible. The tape diagrams provide a more concrete strategy that can be done first and later connected to other strategies, such as setting up an equation.

technology note

For a nice virtual model of a tape or strip diagram, go to Math Playground and scroll down to “Thinking Blocks Model and Solve Ratio and Proportion Word Problems.” This site has instructions and practice that connects the two strips to different types of proportional situations.

Double Number Lines

Double number lines are similar to tape diagrams but may not show partitions. Importantly, as with tape diagrams, labeling the two lines must be emphasized. Labeling the parts and wholes on the number line helps students to keep track of the objects in the problem and how they covary, as well as determine the solution to the problem.

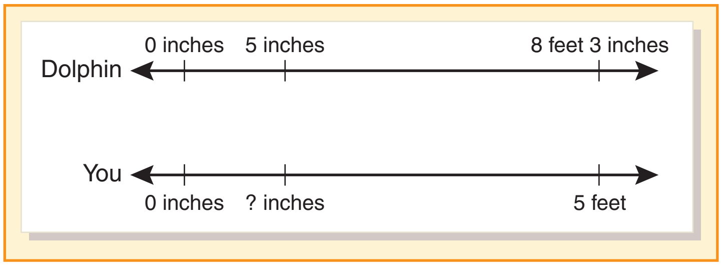
Activity 11.15

CCSS-M: 6.RP.A.3a, d; 7.RP.A. 2b; 7.RP.A.3

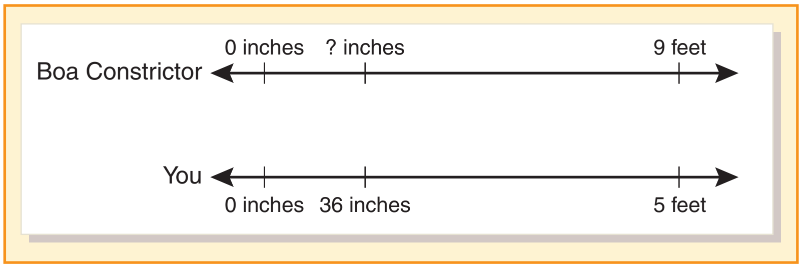
You and the Zoo

Bring in small (several inches in length) plastic animals that are in scale with each other and play dough (construction paper could be used instead). Explain to students that they will use double number lines to prepare a zoo scene—starting with preparing a replica of themselves!

Each pair or group of students is given a plastic animal and asked to make a representation of themselves that is in proportion to the animals. First, students look up the actual height of the animal (e.g., online or provided by you), then they record their actual height and figure out their miniature height. Here is an example:



Once the "You" number line is known, students can identify other animals they would like to create for their zoo. Challenge students to select two favorite animals, or assign animals that vary in size and might not be hard to sketch or mold. Examples include: bald eagle, tree frogs, boa constrictor, panda bear, tiger, and lemur.



Note: Consider which measurement conversions your students need to practice, this activity can be done with metric or U.S. units.

Percents

Percent problems can be solved using any of the strategies in this section (e.g., a ratio table). The double number line described above is an excellent tool for solving percent situations. It is particularly helpful in helping students figure out which part is unknown. Once the double number line is created, students can use this diagram to set up a proportion. The values of one number line correspond to the numbers or measures in the problem. On the second number line, the values indicate the corresponding values in terms of percents (with a whole of 100). The two lines therefore lend to setting up equations that reflect between or within ratios.

Activity 11.16

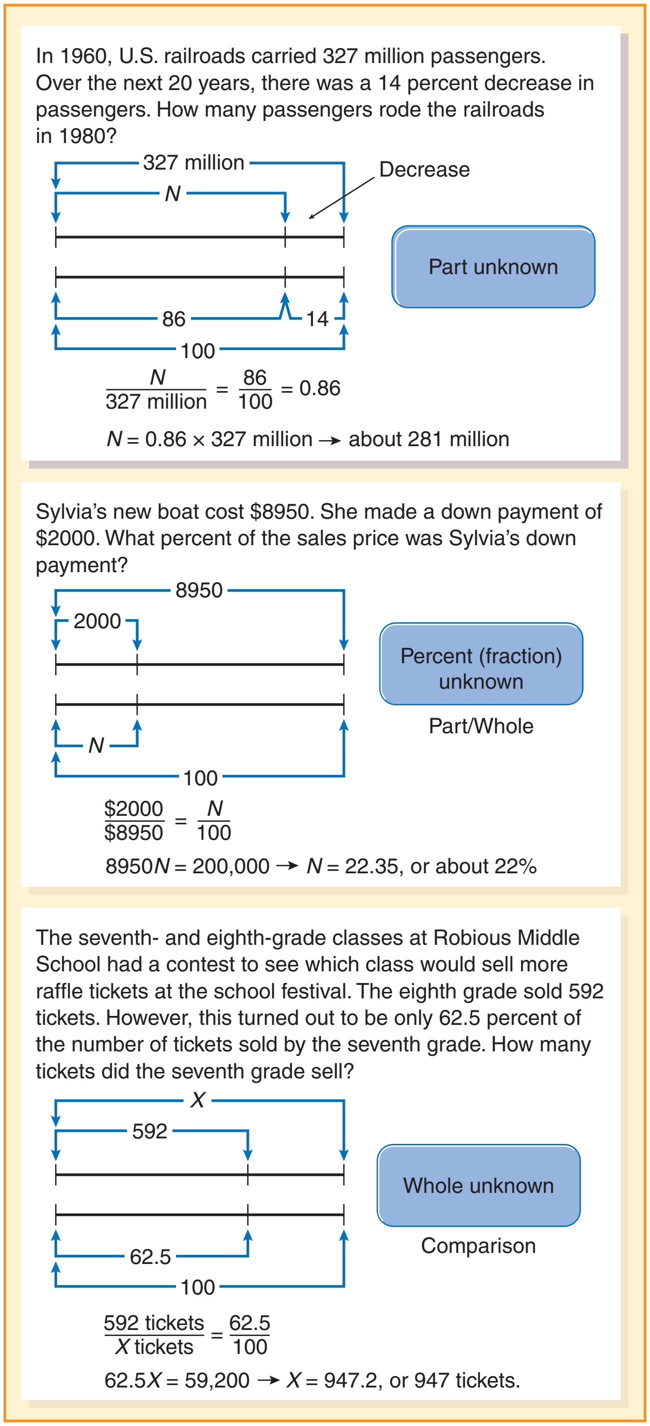
CCSS-M: 6.RP.3c; 7.RP.A. 2 c; 7.RP.A.3

Making Sense of Percent Stories

This activity can be done with any set of traditional missing value percent problems. You can post a story, or you can create a set of cards, each with its own problem on it. Three such stories are provided in Figure 11.14 along with how they can be illustrated on a double number line. In order to make the activity interactive and to connect representations follow these steps: (1) hand out problems (different problems to different groups), (2) have each group set up their double number line on a notecard (don't solve it), (3) trade the double number line sketch with another pair of students, (4) write a proportion that matches the double number line and solve it, (5) return the cards to the original pair to check to see if the answer makes sense.

Figure 11.14

Percentage problems represented as double number lines and equations.



In 1960, U.S. railroads carried 327 million passengers. Over the next 20 years, there was a 14 percent decrease in passengers. How many passengers rode the railroads in 1980?

$$\frac{N}{327\text{ million}} = \frac{86}{100} = 0.86$$

$N = 0.86 \times 327$ million → about 281 million

Sylvia's new boat cost $8950. She made a down payment of $2000. What percent of the sales price was Sylvia's down payment?

$$\frac{\$2000}{\$8950} = \frac{N}{100}$$

$8950N = 200{,}000 \rightarrow N = 22.35$, or about 22%

The seventh- and eighth-grade classes at Robious Middle School had a contest to see which class would sell more raffle tickets at the school festival. The eighth grade sold 592 tickets. However, this turned out to be only 62.5 percent of the number of tickets sold by the seventh grade. How many tickets did the seventh grade sell?

$$\frac{592\text{ tickets}}{X\text{ tickets}} = \frac{62.5}{100}$$

$62.5X = 59{,}200 \rightarrow X = 947.2$, or 947 tickets.

Notice how flexible this double number line representation is for different types of percent problems. It allows modeling of not only part whole scenarios but also increase–decrease situations, and those in which there is a comparison between two distinct quantities. Another advantage of a linear model is that it does not restrict students from thinking about percents greater than 100 since the line can represent more than 100 percent, which is not true for a circle model (Parker, 2004).

Equations (Cross Products)

Cross products are sometimes considered the standard algorithm for solving proportions. Unfortunately, it is sometimes presented as the only method and/or presented in a step-by-step way without connecting to why it works (e.g., "Set up your proportion, cross multiply, and then solve for *x*."). If not well understood, students are more prone to make errors or not use the method. Smith (2002), for example, found that sixth- and seventh-graders rarely use cross multiplication to solve proportion problems, even when that method has been taught. When teaching cross products, make sure students understand why it works and realize that this is one of many ways to find the missing value. Here we share how to help students understand how to use equations to solve proportion problems.

Teaching Tip

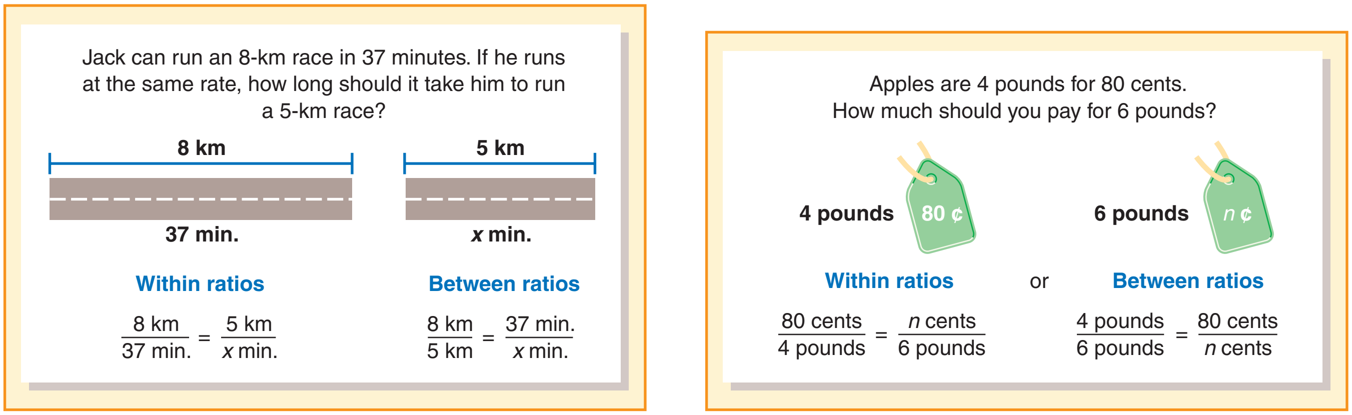
Ask students to create visual cues to set up proportions—it is a very effective way to support a wide range of learners.

Create a Visual

Rather than start with telling students to set up their proportion, ask them to illustrate the problem in a way that shows what is covarying. Figure 11.15 shows two examples where the first representation reflects the context. Both examples include two equations, one that focuses on within ratios and one that focuses on between ratios. Providing visual cues to set up proportions is a very effective way to support a wide range of learners.

Figure 11.15

Drawings that illustrate the varying quantities can help in setting up proportions.



Solve the Proportion

Look at the situations in Figure 11.15. As students (and adults) often do naturally, you can determine the *unit rate* to solve these problems. For example, you can find the price for 1 pound of apples by dividing the $0.80 by 4 and then multiplying this result by 6 to determine the price of 6 pounds. The equation is $(\$0.80 \div 4) \times 6 = \$1.20$. Or you can examine the *scale factor* from 4 to 6 pounds (within ratio), which is 1.5. Multiply $0.80 by the same scale factor to get $1.20. The equation is $(6 \div 4) \times \$0.80 = \$1.20$. One equation uses $0.80 in multiplication, and the other equation uses $0.80 in division. These are exactly the two devices we employed in the other two approaches: (1) *scale factor* and (2) *unit rate*. If you cross multiply the between ratios, you get exactly the same result. Furthermore, you get the same result as if you had written the two ratios inverted—that is, with the reciprocals of each fraction. Try it!

Teaching Proportional Reasoning

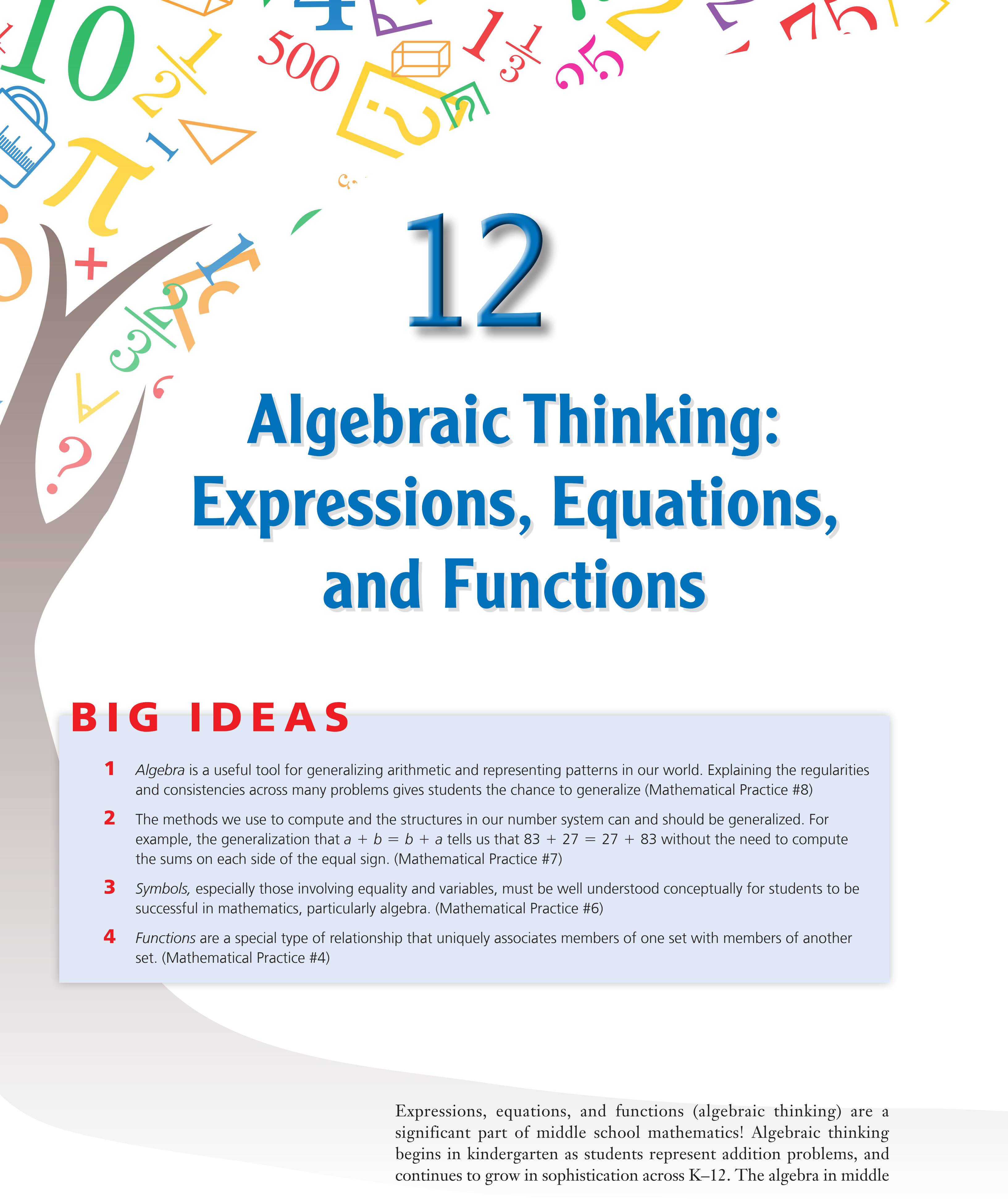
Considerable research has been conducted to determine how students reason in various proportionality tasks and to determine whether developmental or instructional factors are related to proportional reasoning (e.g., see Bright et al., 2003; Lamon, 2007, 2012; Lobato et al., 2010; Siegler et al., 2010). The findings are shared here as a way to summarize the chapter.

1. Use composed-unit and multiplicative-comparison ideas in building understanding of ratio. Learning more about multiplicative comparisons should lead to an understanding of rate, which is a strategy to be applied to proportions.

2. Help students distinguish between proportional and nonproportional comparisons by providing examples of each and discussing the differences.
3. Provide ratio and proportion tasks in a wide range of contexts, including situations involving measurements, prices, geometric and other visual contexts, and rates of all sorts.
4. Engage students in a variety of strategies for solving proportions. In particular, use ratio tables, visuals, (e.g., tape diagrams and double number lines), equations, and graphs to solve problems—always expecting students to apply reasoning strategies.
5. Recognize that symbolic or mechanical methods, such as the cross-product algorithm, for solving proportions do not develop proportional reasoning and should not be introduced until students have had many experiences with intuitive and conceptual methods.

Literature Connections

Literature brings an exciting dimension to the exploration of proportional reasoning. Many books and stories discuss comparative sizes, concepts of scale as in maps, giants and miniature people who are proportional to regular people, comparative rates (especially rates of speed), and so on. For example, Beckman, Thompson, and Austin (2004) explore the popular *Harry Potter* stories, *The Lord of the Rings*, and *The Perfect Storm* for exciting contexts for proportional reasoning activities. Table 11.1 provides a great start, but many more exist beyond this list!



12

Algebraic Thinking: Expressions, Equations, and Functions

BIG IDEAS

1. *Algebra* is a useful tool for generalizing arithmetic and representing patterns in our world. Explaining the regularities and consistencies across many problems gives students the chance to generalize (Mathematical Practice #8)
2. The methods we use to compute and the structures in our number system can and should be generalized. For example, the generalization that $a + b = b + a$ tells us that $83 + 27 = 27 + 83$ without the need to compute the sums on each side of the equal sign. (Mathematical Practice #7)
3. *Symbols,* especially those involving equality and variables, must be well understood conceptually for students to be successful in mathematics, particularly algebra. (Mathematical Practice #6)
4. *Functions* are a special type of relationship that uniquely associates members of one set with members of another set. (Mathematical Practice #4)

Expressions, equations, and functions (algebraic thinking) are a significant part of middle school mathematics! Algebraic thinking begins in kindergarten as students represent addition problems, and continues to grow in sophistication across K–12. The algebra in middle school—and high school—should not be a course dominated by symbolic manipulation for the sake of simplification and artificial applications with little connection to the real world. Algebraic thinking, including functions, must focus on thinking and reasoning that uses mathematics to make sense of the world.

In the *Common Core State Standards* (CCSSO, 2010), the close connections between arithmetic and algebra are explicit in the cluster titles, “Operations and Algebra”. In middle school, students begin to study algebra in more abstract and symbolic ways, focusing on understanding and using variables, expressions, and equations. CCSS-M introduces ***functions*** as a domain in grade 8, though ***functional thinking*** begins several years earlier as students consider situations that co-vary, such as the relationship between number of t-shirts purchased and the cost of those t-shirts. Algebraic thinking is present across content areas and is central to mathematical reasoning, as can be seen from the strong connections to the Standards for Mathematical Practice (noted parenthetically in the Big Ideas at the start of this chapter). A summary of the way algebraic thinking (expressions, equations, and functions) is developed in grades 6–8 looks like this:

> *Grade 6:* Explore ratios and rates (co-variation) using various techniques, including equations that illustrate the relationship between two quantities; they begin to simplify expressions and solve an equation for a missing value represented with a variable.
>
> *Grade 7:* Extend ratio concepts to proportions, writing the relationships as equations and graphing them; solve real world problems writing linear equations or inequalities to represent the situations.
>
> *Grade 8:* Continue to solve linear equations and to write equations to represent a situation that co-varies (two variables), comparing situations to determine which are linear and/or proportional; begin to explore systems of equations; begin explicit explorations of functions.

Algebraic thinking involves forming generalizations from experiences with number and computation, formalizing these ideas with the use of a meaningful symbol system, and exploring the concepts of pattern and functions. Far from being a topic with little real-world use, algebraic thinking pervades all of mathematics and is essential for making mathematics useful in daily life.

Researchers suggest three strands of algebraic reasoning, all infusing the central notions of generalization and symbolization (Kaput, 2008; Blanton, 2008):

1. The study of structures in the number system, including those arising in arithmetic (algebra as generalized arithmetic)
2. The study of patterns, relations, and functions
3. The process of mathematical modeling, including the meaningful use of symbols

These three strands provide the organization for this chapter, with the first and third strand split into two parts.

Structure in the Number System: Connecting Number and Algebra

Algebra is often referred to as generalized arithmetic. For students to generalize an operation or pattern, they must look at several examples and notice what is happening in the problem. In other words, they need to gain insights into the structure of the number system. Here we share ways to connect number and algebra.

Generalizing Number Patterns

Even the most basic arithmetic situation can be extended to look at generalizations about numbers. The following task, based on Neagoy (2012), uses the context of birds to explore ways to decompose a number (elementary concept), a pattern that can be generalized using variables (a middle school concept).

Activity 12.1

CCSS-M: 6.EE.A.2a; 6.EE.C.9; 8.F.B.4

Birds in the Backyard

Distribute a copy of **Birds in the Backyard** to every student (or to pairs of students). Pose the following question:

> *Seven birds have landed in your backyard, some landed on a tree and some are at your feeder. How many birds might be in the tree and how many at the feeder?*

Ask students to list all the ways that are possible for 5, 7, and 10 birds, completing the tables on the Activity Page (note: the tables provide more cells than are needed). After students have completed the three tables, ask, "If I have t birds in the tree, how might you describe how many birds are at the feeder?" Students might answer, "Seven minus t." Write $7 - t$. If the students answer "f" (for the variable to represent the number of birds at the feeder), then ask how t and f are related in an equation. Three equations could describe this situation: $t + f = 7$, $7 - f = t$, and $7 - t = f$. Finally, ask students to answer the question at the bottom of the page: "If you have n birds in the backyard, how many ways might the birds be split between the tree and the feeder?" Ask students to make sense of the answer related to the context and to write the generalization using symbols.

This is a *generalization* for how to determine the number of solutions without listing them. In the *Common Core State Standards* for sixth grade, students are asked to solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$. Using a problem that is concrete and that begins with listing numeric possibilities is a way to help students learn to generalize and use variables.

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

Algorithms

Slight shifts in how arithmetic problems are presented can open up opportunities for generalizations that can help students understand *why* and *how* an algorithm "works" (Blanton, 2008). Instead of a series of unrelated computation problems, prepare a list that can lead to a discussion of a generalizable rule (i.e., pattern or algorithm).

$$\frac{1}{2} \times 12 = \quad \frac{1}{4} \times 12 = \quad \frac{1}{8} \times 12 = \quad \frac{3}{4} \times 12 = \quad \frac{3}{8} \times 12 =$$

Once students have solved sets of related problems, focus attention on what you want students to notice (generalize), asking questions like these:

What do you notice?

When will this be true?

Why is this true?

Use of when and why questions improves students' flexibility, a core component to procedural fluency (Star, 2005). In their own words, students will explain the relationship between the numerator and the denominator and what that means in multiplication situations with a fraction and a whole number.

Standards for Mathematical Practice

7 Look for and make use of structure.

If students are to be successful in algebra, which is more abstract and symbolic than previous work with numbers, such discussions must be a part of the daily middle-school mathematics experience (Mark, Cuoco, Goldenberg, & Sword, 2010). Algebraic thinking is central to mathematical proficiency. Not only are Standards for Mathematical Practice 7 and 8 relevant (see margin notes), but others could be listed here, too (which is true throughout this chapter).

Structure in the Number System: Properties

The importance of the properties of the operations cannot be overstated. Table 12.1 provides a list of the ones students must know, including how students might describe the property. In the CCSS-M, properties of the operations are one of the only things mentioned in each grade for grades 1–8. Importantly, the emphasis is on *using* and *applying* the properties (not identifying them):

> *Grade 6:* Apply the properties of operations to generate equivalent expressions. (p. 44)
>
> *Grade 7:* Apply properties of operations as strategies to add, subtract, multiply and divide rational numbers. (p. 48, 49)
>
> Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients. (p. 49)
>
> *Grade 8:* Know and apply the properties of integer exponents to generate equivalent numerical expressions. (p. 54)

Table 12.1. Properties of the number system.

Name of Property	Symbolic Representation	How Students Might Describe the Pattern or Structure
Addition		
Commutative	$a + b = b + a$	"When you add two numbers in any order, you'll get the same answer."
Associative	$(a + b) + c = a + (b + c)$	"When you add three numbers, you can add the first two and then add the third, or add the last two numbers and then add the first number. Either way, you will get the same answer."
Additive Identity	$a + 0 = 0 + a = a$	"When you add zero to any number, you get the same number you started with."
	$a - 0 = a$	"When you subtract zero from any number, you get the number you started with."
Additive Inverse	$a + (^{-}a) = 0$	"When you subtract a number from itself, you get zero."
Inverse Relationship of Addition and Subtraction	If $a + b = c$ then $c - b = a$ and $c - a = b$	"When you have a subtraction problem you can 'think addition' by using the inverse."

(*continued*)

Table 12.1. Properties of the number system. (continued)

Name of Property	Symbolic Representation	How Students Might Describe the Pattern or Structure
Multiplication		
Commutative	$a \times b = b \times a$	"When you multiply two numbers in any order, you will get the same answer."
Associative	$(a \times b) \times c = a \times (b \times c)$	"When you multiply three numbers, you can multiply the first two and then multiply the answer by the third, or multiply the last two numbers and then multiply that answer by the first number. Either way, you will get the same answer."
Multiplicative Identity	$a \times 1 = 1 \times a = a$	"When you multiply one by any number, you get the same number you started with."
Multiplicative Inverse	$a \times \frac{1}{a} = \frac{1}{a} \times a = a$	"When you multiply a number by its reciprocal, you will get one."
Inverse Relationship of Multiplication and Division	If $a \times b = c$ then $c \div b = a$ and $c \div a = b$	"When you have a division problem, you can 'think multiplication' by using the inverse."
Distributive (Multiplication over Addition)	$a \times (b + c) = a \times b + a \times c$	"When you multiply two numbers, you can split one number into two parts (5 can be 2 + 3), multiply each part by the other number, and then add them together."

Making Sense of Properties

The properties are essential to computation (Blanton, Levi, Crites, & Dougherty, 2011). Traditionally, instruction on the properties has been on matching equations to which property they illustrate. That is not sufficient and should not be the focus of your instruction on the properties. Instead, focus on helping students recognize and understand these important generalizations—and use them to generate equivalent expressions in order to solve problems efficiently and flexibly.

Before middle school, students apply properties of addition and multiplication as they learn basic facts and invented and standard algorithms for the four operations. For example, understanding the commutative property for both addition and multiplication reduces substantially the number of facts to be memorized. In middle school, students examine these properties explicitly and express them in general terms without reference to specific numbers. For example, a student solving $394 + 176 = n + 394$ may say that n must be 176 because $394 + 176$ is the same as $176 + 394$. This is a specific instance of the commutative property. To ensure students recognize the pattern can be generalized to any numbers ask students to share the generalization in using symbols (e.g., $a + b = b + a$). Being able to describe the generalized property with examples, in words and with symbols, is what "making structure explicit" means. When made explicit and understood, these structures not only add to students' tools for computation but also enrich their understanding of the number system, providing a base for even higher levels of abstraction (Carpenter et al., 2003).

Although students may understand the commutative property of multiplication with whole numbers, they may not recognize that the property also applies to fractions (in fact, all real numbers). Ask students "Is this true for fractions?" "Is it true for other types of numbers?" "All numbers?" Just as sets of tasks can be used to generalize an algorithm, *pattern-rich problem sets* can be used to focus on the properties (see Figure 12.1)

After students have a chance to explore the sets of tasks, ask, "What do you notice across the problems?" As students share patterns and relationships, encourage them to use the names of the properties.

Activity 12.2 provides another creative way for students to understand the identity for addition and/or multiplication, as well as other properties.

Figure 12.1

Pattern-Rich Problem Sets.

Set 1:

1. $0.15 \times 42 =$
2. $(0.10 + 0.05)42 =$
3. $0.3 \times 21 =$

Set 2:

1. $\frac{1}{6} \times 12 =$ 2. $12 \times \frac{1}{6} =$
3. $\frac{2}{3} \times 12 =$ 4. $12 \times \frac{2}{3} =$

Set 3: Simplify these expressions for $x = 7$.

1. $15x + 6x$
2. $3(5x + 2x)$
3. $21x$

Activity 12.2

CCSS-M: 6.EE.A.3; 7.NS.A.1; 7.EE.A.1

Different Ways to Zero

Place students in partners. Their task is to use 3 to 5 numbers to generate different ways to get to 0. For example, one answer could be $-8(4 + 2) + 48$ and another answer could be $32 + 16 - 48$. Be sure students are using correct notation and grouping so that their statements are true. After they generate the expressions, ask students how to show they can apply a property to get from one expression on their list to another one on their list. You can incorporate fractions, integers, and decimals by asking students to include these values in at least some of their expressions, or have an entire set with only fractions. Similarly, you can explore linear expressions asking, "Show me 3 equivalent expressions, where all three include a variable." For example, $3x + x$, $4x$, and $x + x + x + x$. Algebra tiles can be a strong support for all students, particularly those that have learning disabilities or that benefit from geometric or concrete representations. After students generate the three expressions, ask students to show or explain which property was applied to get from one expression on their list to another one on their list (or they can trade lists with another group and provide this justification).

Standards for Mathematical Practice

7 Look for and make use of structure.

True/false equations also provide an opportunity to explore properties. Notice that in the discussion, the teacher focuses on investigating the distributive and associative properties, not on whether the equation is true or false:

Ms. J: [*Pointing at* $(2 \times 8) + (2 \times 8) = 16 + 16$ *on the board*] Is it true or false?

LeJuan: True, because two 8's is 16 and two 8's is 16.

Lizett: 2 times 8, plus 2 times 8 is 32 and 16 + 16 is 32.

Carlos: 8 plus 8 is 16, so 2 times 8 is 16, and 8 plus 8 is 16, and 2 times 8 is 16.

Ms. J: [*Writing* $4 \times 8 = (2 \times 8) + (2 \times 8)$ *on the board*] True or false?

Students: True.

Ms. J: What does the 2 stand for?

Reggie: Two boxes of 8.

Ms. J: So, how many boxes are there?

Students: 4.

Ms. J: [*Writing* $32 + 16 = (4 \times 8) + (a \times 8)$ *on the board*] What is a?

Michael: 2, because 4 times 8 is 32, and 2 times 8 is 16.

Ms. J: [*Writing* $(4 \times 8) + (2 \times 8) = (b \times 8)$ *on the board*] What is b?

Students: 6.

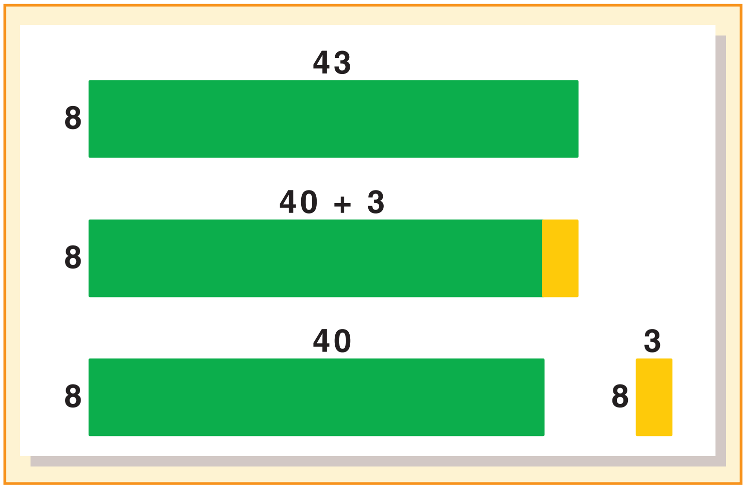
Source: Baek, J. M. (2008). Developing algebraic thinking through exploration in multiplication. In Greenes, C. E., & Rubenstein, R. (Eds.), *Algebra and algebraic thinking in school mathematics: 70th NCTM yearbook* (pp. 151–152). Reston, VA: NCTM.

Notice how the teacher is developing the aspects of these properties in a conceptual manner, focusing on exemplars to guide students to generalize, rather than asking students to memorize the properties as they appear in Table 12.1.

The structure of numbers can sometimes be illustrated geometrically. For example, 43×8 can be illustrated as a rectangular array. That rectangle can be partitioned into two rectangles $(40 \times 8) + (3 \times 8)$, preserving the quantity:

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.



Challenge students to think about this idea *in general.* This may be described in words (at first) and then as symbols: $a \times b = (c \times b) + (d \times b)$, where $c + d = a$. Be sure students can connect the examples to general ideas and the general ideas back to examples. This is the distributive property, and it is perhaps the most important central idea in arithmetic (Goldenberg, Mark, & Cuoco, 2010).

Proving and Justifying Properties

Noticing generalizable properties and attempting to prove that they are true is a significant form of algebraic reasoning and is at the heart of what it means to do mathematics (Ball & Bass, 2003; Carpenter et al., 2003; Schifter, Monk, Russell, & Bastable, 2007). Making conjectures and justifying when conjectures are or are not true are central to fostering algebraic reasoning and sense making (Kieren, 2014). Therefore, when conjectures are made in class, rather than respond by telling students whether that conjecture is true or false, ask, “Do you think that is always true? How can we find out?” Students need to reason through ideas based on their own thinking and not simply rely on the word of others.

A good way to start is to ask students to try to state in words an idea of something they think is always true. For example, when multiplying a number by a second number, you can split the first number and multiply each part by the second number, and you will get the same answer. If a generalization is not clear or entirely correct, have students discuss the wording until all agree that they understand. Write the statement on the board in words and in symbols. Call it a conjecture, and explain that it is not necessarily a true statement until someone either proves it or finds a counterexample.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Activity 12.3

CCSS-M: 5.OA.A.1; 6.EE.A.4; 7.NS.A.1d

Convince Me Conjectures

To begin, offer students a conjecture to test (See **Conjecture Cards** for ideas for the four operations). For example, “If you add one to one addend and take one away from the other addend, the answer will be the same.” Ask students to (1) test the conjecture and (2) prove it is true for any numbers. Point out the difference between testing and proving. Then, invite students to create their own (in words) that they believe is always true. Then, they must prepare a visual or explanation to convince others that it is true. All students, but particularly ELLs, may need support to use precise language. You can “revoice” their ideas using appropriate phrases to help them learn to communicate mathematically, but be careful to not make this the focus—the focus should be on the ideas presented. To add more structure to the task, post the following equation: $2 \times 5 + 5 = 3 \times 5$. Ask students, “Could I trade 5 for another number and still have a true statement? What other numbers will work?” Allow time for students to explore, and then have them share their ideas. Ask, “When does this work?” They will share that it works for any number (even fractions, decimals, and negative numbers). Ask students to write the conjecture using variables ($2 \times n + n = 3 \times n$). Importantly, students with disabilities benefit from the presentation and discussion of counterexamples.

Activity 12.4

CCSS-M: 5.OA.A.1; 6.EE.A.3

Broken Calculator: Can You Fix It?

Distribute calculators to every student. In partners, have students select one of these two problems to explore. They must decide if it is possible, share an example of how to do it (if it is possible), and finally, prepare an illustration and a symbolic justification to describe why it works or doesn’t work (in general).

1. If you cannot use any of the even keys (0, 2, 4, 6, 8), can you create an even number in the calculator display? If so, how?
2. If you cannot use any of the odd keys (1, 3, 5, 7, 9), can you create an odd number in the calculator display? If so, how?

In the follow-up discussion, ask students to compare their illustrations and symbolic justifications.

Using and *applying* the properties is central to mathematical proficiency—it is not only emphasized in the CCSS-M content, but also in the Mathematical Practices (CCSSO, 2010). An explicit focus on seeking generalizations and looking for structure is also important in supporting the range of learners in the classroom, from those who struggle to those who excel (Schifter, Russell, & Bastable, 2009). Doing so requires planning—deciding what questions you can ask to help students think about generalized characteristics within the problems they are doing—across the mathematical strands (not just when they are in an "algebra" unit).

Patterns and Functions

Standards for Mathematical Practice

7 Look for and make use of structure.

8 Look for and express regularity in repeated reasoning.

Patterns are found in all areas of mathematics. Learning to look for patterns and how to describe, translate, and extend them is part of thinking algebraically. A function is a special case of a pattern—a function describes a special relationship between two things that co-vary in such a way that each input has a single output. Looking for a pattern in visuals and contexts that co-vary is a way to build a conceptual understanding of function. Two of the eight Standards for Mathematical Practice begin with the phrase "look for," implying that students who are mathematically proficient pay attention to patterns as they do mathematics. In middle school, the study of patterns and functions is addressed as follows in the *Common Core State Standards*:

- Represent and analyze quantitative relationships between dependent and independent variables (grade 6)
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations (grade 7)
- Understand the connections between proportional relationships, lines, and linear equations (grade 8)
- Define, evaluate, and compare functions (grade 8)
- Use functions to model relationships between quantities (grade 8)

This list is an indication of the importance of focusing on patterns and functions within the algebra curriculum.

Looking for and Analyzing Patterns

Students are surrounded by patterns in the world around them. Keep a look out for patterns that can be analyzed and used to make predictions. Encourage students to do the same. This can be as simple as asking students to think of one situation that covaries (e.g., the time it takes me to get to school is related to how many blocks I walk). Or, they can be assigned to look for patterns or functions in the newspaper (e.g., pricing of food items or graphs illustrating relationship between two things).

Predicting has some interesting real-world contexts appropriate for upper elementary and middle school students. One context is the Olympics (Bay-Williams & Martinie, 2004). The Summer Olympics are held in 2020, 2024, and every four years after that. The Winter Olympics are held in 2018, 2022, and so on. This pattern can be described using variables. The years with summer Olympics are in the form $4n$ and winter Olympics in the form $4n + 2$. Hurricanes, the focus of Activity 12.5, are also in a repeating pattern (Fernandez & Schoen, 2008).

Activity 12.5

CCSS-M: 5.OA.B.3; 6.EE.B.2a

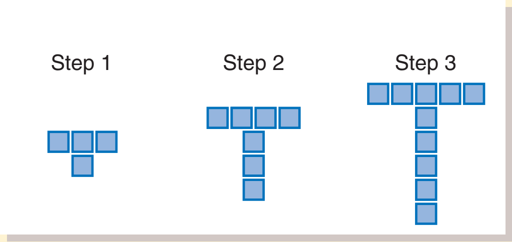
Hurricane Names

Ask students what they know about hurricanes and hurricane names. Hurricanes are named such that the first one of the year has a name starting with A, then B, and so on. For each letter, there are six names in a six-year cycle—an ABCDEF pattern (except that a name is retired when a major hurricane has that name)—and, the gender of the names alternates in an AB pattern. Invite students to select a letter of the alphabet and look up the list of six names. Ask students to answer questions such as these (assume the names don't get retired):

- What years will the hurricanes be named after the first name on your list? The last name? What years will the hurricanes be a girl's name?
- What will the hurricane's name be in the year 2020? 2030? 2050?
- Can you describe in words how to figure out the name of a hurricane, given the year?

Functions and Functional Thinking

A *function* is a rule that assigns to each input exactly one output. An effective way to introduce functions is to explore growing patterns, the focus on the next section. First we share three ways students analyze functional relationships, a developmental process for students as they develop functional thinking (Blanton et al., 2011; Tanish, 2011). Each is shared here, connected to the Growing T Pattern (illustrated below) and the more complicated Dot Pattern. (See Figure 12.2 and the **Predict How Many: Dot Arrays**) Activity Page.



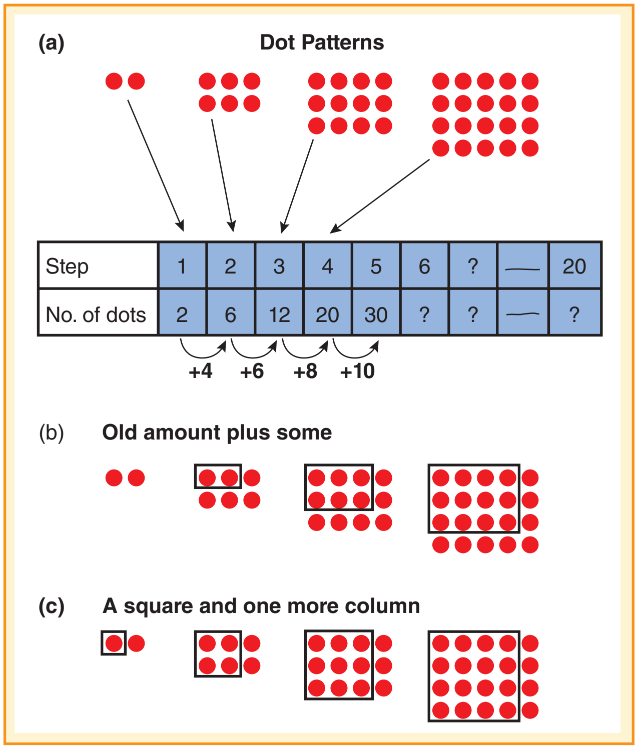
Step Number (*x*)	Number of Tiles (*y*)
1	4
2	7
3	10
4	?

technology note

There are several websites that focus on relationships in functions. NCTM's Illuminations website has a lesson titled, "The Crow and the Pitcher: Investigating Linear Functions Using a Literature-Based Model." PBS Kids' CyberChase has a fun game called "Stop that Creature," where students figure out the rule that runs the game to shut down the creature-cloning machine.

Figure 12.2

Analyzing relationships in the "Dot Pattern."



Recursive Patterns

The description that tells how a pattern changes from step to step is known as a *recursive* pattern (Bezuszka & Kenney, 2008; Blanton, 2008). For most students, it is easier to see the patterns from one step to the next, seeing the increase (or decrease). For the T pattern, this is noticing in the pattern that three more tiles are needed each time, and in the table that the right column goes up by 3 each time the left side goes by 1.

In the Dot Pattern (see Figure 12.2), the recursive pattern is adding successive even numbers, as illustrated in Figure 12.2(a). The recursive pattern can be observed in the physical pattern in at least two ways (see Figure 12.2(b) and 12.2(c)).

Covariational Thinking

Covariational thinking involves noticing how two quantities vary in relation to each other and being explicit in making that connection (Blanton et al., 2011). In the T pattern, a student might say, "As the pattern number grows by 1, the number of tiles needed goes up by 3." Notice that developmentally this is more sophisticated than just noticing a skip pattern as the student is connecting how the change in one quantity impacts the change in the other quantity (i.e., how they covary). In the Dot Pattern, students might note that the pattern goes up two more than the previous step number, so if the last step number went up by 6, the next one will go up by 8.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Explicit Relationship

An *explicit relationship* (also known as a *correspondence relationship*) is a correlation between two quantities expressed as a function rule (Blanton, et al., 2011). In other words, it is being able to look across the table to see how to use the input (x) to generate the output (y). In the T pattern, the rule is $3x + 1$. Imagine that you needed to find the number of tiles needed for the 100th T. If you use recursive thinking, you will need to find all of the prior 99 entries in the table. If you notice how x and y correspond (the explicit relationship), you can use that rule to find how many tiles are needed for the 100th T.

Stop and Reflect

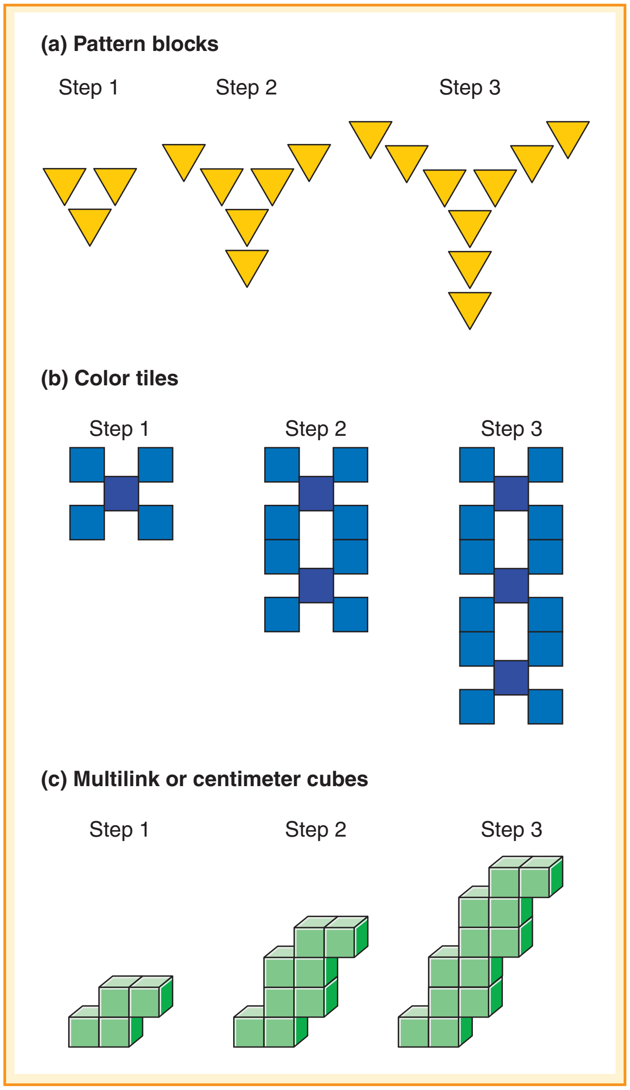
Can you determine an explicit formula for the Dot Pattern in Figure 12.2? How did you find the formula?

Students are likely to discover the explicit rule for a given problem in different ways. Some will analyze the table and notice that if they multiply the step number by the next step number, they will get the number of circles for that step. For the Dot Pattern (Figure 12.2), this leads to the explicit formula: $d = n(n + 1)$ or $d = n^2 + n$, where d is the number of dots and n is the step number. Some will examine the physical pattern to see what is changing. For example, the student who sees it growing like Figure 12.2(c) will notice that the number of dots for one row of the enclosed square is the same as the step number. The column of dots to the right of each enclosed square is also the step number. The related numeric expressions for the growing pattern are $1^2 + 1$, $2^2 + 2$, $3^2 + 3$, and $4^2 + 4$, and that leads to the explicit rule $d = n^2 + n$.

Students will likely be able to describe the explicit rule in words before they can write it in symbols. If the goal of your lesson is to find "the rule," then stopping with the verbal phrase or sentence is appropriate. If your instructional goal is to write the rules with symbols, then ask students first to write the rule in words and then think about how they can translate that statement into numbers and symbols. Conversely, when students share their explicit rules (functions), ask them to connect the symbols back to the physical model and the table.

Figure 12.3

Geometric growing patterns using manipulatives.



Growing Patterns to Functions

Growing patterns are situations that covary, with each step in the pattern being one (independent) variable, and the number of objects (e.g., tiles, cubes, or dots) needed to build it being the second (dependent) variable. Such growing patterns are also called *sequences*. Figure 12.3 shows three different growing patterns, though the possibilities for visuals and patterns are endless. For example, Figure 12.3(a) is a growing pattern in which step 1 requires three triangles, step 2 requires six triangles, and so on. We can say that the number of triangles needed is *a function of* which step it is (in this case, number of triangles = 3 × design number).

With growing patterns, students extend the pattern to see how it is growing, and then analyze the growth in the pattern and in tables and graphs to determine the functional relationship. Analyzing growing patterns should include the developmental progression of looking at the physical pattern, then reasoning about the numerical relationships, and then extending to a larger (or *n*th) case (Friel & Markworth, 2009). Geometric growing patterns are engaging because students can manipulate the objects, and in building they notice how the pattern is changing, helping them to notice recursive and explicit relationships. The questions in Activity 12.6, mapped to the pattern in 12.3(a), can be adapted to any growing pattern, and are good questions to help students reason about functional relationships.

Activity 12.6

CCSS-M: 6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.2; 8.F.A.4

Predict How Many

Working in pairs or small groups, have students explore a geometric growing pattern. Begin with one-step growing patterns like those illustrated in Figure 12.3. Ask students to respond to each of the following or distribute the Predict How Many: Triangle Growing Pattern Activity Page:

- **Complete a table that shows the number of triangles for each step.**

Step	1	2	3	4	5 . . .	10	20
Number of Triangles							

(continued)

- How many triangles are needed for step 10? Step 20? Step 100? Explain your reasoning.
- Write a rule (in words) that gives the total number of pieces to build any step number.
- Write a rule in symbols using the variable *n* for step number.

Keep in mind that ELLs need clarification on the specialized meanings of *step* and *table* because these words mean something else outside of mathematics. [See **Expanded Lesson: Exploring Functions through Geometric Growing Patterns** for more details]. For a two-step pattern, try **Predict How Many: Windows**. For a greater challenge, try **Predict How Many: Dot Arrays**. If you select two or more of the Predict How Many Activity Pages, students can compare these functions the first is linear in the form $y = mx$ the second is linear in the form $y = mx + b$, and the third is quadratic.

Teaching Tip

Where did the multiplication sign × go? Writing equations to describe patterns (functions) is often the first time students see multiplication written in different ways $3 \times d$, $3 \cdot d$, $3d$ Reinforce that all of these are appropriate and can be used interchangeably, though the common form is to drop any multiplication symbol when variables are used ($3d$).

Teaching Tip

Two important questions help students analyze specific examples in order to determine the functional relationship: What is changing? What is staying the same? What is changing (varying) is what becomes the variable.

It is important for students to make connections among representations. So, when a relationship is found in a table, challenge students to see how that plays out in the physical version and vice versa. Students' experiences with growing patterns can begin with fairly straightforward patterns (such as in Figure 12.3) and move to more complicated patterns, such as the Dot Pattern illustrated in Figure 12.2.

These geometric growing patterns are functions. In the dot pattern, we say the number of tiles required is a function of the length of the sides. We also say that s is the independent variable and t is the dependent variable (the number of triangles (t) *depends on* the length of the sides (s)).

Formative Assessment Note

Students can struggle with recognizing the difference between equations and functions (Kieran, 2007). Therefore, it is important that students have opportunities to describe how equations and functions are related, including comparing the goals for each. For example:

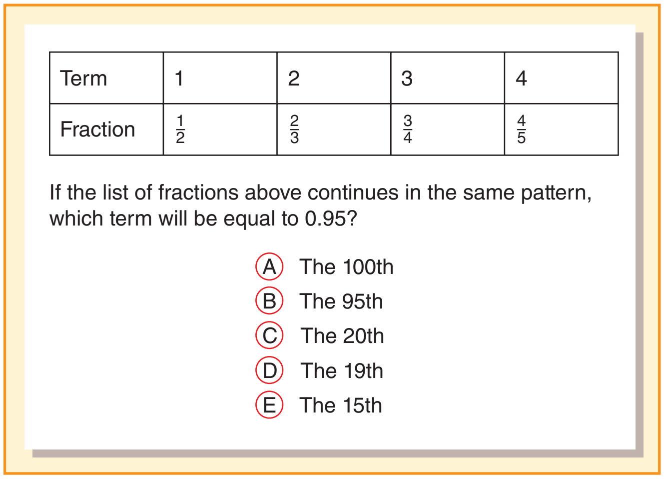
An equation may have any number of variables. When a situation is represented by an equation with one variable, the goal is to determine what the variable is (variable as an unknown), in order to find a solution to the situation.

A function is a situation with two variables. It can be written as an equation (e.g., $y = kx$, $y = mx + b$, $y = x^2$, *and so on). When the situation is a function, the goal is to represent the relationship between the two variables.*

Growing patterns can (and should) include numeric patterns, such as the one illustrated in Figure 12.4 from a NAEP item. In the 2003 NAEP, only 27 percent of 13-year-olds answered this problem correctly (Lambdin & Lynch, 2005). Certainly, we could be incorporating more growing patterns that involve rational numbers, including negative numbers, rather than only exploring geometric or whole number growing patterns. Doing so will strengthen students' fluency with rational numbers and with algebraic thinking!

Figure 12.4

NAEP item for 13-year-olds.



Term	1	2	3	4
Fraction	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$

If the list of fractions above continues in the same pattern, which term will be equal to 0.95?

Ⓐ The 100th
Ⓑ The 95th
Ⓒ The 20th
Ⓓ The 19th
Ⓔ The 15th

Source: Lambdin, D. V. & Lynch, K. (2005). Examining mathematics tasks from the National Assessment of Educational Progress. *Mathematics Teaching in the Middle School, 10*(6), 314–318. Reprinted with permission. Copyright 2005 by the National Council of Teachers of Mathematics. All rights reserved.

The following activity provides an engaging context that can involve any type of numbers (whole numbers, fractions, decimals), and any level of complexity for patterns (e.g., $2n$, $3n + 1$, $\frac{1}{4}n$, etc.).

Activity 12.7

CCSS-M: 5.OA.B.3; 6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.4

Two of Everything

Read *Two of Everything: A Chinese Folktale*, a story about a poor couple's discovery of a magic pot. The pot doubles whatever goes in the pot, including the wife! Begin by exploring the doubling pattern. Use tables of data to help students generalize and write an equation ($y = 2x$ or $f(x) = 2x$) to describe the function. Ask questions like: "What if 200 pencils were dropped in the pot? If 60 tennis balls were pulled out, how many were dropped in the pot?" Second (the next day), explain that the magic pot has been acting up! It is not just doubling; each day, it is using *different rules*. Use the **Magic Pot Mystery Rules** Activity Page or create your own set of mystery rules that best fit your students' needs and grade-level expectations by providing tables with 3–4 rows completed, based on your choice of rules (e.g., $x + 5$, $4x$, or $\frac{1}{2}x$). For each, ask students to add examples to the table, explaining the rule in words and writing the rule as an equation. This activity is well-suited for ELLs because it has a concrete situation that is easily acted out or illustrated, and it connects to a different culture. Additionally, completing the tables for the different rules does not entail an overwhelming use of language, and there is a lot of student interaction (speaking and listening) within the lesson.

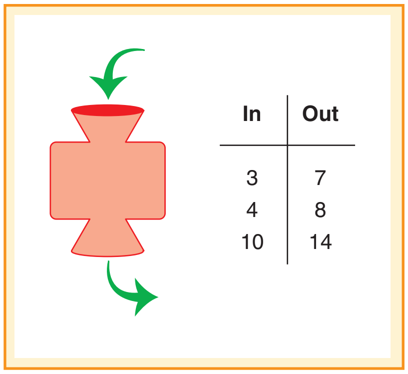
Note the connection to the input-output concept of functions: something goes in (the) pot, and then something comes out (of the) pot (in-pot, out-pot). You can invite students to create a mystery magic pot rule, generate the beginning of a table, give a hint about the rule the pot is using, trade it with another student, and try to figure out the rule.

The input-output idea can be explored without a context, but with a physical "function machine" into which some value goes and another number comes out. Students figure out the function machine rule. This can become a fun daily routine, trading out whoever is the function operator (Figure 12.5).

The choices for function contexts are endless because functions are everywhere in our world. The key is to incorporate a variety of contexts and representations so that students develop a strong understanding of functions.

Figure 12.5

A function machine illustrates the relationship between the independent and the dependent variable in a concrete manner.



In	Out
3	7
4	8
10	14

Teaching Tip

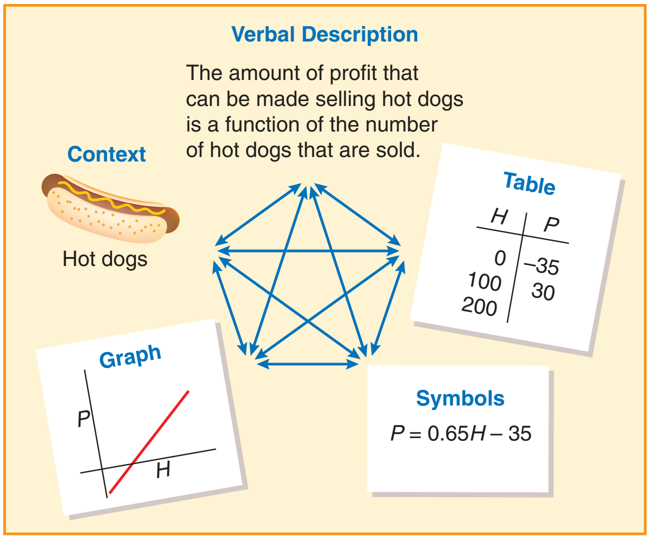
With a large empty box, you can have a student inside a function machine generating the output.

Graphs of Functions

So far, patterns and functions have been represented by (1) physical materials or drawings, (2) tables, (3) words, and (4) symbols. A graph adds a fifth representation (see Blackline Master 42 for a coordinate axis). Figure 12.6 illustrates what these five representations look like with the context of selling hotdogs. Importantly, given any one of these representations, students need to be able to generate the others and understand how they are related.

Figure 12.6

Five representations of a function for the situation of selling hotdogs.



The amount of profit that can be made selling hot dogs is a function of the number of hot dogs that are sold.

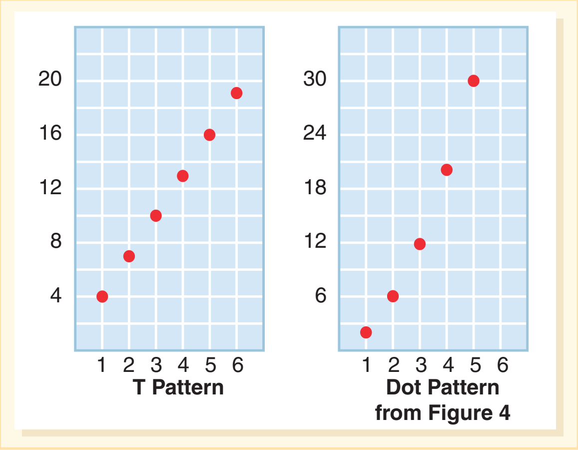
H	P
0	−35
100	30
200	

$P = 0.65H - 35$

Figure 12.7 shows the graph for the T Pattern and the Dot Pattern. Notice that the first is a straight-line (linear) relationship and the other is a curved line that would make half of a parabola if the points were joined.

Figure 12.7

Graphs of two growing patterns.



Activity 12.8

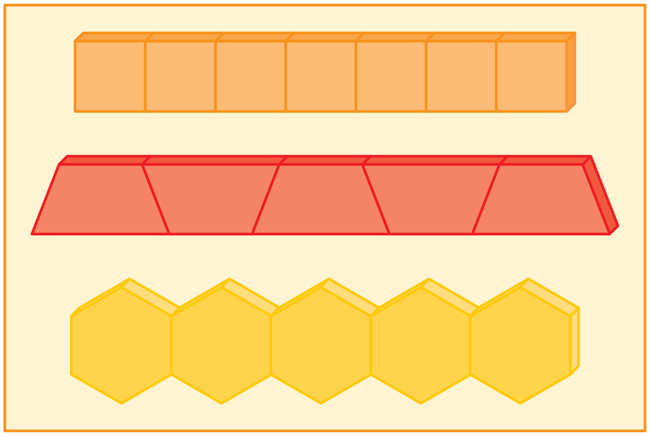
CCSS-M: 6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.2

Perimeter Patterns

Using a document camera or interactive white board, show strings of same-shape pattern blocks (see Figure 12.8). Working in pairs or small groups, have students build each pattern and explore what patterns they notice about how the perimeter grows. Ask: What is the perimeter of the strip for 6 squares? 10 squares? Any number of squares? Repeat the process with trapezoids and hexagons (or have different groups of students working on different shapes). Distribute a **Coordinate Grid**, Blackline Master 22. Ask students to create a graph to illustrate the relationship between **the** number of pattern blocks and the perimeter. After they have generalized the rule for each pattern block shape, ask if they notice a pattern *across* polygons whether they could use that to write a function that describes the perimeter for any length polygon with x sides.

Figure 12.8

Same-color strings of pattern blocks. Can you determine the perimeter for *n* pattern blocks in a string?



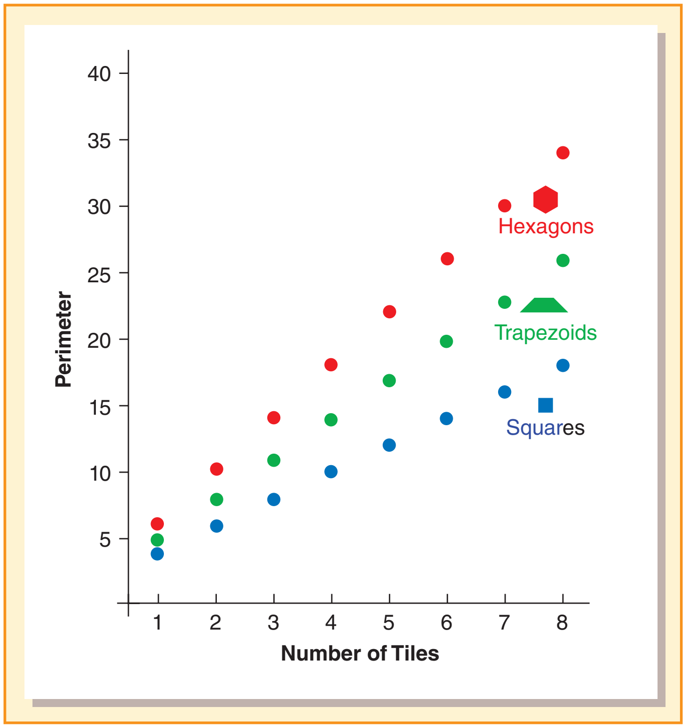
Stop and Reflect

Which representations do you find most useful in determining the explicit rule or function? Which representation do you think students new to exploring patterns will use?

Encourage students to compare and connect the graphs to the physical materials and to the tables (see Figure 12.9).

Figure 12.9

Graphs of the perimeters of three different strings of pattern blocks.



Note: the lines are not drawn because, for this context, there are no solutions between the points. A graph encourages covariational thinking, which can lead to identifying the function. For example, ask students to discuss how to get from one to the next (over 1, up 6), and then ask how that information can be found in the table. Second, identify a particular point on the graph, and ask what it tells about the pattern. Pose the following questions to students to support covariational thinking and help students understand the graph representation of the function:

- How does each graph represent each of the string patterns?
- Why is there not a line connecting the dots?
- Why is one line steeper than the others?
- What does this particular point on the graph match up to in the design and in the table?

Stop and Reflect

How does the graph illustrate the following concepts: Linear or nonlinear? Proportional or nonproportional? How might the graph be connected to the context and to the equation?

To build on the border tiles problem, students can explore other perimeter-related growing patterns, such as a triangle made of dots with 3, 4, or 5 dots on each side. Students should reason that this is the same type of pattern, except that it has three sides, and be able to use their previous generalization to generate and describe the new pattern or function (Steele, 2005).

technology note

Function graphing tools (including graphing calculators) permit users to create the graph of almost any function very quickly. (Check out "Graphsketch," an online demonstration tool for making graphs of equations.) Multiple functions can be plotted on the same axis. It is usually possible to trace along the path of a line or curve and view the coordinates at any point. The dimensions of the viewing area can be changed easily so that it is just as easy to look at a graph for x and y between -10 and $+10$ as it is to look at a portion of the graph thousands of units away from the origin. By zooming in on the graphs, it is possible to find points of intersection without algebraic manipulation or to confirm an algebraic manipulation. Similarly, the point where a graph crosses the axis can be found to as much precision as desired. Digital programs can also be used for these purposes, and they add speed, color, visual clarity, and a variety of other interesting features to help students analyze functions.

Formative Assessment Note

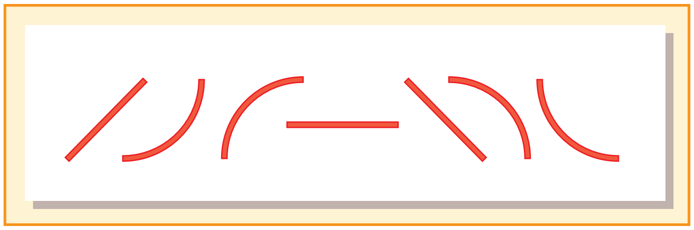
Being able to make connections across representations is important for understanding functions, and the only way to know if a student is seeing the connections is to ask. In a *diagnostic interview*, ask questions like the ones just listed, and look to see whether students are able to link the graph to the context, to the table, and to the formula.

Describing Graphs Qualitatively

Analyzing graphs should focus on how the function is changing—whether it is increasing or decreasing rapidly or gradually. A graph is a picture of the rate of change of one variable in terms of the other. Essentially, graphs can have only one of the seven characteristics shown in Figure 12.10 or some combination of these. These types of change will be seen in the following activity.

Figure 12.10

Seven ways in which graphs can change. A graph may have combinations of these characteristics.



Activity 12.9

CCSS-M: 5.G.A.2; 8.F.B.5

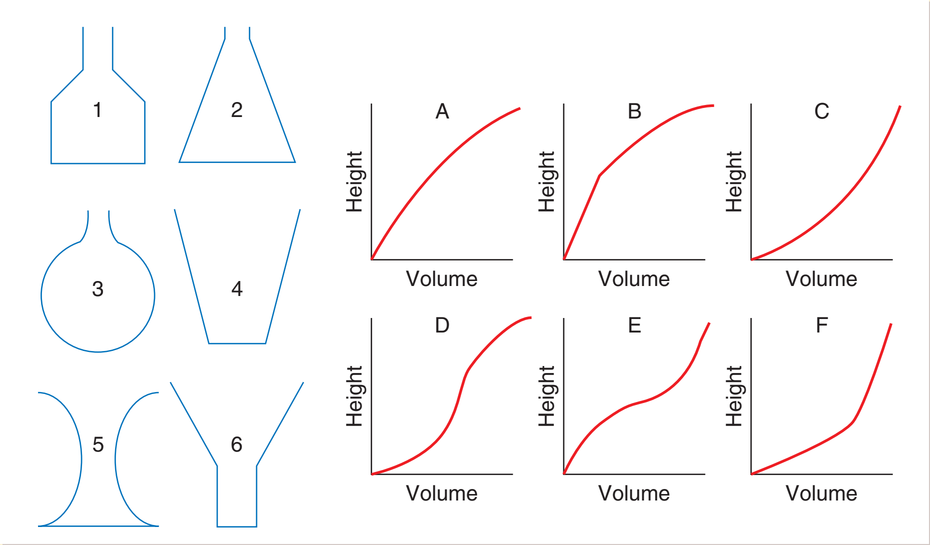
Bottles and Volume Graphs

Figure 12.11 shows six vases and six graphs. Assume the bottles are filled at a constant rate. Because of bottle shape, the height of the liquid will increase either more slowly or more quickly as the bottle gets wider or narrower. Match the graphs with the bottles.

Find some containers that have different shapes. Give each group one vase to use. Fill a small medicine cup with water and empty it into the container, recording in a table the number of little cups used and the height of the water. After each group gathers the data, they graph their findings. Ask students to submit their graphs anonymously, and then redistribute or post the graphs asking other groups to match the graphs with the containers.

Figure 12.11

Assuming the vases are filled at a constant rate, match the graphs with the vases.



Students also benefit from opportunities to estimate, or sketch, graphs. Exploring situations without exact values focuses students' attention on covariation relationships, the focus of Activity 12.10.

Activity 12.10

CCSS-M: 6.EE.D.9; 8.F.B.5

Sketch a Graph

Use the **Sketch-a-Graph** Activity Page, or project the stories as written here. Ask students to sketch a graph for each of these situations. No numbers or formulas are to be used.

1. The temperature of a frozen dinner from 30 minutes before it is removed from the freezer until it is removed from the microwave and placed on the table
2. The value of a 1970 Volkswagen Beetle from the time it was purchased to the present (assuming it was kept in perfect condition)
3. The level of water in the bathtub from when you start filling it to when it is empty after a bath
4. Profit in terms of number of items sold
5. The height of a thrown baseball from when it is released to when it hits the ground
6. The speed of a thrown baseball from when it is released to when it hits the ground

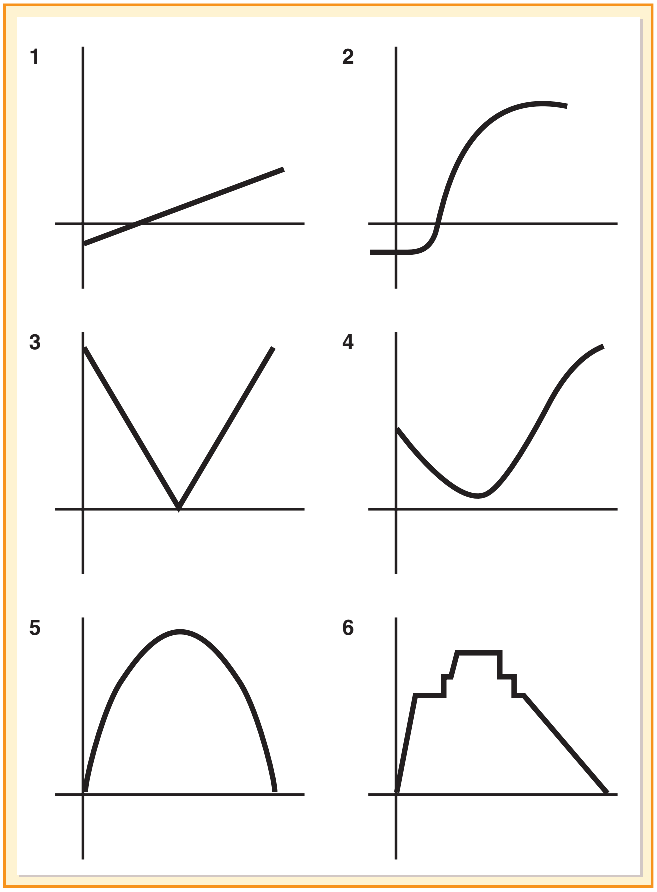
Be sure that the contexts you pick are familiar to ELLs (and other students). If they are not, change the context or provide visuals. For students with disabilities, you may want to have them match graphs (Figure 12.12) before drawing them. Finally, it is also worthwhile to have students look at a graph and write a story to match (see **Create a Journey Story**)

Stop and Reflect

Can you sketch a graph for each situation in the last activity?

Figure 12.12

Match each graph with the situations described in Activity 12.10, “Sketch a Graph.” Talk about what change is happening in each case.



Characteristics of Functions

Importantly, as you explore functions across representations, students, using appropriate terminology, should be able to articulate the patterns and functions that they are noticing. Table 12.2 summarizes important function language. It is not an exhaustive list, but it adds to what has already been discussed in this chapter.

Table 12.2. Function Language.

Concepts	Description	Example
Independent and dependent variables	The *independent variable* is the input, or whatever value is being used to find another value. For example, in the case of the strings of pattern blocks, the independent variable is the number of blocks in the string. The *dependent variable* is the number of objects needed, the output, or whatever value you get from using the independent variable.	In the “Pattern Blocks Perimeter” problem, the dependent variable is the perimeter. You can say that the perimeter of the block structure depends on the number of blocks used.
Discrete and continuous functions	When isolated or selected values are the only ones appropriate for a context, the function is *discrete.* If all values along a line or curve are solutions to the function, then the function is *continuous.*	Discrete: “Pattern Blocks Perimeter” problem (Figures 12.8 and 12.9). Only whole-number values make sense. Continuous: “How Many Gallons Left?” (Activity 12.20). There can be any value for miles and gallons (within the appropriate domain).
Domain and range	The *domain* of a function comprises the possible values for the independent variable. The *range* is the corresponding possible values for the dependent variable.	In the “Pattern Blocks Perimeter” problem, the domain and range are all positive whole numbers. In the “Building a Pen” problem (Figure 12.13), the domain and range are all real numbers between 0 and 12.

Linear Functions

Linear functions are a subset of all functions, but because linearity is a major focus of middle-school mathematics, it appears here in its own section. The *Common Core State Standards* emphasize the importance of linear functions across the middle grades, with a strong focus on linearity in the seventh and eighth grades (CCSSO, 2010).

Stop and Reflect

Think back to many tasks shared in this chapter—birds, border tiles, the magic pot, and the dot pattern. Which are linear functions? Which are not linear, but are still functions?

The examples that involved linear functions include the birds (how many ways they could be on the bush and in the tree), the border tiles problem, $m + n = 1$, the geometric growing patterns, and the "Two of Everything" activity, which can be nonlinear if the rule of the magic pot is something nonlinear (like "square the number"). The dot pattern (Figure 12.10) is nonlinear (it is quadratic).

In middle school, students need to determine whether situations are linear or not. Consider the following example:

You are asked to build a rectangular pen with 24 yards of fence. (1) Write an equation with variables to describe the relationship between the length and the area. (2) Write an equation with variables to describe the relationship between the length and width (Figure 12.13).

An explicit formula for determining the width is $w = 12 - l$ (l is the length), which decreases at a constant rate and is linear. The explicit formula for the area of the pen is $a = l(12 - l)$, which is not linear.

Linear (and nonlinear) situations should be analyzed across representations. In a situation, students should be able to describe a constant rate of change related to the situation. In a graph, this can be established by seeing that the plotted points lie on one line. In a table, the change will be constant (e.g., a recursive pattern is $+ 4$ each time). In the equation, linearity can be determined by looking at the part of the expression that changes and seeing if it represents constant change or not.

Rate of Change and Slope

Rate, whether constant or varying, is a type of change often associated with how fast something is traveling. Rates can be seen in a wide range of contexts, such as the geometric model of the pattern block perimeters or the rate of growth of a plant. Other rate contexts include hourly wages, gas mileage, profit, and cost of an item, such as a bus ticket.

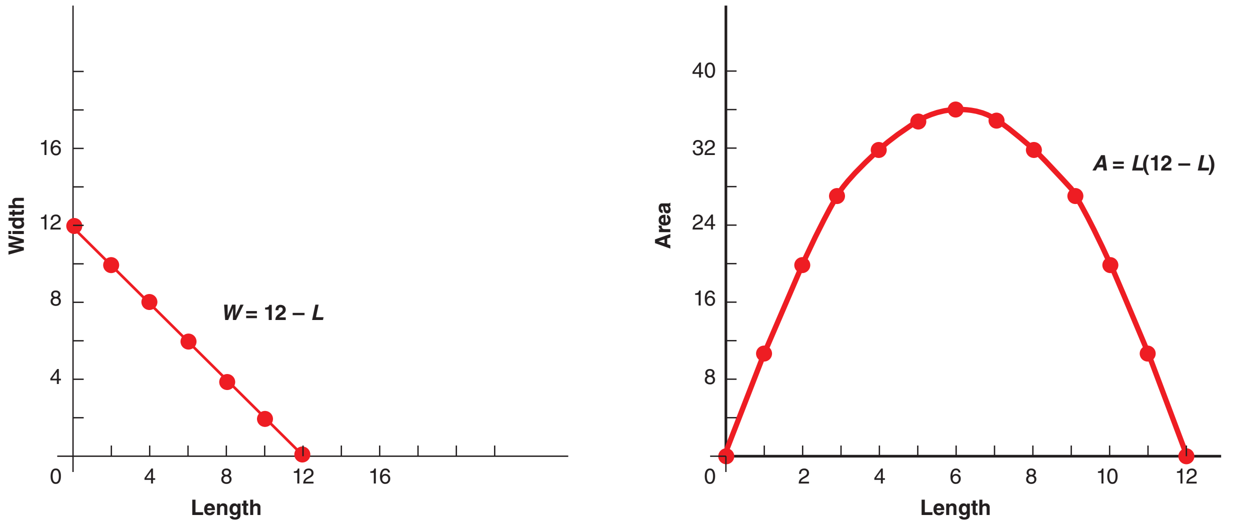
Explorations of linear rate situations develop the concept of *slope*, which is the numeric value that describes the rate of change for a linear function. For example, one of the explicit formulas for the hexagon perimeter growing pattern is $y = 4x + 2$. Note that the rate of change is 4 because the perimeter increases by 4 with each new piece. All linear functions can be written in this form: $y = mx + b$ (including $y = mx$ when $b = 0$).

Figure 12.13

The width and area graphs as functions of the length of a rectangle with a fixed perimeter of 24 units.

Building a Pen

L	0	1	2	3	4	5	6	7	8	9	10	11	12
W	12	11	10	9	8	7	6	5	4	3	2	1	0
A	0	11	20	27	32	35	36	35	32	27	20	11	0



Conceptually, then, slope signifies how much y increases when x increases by 1. If a line contains the points (2,4) and $(3,{}^{-}5)$, you can see that as x increases by 1, y decreases by 9. So the rate of change, or slope, is −9. For the points (4,3) and (7,9), you can see that when x increases by 3, y increases by 6. Therefore, an increase of 1 in x results in a change of 2 in y (dividing 6 by 3). The slope is 2. After further exploration and experiences, your students will begin to generalize that you can find the rate of change or slope by finding the difference in the y values and dividing by the difference in the x values. Exploring this first through reasoning is important for students if they are to be able to make sense of and remember the formula for calculating slope when given two points.

Teaching Tip

For an interactive tool that connects linear equations in the form $y = mx + b$ to graphs, try "Interactive Linear Equation" at Math Warehouse.

Zero Slope and No Slope

Understanding these two easily confused slopes requires contexts, such as walking rates. Consider this story:

> You walk for 10 minutes at a rate of 1 mile per hour, stop for 3 minutes to watch a nest of baby birds, then walk for 5 more minutes at 2 miles per hour.

What will the graph look like for the 3 minutes when you stop? What is your rate when you stop? In fact, your **rate is 0**, and since you are at the same distance for 3 minutes, the graph will be a horizontal line.

Let's say that you see a graph of a walking story that includes a vertical line—a line with **no slope**. What would this mean? It means that there is no change in the x variable—you traveled a distance with no time passing! Now, even if you were a world record sprinter, this would be impossible. Remember that rate is based on a change of 1 in the x value.

Proportional and Nonproportional Situations

Linear functions can be proportional or nonproportional. A babysitter's pay is proportional to the hours he or she works (assuming an hourly rate). Proportional representations are shown in the geometric growing patterns in Figure 12.3. All proportional situations, then, are equations in the form $y = mx$. Notice that the graphs of all situations are straight lines that pass through the origin. Students will find that the slope of these lines is also the rate of change between the two variables.

The Perimeter Patterns, such as the hexagon string, however, are nonproportional. Although you have a constant increase factor of 4, there are 2 extra units of perimeter (the sides of the two end blocks). Said another way, you cannot get from the input (number of blocks) to the perimeter by multiplying by a factor as you can in proportional situations.

In nonproportional situations, one value is constant. In the Growing T Pattern (p. 263), there is always an extra tile, a constant of 1. That leads to the equation $y = 3x + 1$. Other examples include if you were walking at a constant rate but had a head start of 50 meters or if you were selling something and had an initial expense. The constant value, or initial value, can be found across representations beyond the contexts described here. In the table, it is the value when $x = 0$, which means it is the point where the graph crosses the y-axis. Context is important. The **Grocery Store** task, for example, asks students to figure out how long grocery carts are when they are pushed together. Students can see that it grows at a constant rate, but that there is a little extra on the end.

Stop and Reflect

Can you answer the following questions: What is the perimeter for any string of triangles? Any string of squares? Hexagons? What is the same across each of these growing patterns? And the proportional question: Is the number of pieces needed proportional to the string length?

Teaching Tip

Use 0 as an input in tables as students analyze proportional and nonproportional situations—it makes the constant value more visible.

The equation representing the triangles is $y = x + 2$, the equation for the squares is $y = 2x + 2$, and the equation for the hexagons is $y = 4x + 2$. Each string has 2 units of perimeter on each end, so the perimeter is not proportional to the number of blocks used.

Nonproportional situations are more challenging for students to generalize. Students want to use the recursive value (e.g., + 4) as the factor (× 4). Students often make the common error of using the table to find the tenth step and doubling it to find the twentieth step, which works in proportional situations but does not work in nonproportional situations. Having students analyze errors such as these supports their learning of mathematical concepts (Lannin, Arbaugh, Barker, & Townsend, 2006).

Parallel, Same, and Perpendicular Lines

Students in eighth grade should be comparing different linear situations that result in parallel, same, or perpendicular lines (CCSSO, 2010). Using a context helps build understanding.

Larry and Mary each earn \$30 a week for the summer months. Mary starts the summer \$50 dollars in debt, and Larry already has \$20. In week 3, how much more money does Larry have? How much more does he have in week 7? When will Mary and Larry have the same amount of money?

The rates for Larry's and Mary's earnings are the same—and the graphs therefore go up at the same rate—that is, the slopes are the same. The graphs of Larry's earnings ($y = 30x + 20$) and Mary's earnings ($y = 30x - 50$) are parallel. We know this without even making the graphs because the rates (or slopes) are the same.

Can you think of what change in Larry's and Mary's situations might result in the same line? Remember the equivalent expressions discussed earlier? As illustrated on the calculator, they will have the same line. Slopes can also tell us when two lines are perpendicular, but it is less obvious. A little bit of analysis with similar triangles will show that for perpendicular lines, the slope of one is the negative reciprocal of the other.

Meaningful Use of Symbols

Perhaps one reason that students are unsuccessful in algebra is that they do not have a strong understanding of the symbols they are using. For many adults, the word *algebra* elicits memories of simplifying long strings of variable-filled equations with the goal of finding *x*. These experiences of manipulating symbols were often devoid of meaning and resulted in such a strong dislike for mathematics that algebra has become a favorite target of cartoonists and Hollywood writers. In reality, symbols represent real events and should be seen as useful tools for solving important problems that aid in decision making (e.g., calculating how many boxes of cookies we need to sell to make *x* dollars, or the rate at which a given number of employees need to work to finish a project on time). Students cannot make sense of such situations without a strong understanding of mathematical symbols.

Looking at equivalent expressions that describe a context is an effective way to bring meaning to numbers and symbols. The classic task in Activity 12.11 involves such reasoning.

Activity 12.11

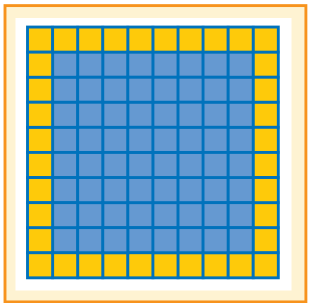
CCSS-M: 6.EE.A.1; 6.EE.A.2a, b, c; 6.EE.A.3; 6.EE.A.4

Border Tiles

Ask students to build an 8 × 8 square array representing a swimming pool with colored tiles such that tiles of a different color are used around the border (Figure 12.14). Challenge students to find at least two ways to determine the number of border tiles used without counting them one by one. Students should use their tiles, words, and number sentences to show how they counted the squares. Ask students to illustrate their solution on **2-Centimeter Grid** paper. For ELLs, the drawing will be a useful support, but be sure the instructions are clear and that they understand that they are counting the outside tiles and need to find more than one way. There are at least five different methods of counting the border tiles around a square other than counting them one at a time. A great tool to help students explore the border tiles problem is the site "Plan Your Room" input your dimensions (e.g., 8 feet, 0 inches × 8 feet, 0 inches) and click "Start with a Room."

Figure 12.14

How many different ways can you find to count the border tiles of an 8 × 8 pool?



Stop and Reflect

See whether you can find four or five different counting schemes for the border tiles problem. Can you see how the different expressions are equivalent? What questions might you pose to students in order to help them focus on these equivalent expressions?

A very common solution to the border tiles problem is to notice that there are 10 squares across the top and also across the bottom, leaving 8 squares on either side. This might be written as follows:

$$10 + 10 + 8 + 8 = 36 \qquad \text{or} \qquad (2 \times 10) + (2 \times 8) = 36$$

Each of the following expressions can likewise be traced to looking at the squares in various groupings:

$$4 \times 9 \qquad 4 \times 8 + 4 \qquad 4 \times 10 - 4 \qquad 100 - 64$$

Standards for Mathematical Practice

4 Model with mathematics.

More equivalent expressions are possible because students may use addition instead of multiplication. In any case, once the generalizations are created, ask students to justify how the elements in the expression map to the physical representation. Ask students to compare the different ways and discuss whether they are all correct (and therefore equivalent) expressions for describing the general rule.

Notice that the task just completed involved numeric expressions—a good place to start making meaning for symbols. These expressions did not involve the two types of symbols that are perhaps the most important to understand—and, unfortunately, among the least well understood by many middle-school students. The equal sign (=) and inequality signs ($<$, $\leq$, $>$, $\geq$, $\neq$) are the first type. Variables are the second type. The sections that follow provide strategies for helping students understand these symbols.

Equal Sign and Inequality Signs

The equal sign is one of the most important symbols in elementary arithmetic, in algebra, and in all mathematics using numbers and operations. At the same time, research dating from 1975 to the present indicates clearly that the equal sign (=) is a very poorly understood symbol (Kieran, 2007; RAND Mathematics Study Panel, 2003) and rarely represented in U.S. textbooks in a way that encourages students to understand the equivalence relationship—an understanding that is critical to understanding algebra (McNeil et al., 2006).

Why is it so important that students correctly understand the equal and inequality signs? First, it is important for students to understand and symbolize relationships in our number system. These signs are how we mathematically represent quantitative relationships. For example, $0.6 \times 11 = 0.6 \times 10 + 0.6 \times 1$ shows a mental strategy for computation based on the distributive property. The equal sign is a principal method of representing these relationships.

Second, when students fail to understand the equal sign, they typically have difficulty with algebraic expressions (Knuth, Stephens, McNeil, & Alibali, 2006). Consider the equation $5x + 24 = 54$. It requires students to see both sides of the equal sign as equivalent expressions. It is not possible to "do" the left-hand side. However, if both sides are understood as being equivalent, students will see that $5x$ must be 24 less than 54 or $5x = 30$. Therefore, x must equal 6.

Stop and Reflect

In the equation below, what number belongs in the box?

$$8 + 4 = \square + 5$$

How do you think students in middle school typically answer this question?

In a classic study, not one sixth grader among 145 put a 7 in the box (Falkner, Levi, & Carpenter, 1999). Try it with your students!

Where do such misconceptions come from? A large majority of equations students encounter prior to algebra look like this: $5 + 7 = \_\_\_\_\_$ or $\frac{3}{4} \times 4\frac{1}{2} = \_\_\_\_\_$. Naturally, students come to see = as a symbol signifying "and the answer is" rather than as a symbol indicating the relationship of equivalence (Carpenter, Franke, & Levi, 2003; McNeil & Alibali, 2005; Molina & Ambrose, 2006).

Subtle shifts in the way you approach teaching computation can alleviate this misconception. For example, rather than ask students to *solve* a problem (e.g., $4.5 + 0.61$ or 0.25×26), ask them to *find an equivalent (simplified) expression*, then write the equivalent expressions as an equation (Blanton, 2008). So, for $4.5 + 0.61$, students might write $4.5 + 0.61 = 4 + 0.6 + 0.5 + 0.01$, illustrating what place values they will add.

Teaching Tip

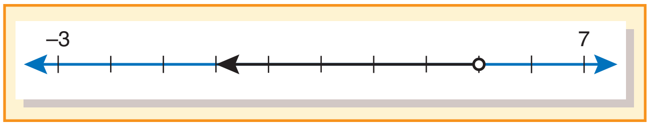
On occasion, rather than have students solve an expression, ask students to rewrite it in an equivalent way—this takes the focus *off* answers and puts the focus *on* equivalence.

Another way to support a stronger understanding of symbols is to encourage students to write their mental math strategies symbolically. For example, a student might write $0.25 \times 26 = \frac{1}{2}(\frac{1}{2} \times 26)$. This practice increases student understanding of the equal sign *and* the relationships among numbers and forms of numbers (e.g., $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, so $0.5 \times 0.5 = 0.25$). Finally, as students are exploring operations with rational numbers, remember to write equations such that the result is on the left (e.g., $0.015 = 1.5 \div 100$).

Inequalities are also poorly understood and have not received the attention that the equal sign has received, likely because inequalities are not as prevalent in the curriculum or in real life. Understanding and using inequalities is important and will require significant time and experiences to develop. The number line is a valuable tool for understanding inequalities. For example, students can be asked to show $x < 5$ on the number line:

Standards for Mathematical Practice

7 Look for and make use of structure.



A context such as money can provide a good way to make sense of inequalities, as in the example below.

You have $100 for purchasing gift cards for your 5 friends. You want to spend the same on each, and you will also need to spend $10 to buy a package of card holders for the gift cards. Describe this situation with symbols.

Stop and Reflect

How would you write this inequality? How might students write it? What difficulties do you anticipate? And, importantly, what questions will you pose to help students build meaning for the inequality symbols?

Students might record the situation in any of the following ways (using *a* for the amount of money for the gift):

$$5a + 10 \leq 100 \qquad 10 + 5a \leq 100 \qquad 100 \geq 10 + 5a \qquad 100 \geq 5a + 10$$

Another set of four possibilities are these four inequalities without the equal signs. Discuss with students what it means to say "less than" or "less than or equal to." Allow them to debate which makes more sense given the situation. In particular, bring attention to the ways in which to write the inequality with a less-than sign and with a greater-than sign.

The dilemma of which way to place the sign is what makes understanding inequalities even more challenging than understanding equations. Invite students to say exactly what each of these statements means. For example, the first statement directly translates to 5 gift cards and $10 for a package of holders must be less than or equal to $100. The final example directly translates to I have $100, which must be more than or the same as the cost of 5 cards and the holders. Ask questions that help students analyze the situation quantitatively, such as, "Which has to be more, the amount you have or the amount you spend?"

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Formative Assessment Notes

Ask students to write a real-life story problem that involves an inequality. You can add expectations such as "It must be multi-step" and "you must illustrate the solution on a number line" (for more details see Whaley, 2012, for a full lesson, examples, rubric and discussion). Writing helps students connect representations and helps you see what misconceptions they might have.

Conceptualizing the Equal and Inequality Signs with a Balance

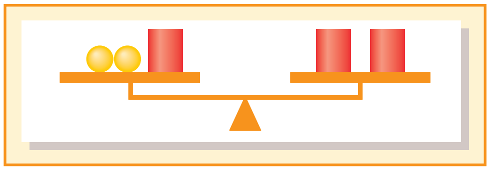
Students' understanding of the idea of equivalence can be developed concretely, as in Activity 12.12.

Activity 12.12

CCSS-M: 6.EE.A.2a; 6.EE.B.5; 6.EE.B.6; 7.EE.B.4a, b

What Do You Know about the Shapes?

Present a scale with objects on both sides. Use a real scale and objects, if possible. If not, use clip art or another picture of a scale, as shown here.

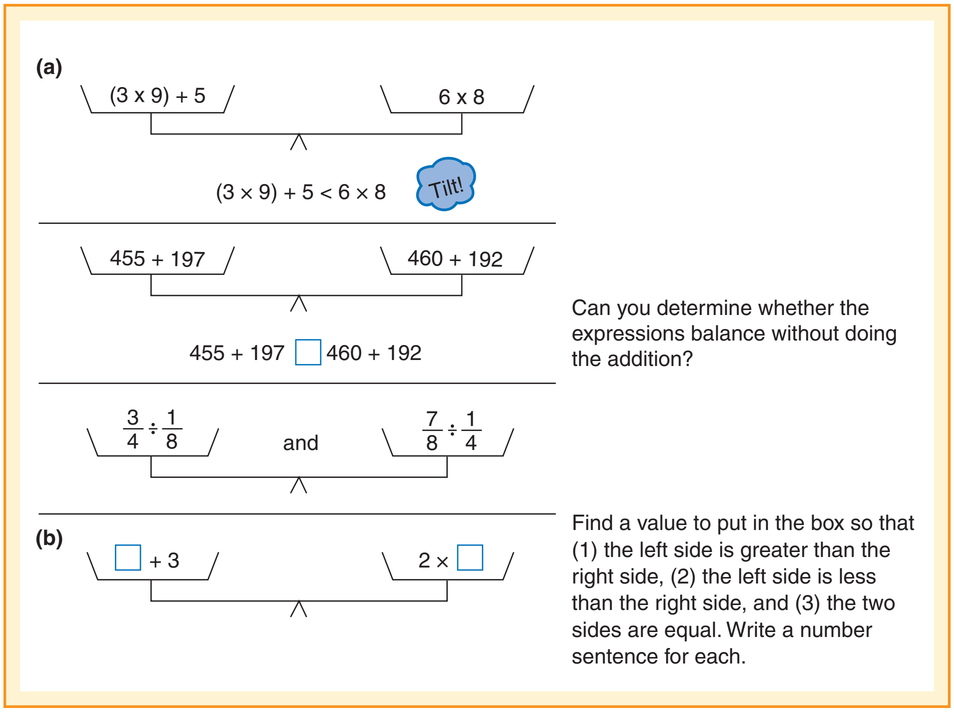


Ask students what they know about the shapes. "The cubes weigh the same. The spheres weigh the same. What do you know about how the weights of the cubes and the spheres compare?" For more explorations like this, see "Pan Balance—Shapes" on the NCTM Illuminations website.

After students have experiences with these shapes, they can then explore numbers and focus on equal and not-equal situations. Figure 12.15 offers examples that connect the balance to the related equation. This two-pan balance also illustrates that the expressions on each side represent a number.

Figure 12.15

Using expressions and variables in equations and inequalities. The two-pan balance helps develop the meaning of =, < , and > .



Can you determine whether the expressions balance without doing the addition?

Find a value to put in the box so that (1) the left side is greater than the right side, (2) the left side is less than the right side, and (3) the two sides are equal. Write a number sentence for each.

Activity 12.13

CCSS-M: 2.NBT.A.4; 4.NBT.A.1; 5.NBT.A.3a, b; 6.EE.A.4

Tilt or Balance?

Post two expressions and ask students whether they will balance or if one will be greater than the other (see Figure 12.15(a)), and how they know. Tilt or Balance Equation Cards Activity Page provides some possible equations to compare, which could be cut out as cards, projected for students, or just used for reference. To make it more concrete, distribute or project two-pan balances, such as the Balance Scales Placemat, which includes one scale that is balanced and one that is tilted. Ask students to write number sentences for each example after they decide. Note that when the scale "tilts," either a "greater-than" or "less-than" symbol (> or <) is used, and if it is balanced, an equal sign (=) is used. Include examples for which students can make the determination by analyzing the relationships (like the second example in 12.15a) or estimation (like the first and third examples). Also, use all kinds of rational numbers. Note: this activity can also serve as a daily warmup, giving students practice with rational numbers, reasoning quantitatively, and a stronger understanding of the equal and inequality signs. For ELLs, have the symbols with the words written next to them for reference and encourage students to read or say their expressions.

As an alternative or extension, use missing-value expressions. Ask students to find a number that will result in one side tilting downward, a number that will result in the other side tilting downward, and one that will result in the two sides being balanced (see Figure 12.15(b)).

Another NCTM Illuminations applet, "Pan Balance—Expressions," provides a virtual balance where students can enter what they believe to be equivalent expressions (with numbers or symbols), each in a separate pan, to see whether the expressions do, in fact, balance.

Figure 12.16 shows solutions for two equations, one in a balance and the other without. Even after you have stopped using the balance, it is a good idea to refer to the two-pan balance concept of equality and the idea of keeping the sides balanced. This use of concrete (an actual balance)

Teaching Tip

Use real scales to show that adding or subtracting the same object (quantity) from both sides preserves the equation or inequality.

or semi-concrete (drawings of a balance) representations helps students develop a strong understanding of the abstract concept of equality, and provides initial experiences in how to preserve equivalence when moving numbers or variables across the equal sign (e.g., subtracting 5 from both sides).

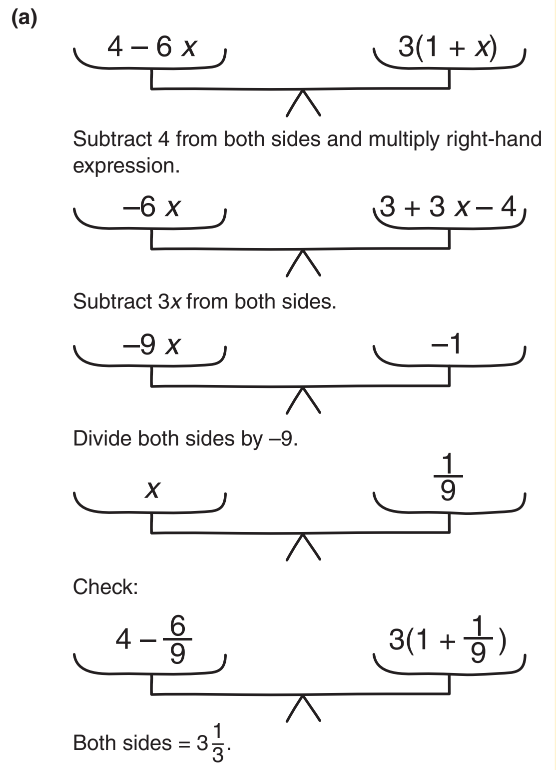
The notion of preserving balance also applies to inequalities—but what is preserved is imbalance. In other words, if one side is 5 more than the other side, then subtracting 5 from both sides results in the same side being 5 more. The exception is multiplying an inequality by -1; in making the values the opposite of what they were, the larger expression becomes the smaller expression.

True/False and Open Sentences

Carpenter and colleagues (2003) suggest that a good starting point for helping students with the equal sign is to explore equations as either true or false. Clarifying the meaning of the equal sign is just one of the outcomes of this type of exploration, as seen in the following activity.

Figure 12.16

Using a balance scale to think about solving equations.



(b)

$4.2n + 63 = \frac{n}{2}$

Subtract 63.

$4.2n = \frac{n}{2} - 63$

Multiply by 2.

$8.4n = n - 126$

Subtract n.

$7.4n = {}^{-}126$

Divide by 7.4. (Use a calculator!)

$n = {}^{-}17.03$ (about)

Activity 12.14

CCSS-M: 5.NF.A.1; 6.EE.A.4; 7.NS.A.1d

True or False?

Introduce true/false equations with simple examples to explain what is meant by a true equation and a false equation. Then put several simple equations on the board, some true and some false. Keep the computations simple so that the focus is on equivalence.

$7 = 5 + 2 \qquad 4 + 1 = 6$

$4 + 5 = 8 + 1 \qquad 8 = 10 - 1$

Ask students to decide which of the equations are true and which are not. For each response, the students must explain their reasoning. The following are appropriate for middle grades:

$120 = 60 \times 2 \qquad 1 = \frac{3}{4} + \frac{2}{1} \qquad 318 = 318$

$\frac{1}{2} = \frac{1}{4} + \frac{1}{4} \qquad 345 + 71 = 70 + 344$

$1210 - 35 = 1310 - 45 \qquad 0.4 \times 15 = 0.2 \times 30$

Listen to the types of reasons that students use to justify their answers, and plan additional equations accordingly. ELLs and students with disabilities will benefit from first explaining (or showing) their thinking to a partner (a low-risk speaking opportunity) and then sharing with the whole group. For false statements, ask students to rewrite the statement with $>$ or $<$ to make the statement true. “Pan Balance—Numbers” on the NCTM Illuminations website can be used to model and/or verify equivalence.

An equation with no operation (318 = 318) can raise questions for students who have not seen the equal sign without an operation on one side. Reinforce that the equal sign means "is the same as" by using that language when you read the symbol. In other words, expressions can be related in one of three ways: an expression is less than, the same as, or greater than another expression.

After students have experienced true/false sentences, introduce an open sentence—one with a box or variable with a missing value. To develop an understanding of open sentences, encourage students to look at a number sentence holistically and discuss in words what the equation represents.

Standards for Mathematical Practice

7 Look for and make use of structure.

Open sentences are an excellent opportunity to explore inequalities. Consider replacing the equal sign in any of the equations in Activity 12.15 with a $>$ or $<$. What is now possible for the missing value or the variable? Not just one answer, but many (in fact, infinitely many). Since students are more used to equations, they are likely to find the value that makes it equal and then struggle to decide how to adjust the answer to create the inequality. They may begin with trial and error, but students eventually should be able to use reasoning, also called relational thinking.

Activity 12.15

CCSS-M: 5.OA.A.2; 6.NS.B.C; 6.EE.A.3; 6.EE.B.5

What's Missing?

Prepare a set of missing value equations, using numbers appropriate for the grade level. A sampling across the grades is provided below. Ask students to figure out what is missing and how they know. Notice that the equations are set up so that students do not always have to perform the operations to figure out what is missing. Encourage them to look at the equation and see if they can figure out what is missing without solving it. Probe to see if there is more than one way to find what is missing. Here is a sampling for operations with decimals:

$0.5 + \square = 5$ $\quad 0.4 + \square = 0.6$ $\quad 4.5 + 5.5 = \square - 1$ $\quad 0.3 \times 7 = 7 \times \square$

$\square \times 4 = 4.8$ $\quad \square = 2.3 - 0.5$ $\quad 2.4 \div \square = 4.8 \div 6$

Discuss how students solved the problems. After using open boxes, begin to use variables so that students can see the variable as representing a missing value:

$3.6 - n = 3.7 - 4$ $\quad n + 0.5 = 0.5 + 4.8$ $\quad 1.5 + 2.7 = n + 2.8$ $\quad -6 \times n = 3 \times 8$ $\quad 15 \times 27 = n \times 27 + 5 \times 27$ $\quad 1 = 0.5 \div n$

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Relational Thinking

Students may think about equations in three ways, each developmental in nature (Stephens et al., 2013). First, they may have an *operational view*, meaning that the equal sign means "do something." Second, students develop a *relational–computational view*. At this phase, students understand that the equal sign symbolizes a relation between answers to two calculations, but they only see computation as the way to determine if the two sides are equal or not. Finally, students develop a *relational–structural view* of the equal sign (we will refer to this as relational). In this thinking, a student uses numeric relationships between the two sides of the equal sign rather than actually computing the amounts.

Teaching Tip

Notice that each open sentence is simultaneously addressing important number relationships and number sense, so they are excellent choices for warm-up problems or learning centers.

Consider two distinctly different explanations for deciding that $n = 3$ for this open sentence: $2.4 \div n = 4.8 \div 6$.

"Since $4.8 \div 6$ is 0.8, then $2.4 \div$ something is also 0.8, so that must be 3."

"I noticed that 2.4 is half of 4.8, so I need to divide by a number half the size of 6 in order to maintain equivalence, so the answer is 3."

Standards for Mathematical Practice

7 Look for and make use of structure.

Both students are reasoning abstractly. The first student computes the result on one side and adjusts the result on the other to make the sentence true (relational–*computational* approach). The second student uses a relationship between the expressions on either side of the equal sign. This student does not need to compute the values on each side (relational–*structural* approach). When the numbers are large, a relational-structural approach is much more efficient and useful.

Formative Assessment Note

As students work on these types of tasks, you can conduct a *diagnostic interview* (though you may not get to everyone, so you may want to start with students you think are having difficulty). Listen for whether they are using relational thinking. If they are not, ask, "Can you find the answer without actually doing any computation?" This questioning nudges students toward relational thinking and helps you decide what instructional steps are next.

Students need many ongoing opportunities to explore problems that encourage relational thinking (Stephens et al., 2013). Explore increasingly complex true/false and open sentences with your class, including multi-step problems, as daily problems, warm-ups, enrichment, or at stations. Use numbers that make actual computation difficult (not impossible) as a means to encourage relational thinking. Here are some examples:

True/False

$$6.74 - 3.89 = 6.64 - 3.79 \qquad 42 = 0.5 \times 84$$

$$\tfrac{2}{5} = \tfrac{1}{2} + \tfrac{1}{3} \qquad 64 \div 14 = 32 \div 28$$

Open Sentences

$$7.03 + 0.056 = 7.01 + \square \qquad 0.126 - 0.37 > \square - 0.4$$

$$20 \times 4.8 = n \times 2.4 \qquad \tfrac{2}{5} + \tfrac{1}{5} < \tfrac{3}{10} + \tfrac{1}{10} + n$$

Multistep Sentences

$$512 \times 5 \times 20 = n \qquad 68 + 58 = 57 + 69 + n$$

$$\tfrac{3}{10} + n + \tfrac{1}{10} = \tfrac{2}{5} + \tfrac{1}{5} \qquad 37 \times 18 \div 37 = n$$

Stop and Reflect

Look at each of the multi-step sentences and think about the order in which you can solve each one. Can relational structural thinking help make the problem easier to solve?

Notice that the careful construction of the multistep equations encourages relational thinking. Each of the problems can be solved mentally if the first step of solving is chosen carefully. Selecting equations such as these encourage students to look at equations *in their entirety* rather than just jumping right into a series of computations, an important aspect of algebraic thinking (Blanton et al., 2011).

Standards for Mathematical Practice

7 Look for and make use of structure.

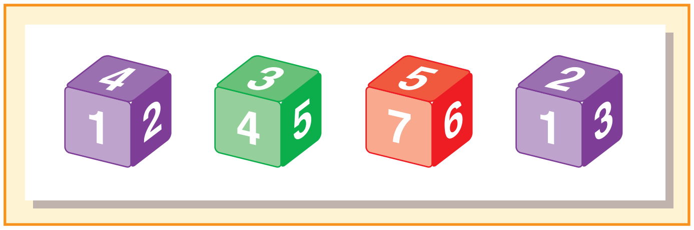
Molina and Ambrose (2006) found that asking students to write their own open sentences was particularly effective in helping them solidify their understanding of the equal sign.

Activity 12.16

CCSS-M: 5.OA.A.2; 6.NS.B.C; 6.EE.A.3; 6.EE.B.5

Writing True/False Statements

Ask students to make up their own true/false sentences that they can use to challenge their classmates. To support student thinking, provide dice with numerals on them. They can turn the dice to different faces to try different possibilities.



Each student should write a collection of three or four equations, with at least one true and at least one false sentence. Encourage them to include one "tricky" one. Their equations can either be traded with a partner or used in full-class discussions.

Repeat for open-sentence problems. For students needing additional structure, in particular students with disabilities, consider providing frames such as these:

_______ + _______ = _______ + _______   _______ + _______ > _______ + _______

_______ − _______ = _______ − _______   _______ − _______ < _______ − _______

_______ + _______ = _______ − _______   _______ + _______ < _______ − _______

(or use multiplication and division)

Students can trade their set of statements with other students to find the False Statement. Interesting equations/inequalities can be the focus of a follow-up full-class discussion.

When students write their own true/false sentences, they often are intrigued with the idea of using large numbers or lots of numbers in their sentences. This encourages them to create sentences involving relational structural thinking.

The Meaning of Variables

Variables can be interpreted in many ways. Variables are first mentioned in the CCSS-M standards in grades 3 and 4, where students are encouraged to use letters to represent numbers, and are emphasized in grades 6–8, where students evaluate expressions with variables and solve one-variable equations and inequalities. Starting early with variables is important so that students are more adept at using variables when they encounter more complex mathematical situations in middle school (Blanton et al., 2011). Variables can be used to represent a unique but unknown quantity or represent a quantity that varies. Unfortunately, students often think of the former (the variable is a placeholder for one exact number) and not the latter (that a variable could represent multiple, even infinite values). As discussed in the section on functions, variables are also used to describe a pattern. Experiences in elementary and middle school should focus on building meaning for both, as delineated in the next two sections.

Variables Used as Unknown Values

In the open-sentence explorations, the □ is a precursor of a variable used in this way. In grades 4 and 5, students begin using variables to represent missing values, so students should have some initial experiences with variables.

Teaching Tip

Write variables as lowercase letters in italics. This will help distinguish the multiplication symbol × from the variable x.

One-Variable Situations. Initial work with finding the value of the variable that makes the sentence true should rely on relational thinking (reasoning). Starting in sixth grade, students will begin developing specific techniques for solving equations when relational thinking or reasoning is insufficient.

Context can also help students develop meaning for variables. Many story problems involve a situation in which the variable is a specific unknown, as in the following basic example:

Gary ate 14 strawberries, and Jeremy ate some, too. The container of 25 strawberries was gone! How many did Jeremy eat?

Although students can solve this problem mentally without using algebra, they can begin to learn about variables by expressing it in symbols: $14 + n = 25$, where the n represents the number of strawberries that Jeremy ate. These problems can grow in difficulty over time.

Activity 12.17

CCSS-M: 6.EE.A.2a; 6.EE.B.6; 7.EE.B.4

Telephone: Transforming Words to Equations/Inequalities

This activity is based on the classic game of "telephone," in which a statement is whispered to a person and passed from that person to another. The goal is for the phrase to stay the same all the way around the group (see Bay & Ragan [2000] for details). Place students in small groups. Prepare simple situations (one for each group) like the one in the example above with a missing value. Label each one with a number (story 1). In round 1, students write an equation that communicates exactly what the story states. They label their equation (e.g., story 1). At your direction, they pass that equation to the next group (keeping the original story), which will recreate a story that exactly matches the equation (preserving the missing value). The group labels the story as story 1 (or whatever number they received). The group passes the story to the next group (keeping the equation the group had received for story 1). This continues until the stories make it back to the original groups. Each group has all the notes it received (all labeled by story number). Give all the story 1 cards to group 1, story 2 cards to group 2, and so on. Each group is to see if its story made it around correctly, and if not, where it got mixed up (and why). Each group shares what happened with its story as it passed through the groups. The process is the same whether the stories involve equations or inequalities. If you have ELLs or students with disabilities, rather than have students write stories about any context, have them write stories about a given situation (like shopping for particular items at the store with a particular budget).

This activity helps students go both ways with variables and words, deepening the connection between the two. In addition, it helps students find errors and misconceptions. Finally, it helps students recognize equivalent expressions versus not-equivalent expressions because groups may have written an equation that is correct but written it slightly differently.

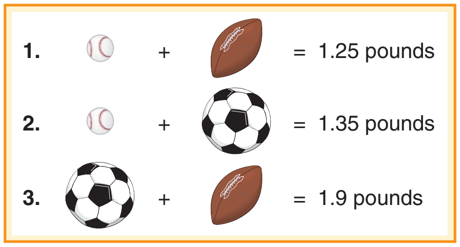
Two-or-More-Variables Situations. Systems of equations (or inequalities) are examples of two or more variables serving as placeholders for missing values or unknowns. With a context, students can explore and solve such situations by using relational understanding, as in Activity 12.18 (based on Maida, 2004).

Activity 12.18

CCSS-M: 6.EE.A.4; 7.EE.A.2; 8.EE.C.8b

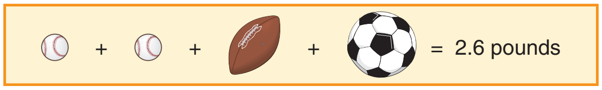
Ball Weights

Students will figure out the weight of three balls, given the following three facts:



Ask students to look at each fact and make observations that help them generate other facts. For example, they might notice that the soccer ball weighs 0.1 pound more than the football. Write this in the same fashion as the other statements. Continue until these discoveries lead to finding the weight of each ball. Encourage students to use tools or pictures to represent and explore the problem.

One possible approach: Add equations 1 and 2.



Then take away the football and soccer ball, reducing the weight by 1.9 pounds (based on the information in equation 3), and you have two baseballs that together weigh 0.7 pound. Divide by 2, so one baseball is 0.35 pound.

This activity is an appropriate sixth- or seventh-grade task to be solved by reasoning, or a great introduction for eighth graders as they begin work with systems of equations.

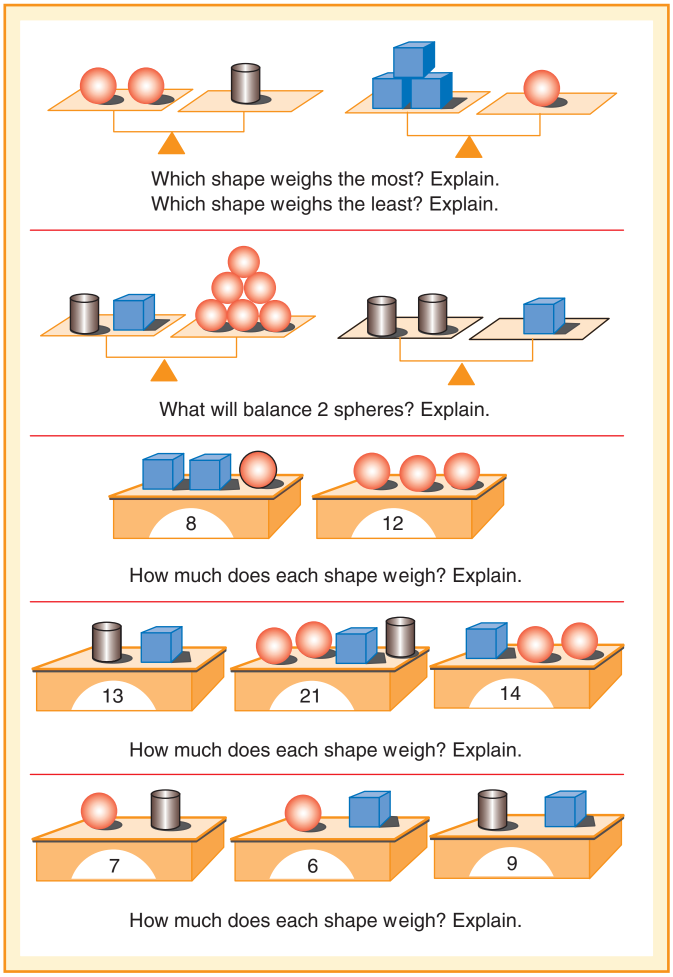
Another concrete way to work on systems of equations is through balancing. Notice that the work done in building the concept of the equal sign is now applied to understanding and solving for two or more variables.

In Figure 12.17, a series of examples shows problems in which each shape on the scales represents a different value. Two or more scales representing a single problem provide different information about the shapes or variables. The NLVM applet Algebra Balance Scales and Algebra Balance Scales—Negative is an excellent tool for learning about balancing equations.

When no numbers are involved, as in the top two examples of Figure 12.17, students can find combinations of numbers for the shapes that make all of the examples balance. If an arbitrary value is given to one of the shapes, then values for the other shapes can be found accordingly. In the second example, if the sphere equals 2, then the cylinder must be 4 and the cube equals 8. If a different value is given to the sphere, the other shapes will change accordingly.

Figure 12.17

Examples of problems with multiple variables and multiple scales.



Which shape weighs the most? Explain.
Which shape weighs the least? Explain.

What will balance 2 spheres? Explain.

How much does each shape weigh? Explain.

How much does each shape weigh? Explain.

How much does each shape weigh? Explain.

Ask middle school students what the purpose is of solving a system of equations, and at best they might say they are solving for x and for y. Students should be able to explain what x and y tell you (as it connects to a situation and in general) and why this might be important. Eighth graders must "understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously" (CCSSO, 2010, p. 55).

Simplifying Expressions and Solving Equations

Simplifying equations and solving for x have often been meaningless tasks, and students are unsure of why they need to know what x is or what steps to do and in what order. This must be taught in a more meaningful way! Knowing how to simplify and recognizing equivalent expressions are essential skills for working algebraically. Students are often confused about what the instruction "simplify" means. (Imagine an ELL wondering why the teacher is asking students to change the original problem to an easier one.) The border tiles problem in Activity 12.11 provides a good context for thinking about simplifying and equivalence. Recall that there are at least 5 possibilities for finding the number of border tiles. If the pool had dimensions other than 8 × 8, those equations would be structurally the same, but with different values. If the square had a side of length p, the total number of tiles could be found in similar ways:

$10 + 10 + 8 + 8$	$(2 \times 10) + (2 \times 8)$	4×9	$100 - 64$
$(p + 2) + (p + 2) + p + p$	$2 \times (p + 2) + (2 \times p)$	$4 \times (p + 1)$	$(p + 2)^2p^2$

Stop and Reflect

How would you solve the last problem in Figure 12.17? Can you solve it in two ways? How can you help students bridge the connection from these informal investigations with variables to a more formal understanding of systems of equations?

Invite students to enter these expressions into the TABLE function on their graphing calculator and graph them to see whether they are equivalent (Brown & Mehilos, 2010). Looking at these options, the connection can be made for which one is stated the most simply (briefest or easiest to understand).

Students need an understanding of how to apply mathematical properties and how to preserve equivalence as they simplify. (This is one of the *Common Core State Standards* in grade 7.) In addition to the ideas that have been offered (open sentences, true/false sentences, etc.), one way to do this is to have students look at simplifications that have errors and explain how to fix the errors (Hawes, 2007). Figure 12.18 shows how three students have corrected the simplification of $(2x + 1) - (x + 6)$.

Activity 12.19 provides an engaging way for students to explore properties and equivalent expressions.

Teaching Tip

Create your own examples of simplified expressions that have an error in them—select an error that is commonly made in your classroom so that the class can discuss why the expression is not correct.

Activity 12.19

CCSS-M: 5.OA.A.2; 6.EE.A.2a; 7.EE.A.2

Solving the Mystery

Begin by having students do the following sequence of operations:

Write down any number.

Add to it the number that comes after it.

Add 9.

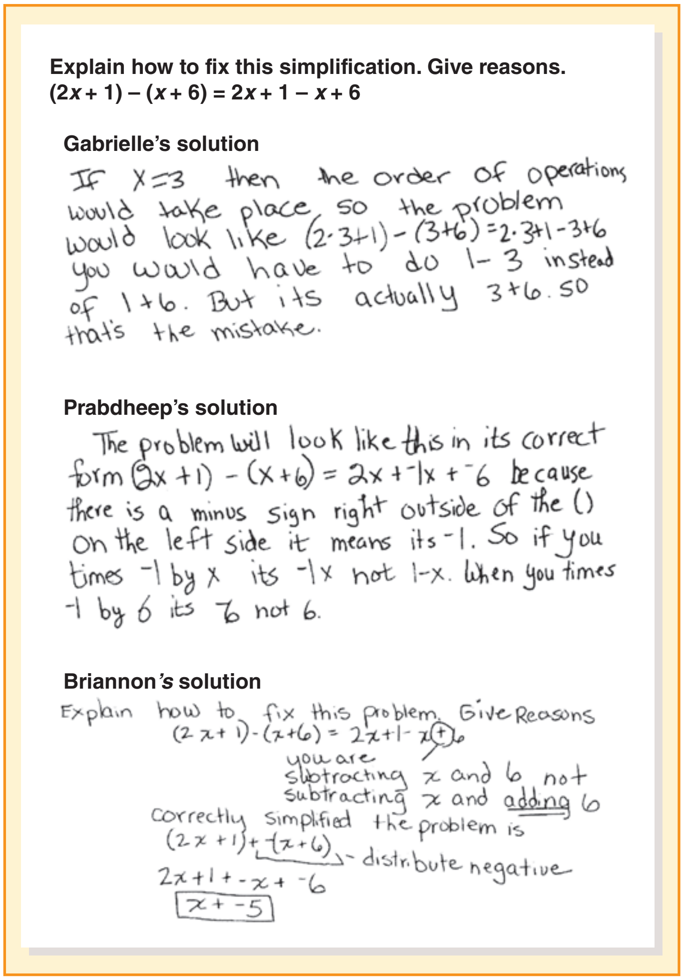
Divide by 2.

Subtract the number you began with.

Now you can “read their minds.” Everyone ended up with 5! Ask students, “How does this trick work? Start with n. Add the next number: $n + (n + 1) = 2n + 1$. Adding 9 gives $2n + 10$. Dividing by 2 leaves $n + 5$. Now, subtract the number you began with, leaving 5. For students with disabilities or students who struggle with variables, suggest that instead of using an actual number they use an object, such as a cube, and physically build the steps of the problem, as illustrated in Figure 12.19. See also a second **Solving the Mystery** task. In this Mystery, the result is a two-digit number where the tens place is the first number selected and the ones place is the second number selected (ask students to explain how this happened). As a follow-up or for enrichment, students can generate their own number tricks.

Figure 12.18

Three students provide different explanations for fixing the flawed simplification given.



Explain how to fix this simplification. Give reasons.
$(2x + 1) - (x + 6) = 2x + 1 - x + 6$

Gabrielle’s solution

If $X=3$ then the order of operations would take place, so the problem would look like $(2\cdot3+1)-(3+6)=2\cdot3+1-3+6$ you would have to do $1-3$ instead of $1+6$. But its actually $3+6$. so that's the mistake.

Prabdheep’s solution

The problem will look like this in its correct form $(2x+1)-(x+6)=2x+{}^{-}1x+{}^{-}6$ because there is a minus sign right outside of the () On the left side it means its $^{-}1$. So if you times -1 by x its $^{-}1x$ not $1-x$. When you times -1 by 6 its $^{-}6$ not 6.

Briannon’s solution

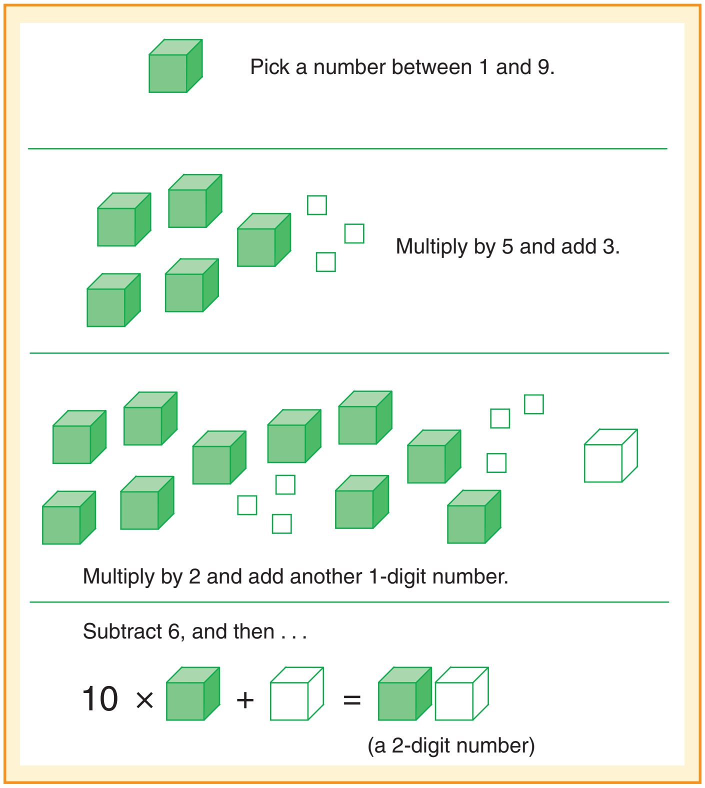
Explain how to fix this problem. Give Reasons
$(2x+1)-(x+6)=2x+1-x\oplus6$
you are subtracting x and 6 not subtracting x and <u>adding</u> 6
correctly simplified the problem is
$(2x+1)+\underbrace{-(x+6)}$ - distribute negative
$2x+1+-x+{}^{-}6$
$\boxed{x+-5}$

Source: Figure 3 from Hawes, K. (2007). Using Error Analysis to Teach Equation Solving. *Mathematics Teaching in the Middle School, 12*(5), p. 241. Reprinted with permission. Copyright 2007 by the National Council of Teachers of Mathematics. All rights reserved.

Solving systems of equations has also been presented in a way that includes a series of procedures with little attention to meaning (e.g., by graphing, by substitution, and simultaneously). Mathematically proficient students should have access to multiple approaches, including these three. However, rather than learn one way by rote each day or be tested on whether they can use each approach, they should encounter a system and be guided by the question when you ask, “How can we determine the point of intersection of this system?” And, students should be able to describe *when* one of the approaches might be the best choice, given the context, the numbers in the situation, or the information they are asked to provide.

Figure 12.19

Cubes can illustrate the steps in "Number Tricks."



Teaching Tip

Just as they did with the operations, students should *choose* a method for solving a system of equations that fits the situation, using appropriate tools.

Graphing calculators make the choice of using a graph to determine the point of intersection an efficient option, whereas graphing by hand used to be one of the most tedious methods. And among the strategies that students must try is observation. Too often, students leap into solving a system algebraically without stopping to observe the values in the two equations. Look at the systems of equations here, and see which ones might be solved for x or y without using algebra. $x + y = 25$ and $x + 2y = 25$ $3x + y = 20$ and $x + 2y = 10$ $8x + 6y = 82$ and $4x + 3y = 41$ $\frac{y}{3} = 5$ and $y + 5x = 60$

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Variables Used as Quantities That Vary

Variables that vary are most commonly used to describe functions (e.g., $y = 3x - 5$). This shift from the variable as an unknown in equations/inequalities to a variable that represents a functional relationship is a critical distinction that must be made; it is also one of the things that is a particular struggle for students in algebra (Kieran, 2007). The difficulty can be alleviated if students have experiences with variables that vary throughout middle school—certainly in sixth grade as they focus on the meaning and use of variables and as they explore patterns. The **Birds in the Backyard** activity discussed at the beginning of this chapter is an excellent way to introduce variables that vary in two ways: first, in representing the way birds can be on the feeder and in the tree as $t + f = 7$, and second, as the general case of number of birds (b) to number of ways they can be in the tree (w) ($w = b + 1$). Importantly, you must emphasize that the variable stands for *the number of* because students can confuse the variable for a label (Blanton et al., 2011).

Help students recognize that when there is more than one variable in a single equation, each variable can represent many, even infinitely many, numbers (or at times the same number). And, at the same time, each variable may represent numerous but limited values. In the example just given, there are only eight things that t can be. In the second equation, b can be infinitely many numbers, but they must be whole numbers. This discussion builds important ideas of range and domain of functions in a concrete manner.

Teaching Tip

When working with variables that can represent more than one number, ask students, "What are possible values for the variable?" and "What values are *not* possible for the variable?" Highlight the fact that when a variable is used more than once in an equation, it represents the same value.

Build on students' understanding of rational numbers, and contrast variables-as-unknowns equations with variables-that-vary equations. Compare the two problems below.

$$\frac{1}{4} + n = 1 \qquad m + n = 1$$

Ask students to find one answer for n for each problem. Then ask if they think they can find a different answer. In the first case, they may be able to find equivalent forms for n, but n is still the same number (e.g., 0.75, $\frac{3}{4}$, $\frac{6}{8}$). But, in the second example, there are literally infinitely many values that n can be. Students can be challenged to create at least 10 expressions with decimals, integers, and fractions—enhancing their understanding of rational numbers while expanding their understanding of what a variable can mean in an expression.

Another difficulty with variables that vary is determining how to represent situations to show the relationship between the two variables correctly.

Stop and Reflect

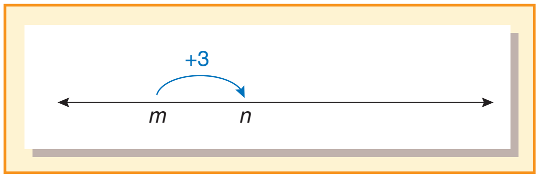
We know there are 3 feet in a yard. Using f for number of feet and y for number of yards, which of these equations correctly expresses the relationship: $3f = y$ or $f = 3y$? Make your choice before reading on.

Many students will (incorrectly) pick $3f = y$ because there are 3 feet in a yard. But, try putting some numbers in $3f = y$. For example, if f is 5, then y will be 15, but 5 feet is not equal to 15 yards. Recall that students confuse variables with labels, a particular challenge when using a measurement context. One instructional strategy is to choose variables of x and y to remove the confusion with measurement errors. Another strategy is to analyze the situation by testing for specific examples (as was done here by trying 5 feet).

Teaching Tip

Rather than saying, "Use x for feet and y for yards," say the precise statement, "Use x for the number of feet and y for the number of yards." This helps students to remember that the variable is the "number of" units.

Slightly altering number tasks can help students figure out how to relate two variables. Instead of "Marta has \$6 and Nathan has \$3 more than Marta; how much more money does Nathan have?" use "If Marta has some money in her bank and Nathan has \$3 more than Marta, how much more money does Nathan have?" (Blanton, 2008, p. 18). Symbolically, this is Marta + 3 = Nathan, or $m + 3 = n$. Picture this on the number line. It can start anywhere (m), jump 3 to the right, and land on anything (n):

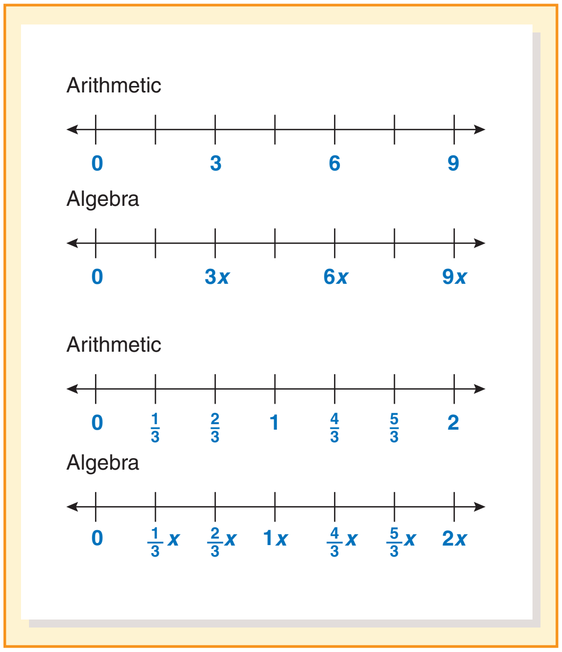


Standards for Mathematical Practice

6 Attend to precision.

Figure 12.20

Number lines for arithmetic and algebra.



The number line is an important model in developing the concept of variable. As illustrated in Figure 12.20, finding where variables are in relation to numbers and in relation to other variables helps to build meaning (Darley, 2009).

When students are looking at the number line, ask questions like these: "What is the value of x? Can it be any number? If we don't know what x is, how can we place $\frac{1}{3}x$ on the number line? Think of a value that x cannot be." Notice that in the two examples, x really can be any positive value. However, if you place $x + 2$ on the number line somewhere close to x, the space between these is 2, and you can use this distance as a "measure" to approximate the size of x. Because students use the number line extensively with whole numbers, it is a good way to bridge to algebra. An open number line posted in your room, where you can post values, and then trade out for new values, can provide many opportunities to think about the relative value of variables.

Context is important in writing equations with variables. The following example is appropriate for middle-school students as a context for exploring variables that vary:

If you have \$10 to spend on \$2 granola bars and \$1 fruit bars, how many ways can you spend all your money without receiving change?

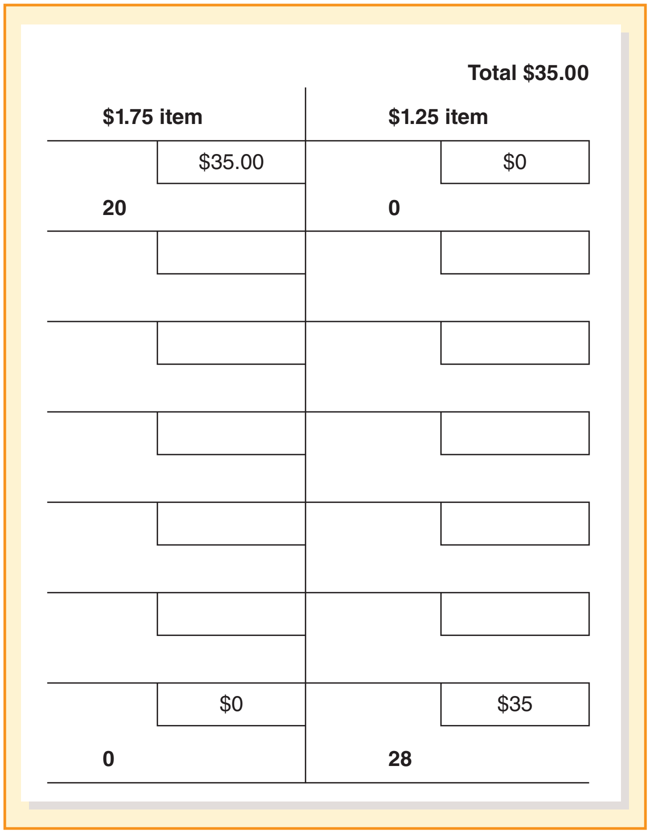
To begin exploring this problem, students record data in a table and look for patterns. They notice that when the number of granola bars changes by 1, the number of fruit bars changes by 2. Symbolically, this representation is $2g + f = \$10$, where g is the number of granola bars and f is the number of fruit bars.

Include decimal and fraction values in the exploration of variables. As any algebra teacher will confirm, students struggle most with these numbers—likely a result of limited concrete and visual experiences in mixing fractions and decimals. The following example includes a context to support student reasoning.

You bought \$1.75 pencils and \$1.25 erasers from the school store, and you spent exactly \$35.00. What might you have purchased? What equation represents this situation?

Figure 12.21

A table adapted to include how many and how much for each row.



			Total $35.00
$1.75 item		$1.25 item	
20	$35.00	0	$0
0	$0	28	$35

Source: Hyde et al., (2006). Creating Multiple Representations in Algebra: All Chocolate, No Change, *Mathematics Teaching in the Middle School, 11*(6), 262–268. Reprinted with permission. Copyright © 2006 by the National Council of Teachers of Mathematics. All rights reserved.

For students with disabilities or students who might be unfamiliar with using a table, it is helpful to adapt the table to include both how many and how much, as shown in Figure 12.21 (Hyde et al., 2006). Reinforce the two elements with each entry (how many? how much?). In addition, calculators can facilitate the exploration of possible solutions. To increase the challenge for advanced or gifted students, ask students to graph the values on a **Coordinate Grid** or to consider more complex situations.

Once students have the expression in symbols (in this case, $1.75x + 1.25y = 35.00$), ask them to tie each number and variable back to the context. In this way, students can make sense of what is often poorly understood and develop a strong foundation for algebraic situations with no context.

Mathematical Modeling

Modeling with mathematics is one of the eight Standards for Mathematical Practice in the *Common Core State Standards*.

> Modeling links classroom mathematics and statistics to everyday life, work, and decision making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions (CCSSO, 2010, p. 72).

Standards for Mathematical Practice

4 Model with mathematics.

We have already seen many examples of mathematical models (e.g., the model or equation for describing the number of border tiles required for different-sized pools). How is modeling used to make decisions? Take the example of selling widgets marked up at some percentage over wholesale. Once a formula is derived for a given price and markup, it can be used to determine the profit at different sales levels. The model (equation) can be adjusted in the price and markup percentage in order to adjust the profit.

You can create a mathematical model to describe the depreciation of a car at 20 percent each year. If the car loses 20 percent of its value in 1 year, then it must be worth 80 percent of its value after a year. So, after 1 year, the \$15,000 car is worth \$15,000 × 0.8. In the second year, it loses 20 percent of that value, so it will be worth only 80 percent of its value at the end of year 1, which was \$15,000 × 0.8. The value at the end of year 2 would be (\$15,000 × 0.8) × 0.8, and so on. At the end of y years, the value of the car can be expressed in this equation: value $= \$15{,}000 \times 0.8^y$. Figure 12.22 shows the graph and the table of values on a graphing calculator.

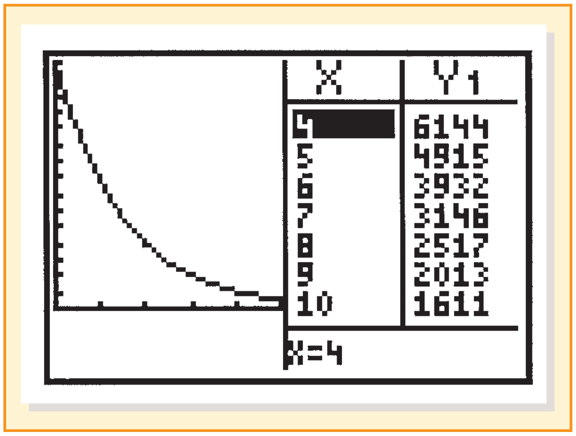
The next activity provides another context appropriate for developing a mathematical model.

Teaching Tip

Mathematical models are not to be confused with tools (such as manipulatives or visuals for building a pattern), and, as *Common Core State Standards* indicate, modeling with mathematics is something students do across the curriculum as they think numerically and symbolically about the mathematics they are doing.

Figure 12.22

The graph and table for $V = 15{,}000 \times 0.8^y$.Year, the independent variable, is shown under *X*, and value of the car, the dependent variable, is shown under *Y*1.



X	Y1
4	6144
5	4915
6	3932
7	3146
8	2517
9	2013
10	1611

X=4

Activity 12.20

CCSS-M: 7.EE.A.2; 7.EE.B.4a; 8.F.B.4

How Many Gallons Left?

Ask students to create equations to describe the gallons left for given miles traveled (assuming that you started the trip with a full tank of gasoline).

(*continued*)

For example:

- A car gets 27 miles per gallon of gas. It has a gas tank that holds 15 gallons.
- A van gets 18 miles per gallon and has a tank that holds 20 gallons.

The mathematical model or equation in the first situation is $g = 15 - \frac{m}{27}$. After students write an equation, use it to answer questions about the trip. For example, "How can you tell from the model how much gas will be left after driving 300 miles?" "How many miles can you drive before the gas tank has only 3 gallons left?" ELLs may be more familiar with kilometers per liter, which means you can adapt the problem to those units or connect the meaning of the two.

Creating equations to describe situations is a very important skill and students need multiple opportunities to translate situations into equations where the goal is the mathematical model (and not a solution). Two more engaging contexts are provided in Figure 12.23.

Figure 12.23

Mathematical modeling problems for further exploration.

1. Pleasant's Hardware buys widgets for $4.17 each, marks them up 35 percent over wholesale, and sells them at that price. Create a mathematical model to relate widgets sold (w) to profit (p). The manager asks you to determine the formula if she were to put the widgets on sale for 25 percent off. What is your formula or mathematical model for the sale, relating widgets sold (s) to profit (p)?
2. In Arches National Park in Moab, Utah, there are sandstone cliffs. A green coating, caused by cyanobacteria, covers some of the sandstone. Bacteria grow by splitting in two (or doubling) in a certain time period. If the sandstone started with 50 bacteria, create a mathematical model for describing the growth of cyanobacteria on the sandstone (Buerman, 2007)

Sometimes a model is provided, and the important task is for students to understand and use the model. Consider the following water pumping problem and related equation from the Michigan Algebra Project (Herbel-Eisenmann & Phillips, 2005):

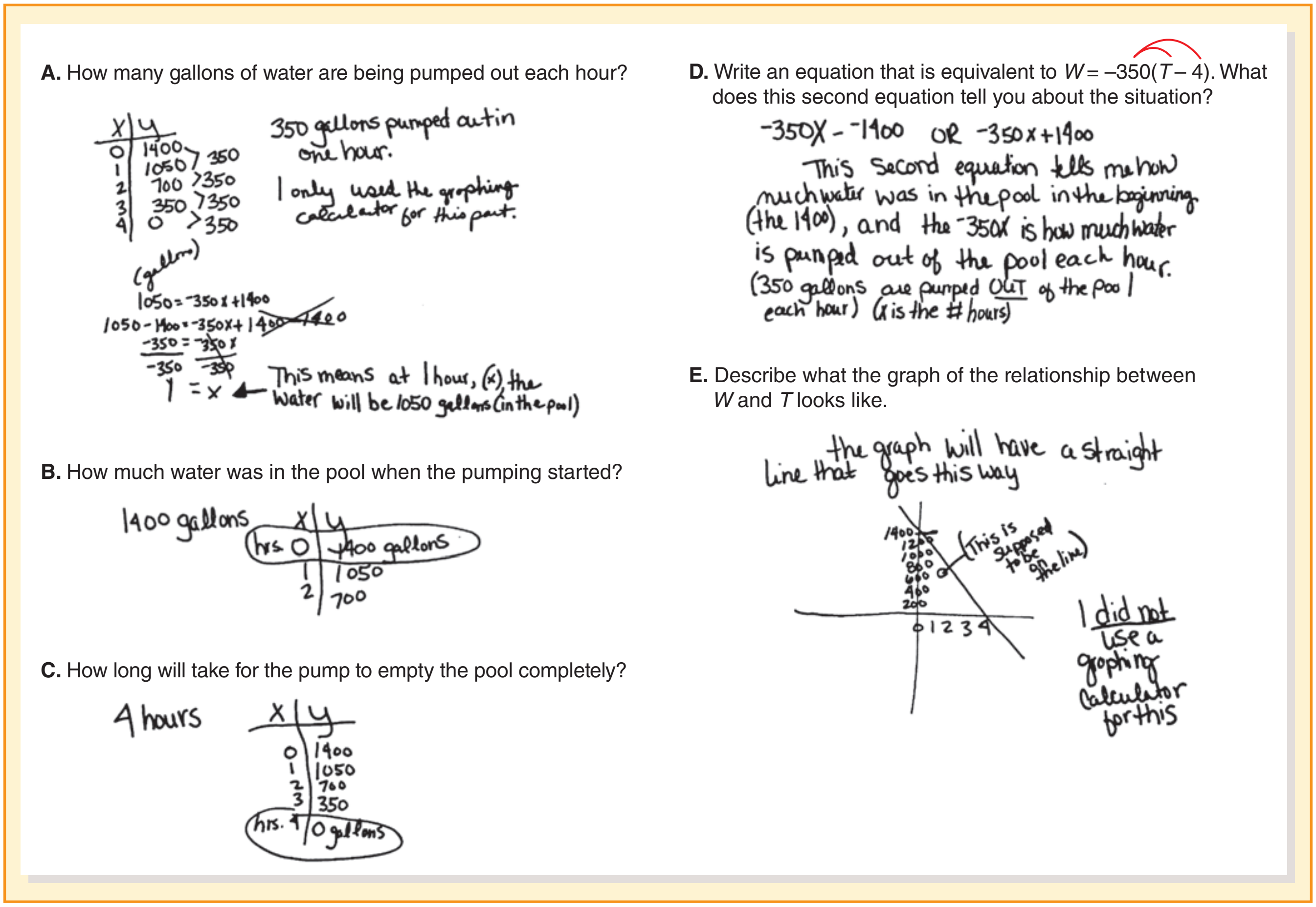
You turn a pump on to empty the water from a swimming pool. The amount of water in the pool (W, measured in gallons) at any time (T, measured in hours) is given by the following equation: $W = {}^{-}350(T - 4)$.

Stop and Reflect

What questions might you pose to middle-school students to help them make sense of this equation? Try to think of three.

In the Michigan Algebra Project, students were asked to solve several problems and explain how the equation was used to find the answer. Those questions and one student's responses are provided in Figure 12.24.

Figure12.24 One student's explanations of questions regarding what a mathematical model means.



A. How many gallons of water are being pumped out each hour?

x	y	
0	1400	> 350
1	1050	> 350
2	700	> 350
3	350	> 350
4	0	> 350

(gallons)

350 gallons pumped out in one hour.

I only used the graphing calculator for this part.

$1050 = {}^{-}350x + 1400$

$1050 - 1400 = -350x + \cancel{1400} \cancel{1400}$

$\frac{-350}{-350} = \frac{\cancel{-350}x}{\cancel{-350}}$

$1 = x$ ← This means at 1 hour, (x) the water will be 1050 gallons (in the pool)

B. How much water was in the pool when the pumping started?

1400 gallons

x	y
hrs. 0	1400 gallons
1	1050
2	700

C. How long will take for the pump to empty the pool completely?

4 hours

x	y
0	1400
1	1050
2	700
3	350
hrs. 4	0 gallons

D. Write an equation that is equivalent to $W = -350(T - 4)$. What does this second equation tell you about the situation?

$-350x - {}^{-}1400$ OR $-350x + 1400$

This second equation tells me how much water was in the pool in the beginning (the 1400), and the ⁻350x is how much water is pumped out of the pool each hour. (350 gallons are pumped OUT of the pool each hour) (x is the # hours)

E. Describe what the graph of the relationship between W and T looks like.

the graph will have a straight line that goes this way

I did not use a graphing calculator for this

Source: Figure 3 from Herbel-Eisenmann, B. A., & Philips, E. D. (2005). Using Student Work to Develop Teachers' Knowledge of Algebra. *Mathematics Teaching in the Middle School, 11*(2), p. 65. Reprinted with permission. Copyright 2005 by the National Council for Teachers of Mathematics. All rights reserved.

Algebraic Thinking across the Curriculum

One reason the phrase "algebraic thinking" is used instead of "algebra" is that the practice of looking for patterns, regularity, and generalizations goes beyond curriculum topics that are usually categorized as algebra topics. You have already experienced some of this integration—looking at geometric growing patterns and working with perimeter and area. Here we share a few more.

Geometry, Measurement and Algebra

Measurement formulas relate various dimensions, areas, and volumes of shapes. Each of these formulas involves at least one functional relationship. Consider any familiar formula for measuring a geometric shape. For example, the circumference of a circle is $c = 2\pi r$. The radius is the independent variable, and circumference is the dependent variable. We can say that the circumference is dependent on the radius. Even nonlinear formulas like volume of a cone ($V = \frac{1}{3}\pi r^2 h$) are functions. Here the volume is a function of both the height of the cone and the radius. If the radius is held constant, the volume is a function of the height. Similarly, for a fixed height, the volume is a function of the radius.

The following activity explores how the volume of a box varies as a result of changing the dimensions.

Activity 12.21

CCSS-M: 5.MD.C.5b; 6.EE.A.2c; 6.G.A.2

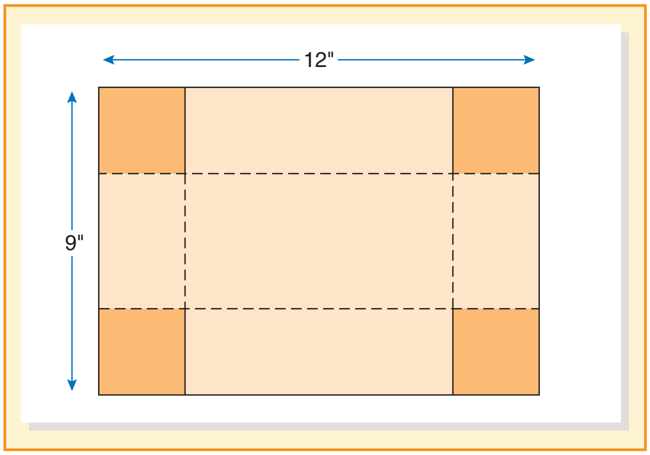
Designing the Largest Box

Give each student or pair of students a piece of card stock. Explain that they are to cut out a square from each corner using an exact measurement. All four squares must be the same size. Assign different lengths for the squares that are cut out (e.g., 2 cm, 2.5 cm, 3 cm, etc.). Explain to students that after they cut out their four squares, they will fold up the four resulting flaps and tape them together to form an open box. Have students calculate the volume of their box. Then, have students trade boxes and determine the volume of other boxes. (The volume of the box will vary depending on the size of the squares [see Figure 12.25].) Ask students to record their data in a table.

After they have recorded the data for several boxes, challenge students to write a formula that gives the volume of the box as a function of the size of the cut-out squares. Use the function to determine what size the squares should be to create the box with the largest volume. Alternatively, make origami boxes using squares with various side lengths, and see what the relationship is between the side length and the volume of the open box. (See DeYoung [2009] for instructions for making the box and more on this idea.)

Figure 12.25

Cutting squares from cardstock. What size squares will result in an open box with the largest volume?



Data and Algebra

Data can be obtained from sports records, census reports, the business section of the newspaper, and many other sources. Students can gather data such as measurement examples or survey data. The Internet has many sites where data can be found.

Experiments. There are many experiments that students can explore to see the functional relationships, if any, that exist between two variables. Gathering real data is an excellent way to engage a range of learners and to see how mathematics can be used to describe phenomena.

Data should be collected and then represented in a table or on a graph. The goal is to determine whether there is a relationship between the independent and dependent variables, and if so, whether it is linear or nonlinear, as in the following engaging experiments:

- How long would it take for 100 students standing in a row to complete a wave similar to those seen at baseball games? Experiment with different numbers of students from 5 to 25. Can the relationship predict how many students it would take for a given wave time?
- How far will a Matchbox car roll off of a ramp, based on the height the ramp is raised?
- How is the flight time of a paper airplane affected by the number of paper clips attached to the nose of the plane?
- What is the relationship between the number of dominoes in a row and the time required for them to fall over? (Use multiples of 100 dominoes.)
- Make wadded newspaper balls using different numbers of sheets of newspaper, using a constant number of rubber bands to help hold the paper in a ball. What is the relationship between the number of sheets and the distance the ball can be thrown?

- What is the relationship between the number of drops of colored water dropped on a paper towel and the diameter of the spot? Is the relationship different for different brands of towels?
- How much weight can a toothpick bridge hold? Lay toothpicks in a bunch to span a 2-inch gap between two boards. From the toothpicks, hang a bag or other container into which weights can be added until the toothpicks break. Begin with only one toothpick (McCoy, 1997).

Experiments like these are fun and accessible to a wide range of learners. They also provide an opportunity for students to engage in experimental design—a perfect blend of mathematics and science.

Scatter Plots. Often in the real world, phenomena are observed that seem to suggest a functional relationship but not necessarily as clean or as well defined as some of the situations we have described so far (like the experiments described above). In such cases, the data are generally plotted on a graph to produce a scatter plot of points. Two very good scatter plot generators can be found online at NLVM and NCES Kids' Zone.

A visual inspection of the scatterplot may suggest what kind of relationship, if any, exists. If a linear relationship seems to exist, for example, students can approximate a line of best fit or use graphing technology to do a linear regression to find the line of best fit (along with the equation).

Literature Connections

The following books are excellent beginnings for patterns and functions.

Bats on Parade *Appelt and Sweet, 1999*

This story includes the pattern of bats walking 1 by 1, then 2 by 2, and so on. One activity from this enjoyable book is determining the growing pattern of the number of bats given the array length (e.g., 3 for the 3 × 3 array). There is also one mouse, so this can be included in a second investigation. Activity Pages for these two ideas and two others can be found in Roy and Beckmann (2007).

Two of Everything: A Chinese Folktale

The magic pot discovered by Mr. Haktak doubles whatever goes in it, including his wife! This idea of input–output is great for exploring expressions, equations and functions in grades 6 through 8; just vary the rule of the magic pot from doubling to something more complex to fit your grade level standards (e.g., use rules with negative numbers or exponents). See Activity 12.7 and, for more ideas and handouts, see Suh (2007a) and Wickett and colleagues (2002).

13

Developing Geometry Concepts

BIG IDEAS

1 What makes shapes alike and different can be determined by geometric properties. For example, shapes have sides that are parallel, perpendicular, or neither; they have line symmetry, rotational symmetry, or neither; they are similar, congruent, or neither.

2 Transformations provide a significant way to think about the ways properties change or do not change when a shape is moved in a plane or in space. These changes can be described in terms of translations, reflections, rotations, and dilations.

3 Shapes can be described in terms of their location in a plane or in space. Coordinate systems can be used to describe these locations precisely. Coordinates can be used to measure distance, an important application of the Pythagorean theorem.

4 Three-dimensional shapes can be seen from different viewpoints, which help us understand relationships between two- and three-dimensional figures and mentally change the position and size of shapes.

Geometry is a "network of concepts, ways of reasoning and representation systems" used to explore and analyze shape and space (Battista, 2007, p. 843). This critical area of mathematics appears in everything from global positioning systems to computer animation. Unique to the *Common Core State Standards* geometry appears as a domain across all grades, K–12: "The notion of building understanding in geometry across the grades, from informal to more formal thinking, is consistent with the thinking of theorists and researchers" (CCSSO, 2010, p. 41). The big ideas in geometry across the grades are:

> *Grade 6 and 8:* Explore shapes on the coordinate plane (drawing them in grade 6 and performing transformations on them in grade 8); find lengths of line segments on the coordinate plane (vertical and horizontal lines in grade 6 and all lines in grade 8 with the Pythagorean theorem).
>
> *Grade 7:* Construct triangles with given conditions, with and without technology, and explore properties of triangles; exploring relationships between two-dimensional and three-dimensional shapes—for example, by slicing three-dimensional figures; and exploring congruence and similarity with physical models, visuals, and technology, focusing on scale drawings.
>
> *Grade 8:* Explore congruence and similarity through transformations and analyze the relationship of angles related to triangles and parallel lines.

For too long, the geometry curriculum in the United States has emphasized learning terminology and labeling in low-level tasks, such as "Is this an obtuse, right, or acute angle?" Geometry is much more than this.

First, it involves developing and applying *spatial sense*. Spatial sense is an intuition about shapes and the relationships among shapes, and is considered a core area of mathematical study, like number (Sarama & Clements, 2009). Spatial sense includes the ability to visualize objects and spatial relationships mentally—to turn things around in one's mind. It includes a comfort with geometric descriptions of objects and position. People with well-developed spatial sense appreciate geometric forms in art, nature, and architecture, and they use geometric ideas to describe and analyze their world.

Second, geometry involves significantly more content than shapes, as indicated by the four major geometry strands:

- *Shapes and properties:* the properties of shapes, as well as the relationships built on properties
- *Transformations:* translations, reflections, rotations, and dilations
- *Location:* coordinate geometry and other ways of specifying how objects are located in a plane or in space
- *Visualization:* the recognition of shapes in the environment, the development of relationships between two- and three-dimensional objects, and the ability to construct and draw figures from different viewpoints

All four of these are significant in the middle-school curriculum. Therefore, the content in this chapter is divided according to these four categories, with the discussion of each category beginning with foundational experiences and moving through more challenging experiences.

Developing Geometric Thinking

All students can develop the ability to think and reason in geometric contexts, but this ability requires ongoing and significant experiences across a developmental progression. The research of two famous Dutch educators, Pierre van Hiele and Dina van Hiele-Geldof (husband and wife) (1984), provides insights into the differences in geometric thinking through the description of different levels of thought. The van Hiele levels of geometric thought have been a major influence in mathematics curriculum worldwide.

The van Hiele Levels of Geometric Thought

The van Hiele model is a five-level hierarchy (see Figure 13.1). Each level describes what types of geometric ideas we think about (called *objects of thought*) and what students can do (called *products of thought*). The levels are developmental in nature—students (regardless of age) begin at level 0 and through experiences progress to the next level. The work in middle school is targeted primarily at level 2, but you may need to pull in level 0 and level 1 experiences for students who are not ready to reason at level 2. Characteristics of the van Hiele levels of geometric thought are provided in Table 13.1.

Figure 13.1

The van Hiele levels of geometric thought.

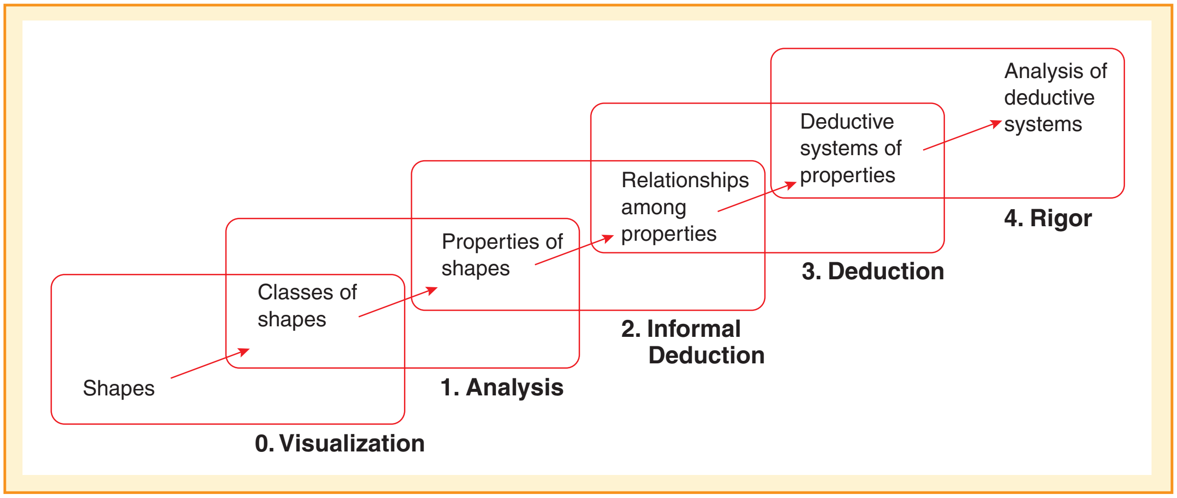


Table 13.1. Characteristics of the van Hiele levels.

Characteristic	Implication
1. Sequential	To arrive at any level, students must move through all prior levels. For students to be ready for high school geometry, middle school must develop through level 2 geometric thought.
2. Developmental	When instruction or language is at a level higher than that of the students, students cannot understand the concept being developed. A student can, for example, memorize a fact (e.g., all squares are rectangles) but not mentally construct the actual relationship of how the properties of a square and rectangle are related.
3. Age-independent	Although a seventh-grader should be working at level 2, he or she may be at level 0 or 1. Some adults remain forever at level 0, and a significant number of adults never reach level 2.
4. Experience-dependent	Advancement through the levels requires geometric experiences. (e.g., an eighth-grader operating at level 0 needs numerous and carefully selected activities to help him or her move to level 1, and then additional carefully selected experiences to move to level 2). Students should explore, talk about, and interact with content at the next level while increasing experiences at their current level.

Level 0: Visualization

The objects of thought at level 0 are shapes and what they "look like."

Students at level 0 recognize and name figures based on the global visual characteristics of the figure. For example, a square is defined by a level 0 student as a square "because it looks like a square." Appearance is dominant at level 0 and can therefore overpower the properties of a shape. A level 0 thinker, for example, may see a square with sides that are not horizontal or vertical (it appears tilted) and believe it is a diamond and no longer a square.

The products of thought at level 0 are classes or groupings of shapes that seem to be "alike."

The emphasis at level 0 is on shapes that students can observe, feel, build, take apart, or work with in some manner. How are shapes alike and different? Some of these classes of shapes have names—rectangles, triangles, rhombi, and so on. Properties of shapes (e.g., parallel sides, right angles) are included at this level, but only in an informal, observational manner.

Level 1: Analysis

The objects of thought at level 1 are classes of shapes rather than individual shapes.

You will know your students are at the analysis level if they are able to consider all shapes within a class rather than just the single shape on their desk. Instead of talking about *this* particular rectangle, they can talk about *all* rectangles. Students focus on what makes a rectangle a rectangle (four sides, four right angles, etc.). The irrelevant features (e.g., size or orientation) fade into the background. If a shape belongs to a particular class, it has the corresponding properties of that class. "All cubes have six congruent faces, and each of those faces is a square." Students operating at level 1 may be able to list all the properties of squares, rectangles, and parallelograms but may not see that each of these classes is a subclass of the next (e.g., that all squares are rectangles, and all rectangles are parallelograms). In defining a shape, level 1 thinkers are likely to list as many properties of a shape as they know.

The products of thought at level 1 are the properties of shapes.

Level 1 students continue to use manipulatives and drawings of shapes, but they also see these individual shapes as representatives of classes of shapes. Their understanding of the properties of shapes—such as symmetry, perpendicular and parallel lines, and so on—continues to be refined. This identification of geometric properties is an important cognitive activity (Yu, Barrett, & Presmeg, 2009).

Standards for Mathematical Practice

7 Look for and make use of structure.

The following activity is targeted at developing level 1 thinkers because the goal is to explore and conjecture about the properties of quadrilaterals.

Activity 13.1

CCSS-M: 3.G.A.1; 4.G.A.2; 4.G.A.3; 7.G.A.2

Property Lists for Quadrilaterals

Prepare handouts (Property Lists) for **Parallelograms**, **Rhombuses**, **Rectangles**, and **Squares**. Assign groups of three or four students to work with one type of quadrilateral (for English language learners [ELLs] and students with disabilities, post labeled shapes as a reference). Their task is to list as many properties as they can that apply to all of the example shapes on their sheet. They will need tools such as index cards (to check right angles, compare side lengths, and draw straight lines); mirrors (to check line symmetry); and tracing paper (for angle congruence). Encourage students to use the

(*continued*)

words *at least*, *only*, *at most*, and *always* to describe how many of something. For example, "rectangles have at least two lines of symmetry"—because squares, included in the category of rectangles, have four.

Have students prepare their **Property List for Sides, Angles, Diagonals, Symmetries** under these headings: Sides, Angles, Diagonals, and Symmetries. Groups then share their lists with the class, and eventually, a class list for each category of shape will be developed.

For ELLs, emphasizing these words, having students say the words aloud, and having students point to each word as you say it are ways to reinforce meaning and support their participation and comprehension during the sharing time. For students with disabilities, provide a structured recording sheet with a table listing the headings. This will help organize their thinking around the many diverse possibilities.

Teaching Tip

Activity 13.1 can be extended to exploring polygons on the coordinate plane by having the shapes positioned on a coordinate axis and asking students to use information from the coordinate axis as part of what they are noticing and comparing.

Notice that students must assess whether the properties apply to all shapes in the category. If they are working on squares, for example, their observations must apply to a square mile as well as a square centimeter.

Level 2: Informal Deduction

The objects of thought at level 2 are the properties of shapes.

After students are able to think about the properties of geometric objects, they are ready to develop relationships between and among these properties. "If all four angles are right angles, the shape must be a rectangle. If it is a square, all angles are right angles. If it is a square, it must be a rectangle." Once students have greater ability to engage in "if–then" reasoning, they can classify shapes with a minimal set of defining characteristics. For example, four congruent sides and at least one right angle are sufficient to define a square. Observations extend beyond properties themselves and begin to focus on logical arguments *about* the properties. When your students are at level 2, they will be able to follow and appreciate informal deductive arguments about shapes and their properties.

The products of thought at level 2 are relationships among properties of geometric objects.

The hallmark of level 2 activities is the inclusion of informal logical reasoning. Since your students have developed an understanding of the various properties of shapes, it is now time for you to encourage conjecture and to ask "why?" or "what if?"

Activity 13.2

CCSS-M: 4.G.A.2; 5.G.B.3; 7.G.A.2

Minimal Defining Lists

This activity is a sequel to Activity 13.1, "Property Lists for Quadrilaterals" and has students determine the unique conditions for each polygon. Once the class has agreed on **Property List for Sides, Angles, Diagonals, Symmetries** for the parallelogram, rhombus, rectangle, and square (and possibly the kite and trapezoid), post the lists. Have students work in groups to find the minimal defining lists (unique features) for each shape. The term *defining* here means that any shape that meets these conditions is this shape. The term *minimal* means that if any single property is removed from the list, the list is no longer defining. For example, a minimal defining list for a square is a quadrilateral with (1) four congruent sides and (2) four right angles (two items on the list). If a shape has these two properties, it must be a square. But there are other minimal defining lists for a square, such as (1) all sides the same length and (2) perpendicular diagonals. Challenge students to find more than one minimal defining list for their shape.

After students have created their lists, have them exchange lists with another group and draw shapes that meet the given conditions. For an extension, explore different types of triangles in this same way (creating a list of characteristics, determining the minimal list of unique features, then exchanging with others to draw the triangles described by that group).

A proposed list can be challenged as being either not minimal or not defining. A list is not defining if a counterexample—a shape other than one being described—can be produced by using only the properties on the list.

The hallmark of this and other level 2 thinking activities is the emphasis on logical reasoning. "*If* a quadrilateral has these properties, *then* it must be a square." Logic is also involved in proving that a list is faulty—either not minimal or not defining. Here, students begin to learn the nature of a definition and the value of counterexamples. The other aspect of this activity that clearly involves level 2 thinking is that students focus on analyzing the relationships between properties (e.g., if a quadrilateral has four right angles, it also has diagonals of the same length).

Teaching Tip

Use "if–then" sentence frames to help students analyze the relationships between properties. Or, create a matching game of "if" statements and "then" statements.

Level 3: Deduction

The objects of thought at level 3 are relationships among properties of geometric objects.

At level 3, students analyze informal arguments; the structure of a system complete with axioms, definitions, theorems, corollaries, and postulates begins to develop; and they begin to appreciate the necessary means of establishing geometric truth. The student at this level is usually in high school and is able to work with abstract statements about geometric properties and make conclusions based on logic.

The products of thought at level 3 are deductive axiomatic systems for geometry.

Level 4: Rigor

The objects of thought at level 4 are deductive axiomatic systems for geometry.

At the highest level of the van Hiele hierarchy, the objects of attention are axiomatic systems themselves, not just the deductions within a system. This is generally the level of a college mathematics major who is studying geometry as a branch of mathematical science.

The products of thought at level 4 are comparisons and contrasts among different axiomatic systems of geometry.

Formative Assessment Note

How do you discover the van Hiele level of each student? Once you know, how will you select the right activities to match your students' levels? As you conduct an activity, listen to the types of observations that students make and record them on an **Observation Checklist**. Noting if they:

- Describe all features of the group of quadrilaterals? Triangles?
- Talk about the conditions that are required for each shape?
- Compare groups of shapes (e.g., rectangles) to other groups of shapes (e.g., squares)?

With observations such as these, you will be able to distinguish between levels 0, 1, and 2. If students are not able to follow logical arguments or make conjectures about the properties of shapes, they are likely at level 1 or below and will need interventions to prepare them for level 2.

Implications for Instruction

Recall that geometry thinking is sequential and developmental (see Table 13.1)! Therefore, carefully selected and implemented tasks are the way to move students up through the levels of geometric thought. The geometry taught in high school is primarily at level 3, so a priority in middle school is to provide instruction to students so that they are at least strong level 2 thinkers, ready for success in high school and beyond. Many activities can be implemented to span two levels of thinking, helping students move from one level to the next.

Teaching Tip

The geometric experiences you provide are the single most important factor in moving students up the developmental ladder.

Moving from Level 0 to Level 1

If middle-school students are still at level 0, they need targeted interventions that will move them to level 1. Memorization or drill is not the answer. Students need experiences that start at level 0 thinking and include introductory level 1 thinking. Instructional activities that support students' movement are as follows:

- *Challenge students to test ideas about shapes by using a variety of examples from a particular category*. Say, "Let's see if that is true for other rectangles," or "Can you draw a triangle that does *not* have a right angle?" In general, question students to see if the observations they make about a particular shape apply to other shapes of a similar kind.
- *Focus on the properties of figures rather than on simple identification*. As new geometric concepts are learned, students should be challenged to use these features to classify shapes.
- *Provide ample opportunities to draw, build, make, put together (compose), and take apart (decompose) shapes in both two and three dimensions*. These activities should be built around the understanding and use of specific characteristics or properties.
- *Apply ideas to entire classes of figures* (e.g., all *rectangles*, all *prisms*) *rather than to individual shapes in a set*. For example, find ways to sort all possible triangles into groups. From these groups, define types of triangles.

technology note

Dynamic geometry software, such as *The Geometer's Sketchpad* (from Key Curriculum Press) or the free public domain software from GeoGebra, is especially useful for exploring many examples of a class of shapes and is appropriate for middle-school students.

Moving from Level 1 to Level 2

Level 2 thinking should begin about grade 5, when students begin to classify two-dimensional figures based on their properties in large categories and subcategories (CCSSO, 2010). Middle-school students who are still at level 1 need to transition to level 2. The following strategies are effective interventions for moving from level 1 to level 2:

Teaching Tip

Ask questions that use the language of informal deduction: *all, some, none, if–then, what if?* Place these prompts prominently on your walls (on posters) to encourage students to ask these questions as they work.

- *Challenge students to explore or test examples*. Ask questions such as these: "If the sides of a four-sided shape are all congruent, will you always have a square? Can you find a counterexample?"

- *Encourage the making and testing of hypotheses or conjectures*. "Do you think that will work all the time? Is that true for all triangles, or just equilateral triangles?"
- *Examine the properties of shapes to determine the necessary and sufficient conditions for a shape to be a particular shape*. "What properties must diagonals have to guarantee that a quadrilateral with these diagonals will be a square?"
- *Encourage students to attempt informal proofs*. As an alternative, require them to make sense of informal proofs that you or other students have suggested.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

In each of the sections in this chapter ("Shapes and Properties," "Location," "Transformations," and "Visualization"), many activities are shared, and they follow the van Hiele levels of geometric thought. If one activity is beyond the reach of a student or a class, drop back to an earlier one. These four sections are quite fluid—that is, the content areas overlap and build on one another. Activities in one section may help develop geometric thinking in another area.

Shapes and Properties

This is the content area most often associated with geometry in pre-K–8 classrooms, when young students begin to "perceive, say, describe/discuss, and construct objects in 2-D space" (National Research Council, 2009, p. 177).

Middle-school students need experience with a wide variety of two- and three-dimensional shapes. Polygons should not always be regular (all sides the same), and they should not always be shown with a base horizontal to the line on the paper in which it appears. (If you have students say a triangle is upside down, it is because they have rarely seen triangles illustrated differently.) Shapes should have curved sides, straight sides, and combinations of these. Along the way, as students describe the shape or property, the terminology can be introduced.

Categories of Two-Dimensional Shapes

Even before children enter school, they begin sorting experiences. As they get older, they begin to classify the categories they have sorted. In the *Common Core State Standards*, classifying two-dimensional shapes into categories based on their properties (a level 1 thinking task) is a fifth-grade concept (with some sorting by properties in grades 3 and 4). In middle school, the categories of two-dimensional shapes are analyzed by determining the relationship between the sides and angles of a given shape, particularly triangles and quadrilaterals. The categories of two-dimensional shapes are provided in Table 13.2. Or, you can download **Categories of 2- and 3-Dimensional Shapes** reference pages for you or your students. Illustrations of how these shapes are classified and how they are nested can be found in Figure 13.2.

Activity 13.3 is a describing-and-classifying activity that can be at level 0 or level 1, depending on the shapes you place in the folder. It is therefore a good task for students who need more experiences in moving toward level 2 thinking.

Activity 13.3

CCSS-M: 4.G.A.2; 7.G.A.2

What's My Shape?

Make and cut out sets of two-dimensional shapes (on card stock, one set per group) (see Figure 13.3). Make one additional shape set so that you can create the 'hidden' shapes. To do this, glue one shape inside a file folder (or folded construction paper). Give each

(continued)

small group one set of pieces. Designate one student as the leader; he or she gets one of the folders that holds the secret shape. The other students are to determine which shape in their set matches the shape in the folder by asking the leader "yes" or "no" questions. Students cannot point to a piece and ask, "Is it this one?" Instead, they ask questions such as these: "Does it have all straight sides? Is it concave?" The group looks at the shapes in the full set and eliminates shapes as they ask questions to narrow the possibilities. The final piece is checked against the one in the leader's folder. The folder is returned to the teacher, and another student in the group selects a new folder and is the leader for the next turn. Students with disabilities or students who are not yet at level-1 thinking may need a list of possible properties and characteristics (e.g., number of sides) for reference as they ask questions.

Table 13.2. Classifications of two-dimensional shapes.

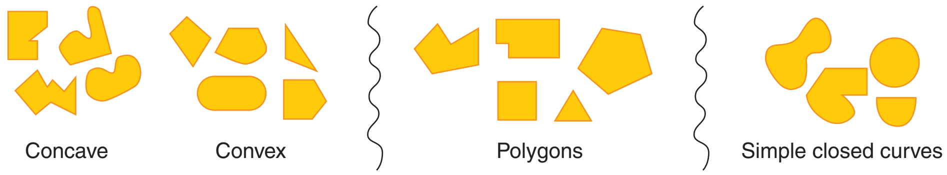
Shape	Description
(A) Simple Closed Curves	
Concave, convex	An intuitive definition of *concave* might be "having a dent in it."
Symmetric, nonsymmetric	Shapes may have one or more lines of symmetry and may or may not have rotational symmetry; these concepts will require more detailed investigation.
Polygons	Simple closed curves with all straight sides.
Regular	All sides and all angles are congruent.
(B) Triangles	
Triangles	Polygons that has exactly three sides.
Classified by sides	
Equilateral	All sides are congruent.
Isosceles	At least two sides are congruent.
Scalene	No two sides are congruent.
Classified by angles	
Right	One angle is a right angle.
Acute	All angles are smaller than a right angle.
Obtuse	One angle is larger than a right angle.
(C) Convex Quadrilaterals	
Convex quadrilaterals	Convex polygons with exactly four sides.
Kite	Two pairs of adjacent sides are congruent.
Trapezoid*	At least one pair of opposite sides are parallel.
Isosceles trapezoid	A pair of opposite sides is congruent.
Parallelogram	Two pairs of sides are parallel.
Rectangle	Parallelogram with angles that are 90°.
Rhombus	Parallelogram where all sides are congruent.
Square	Parallelogram with right angles and sides that are congruent.

*Some definitions of trapezoid specify exactly one pair of parallel sides, in which case parallelograms would not be trapezoids. Consult your local standards and curriculum to determine which definition to use (Manizade & Mason, 2014).

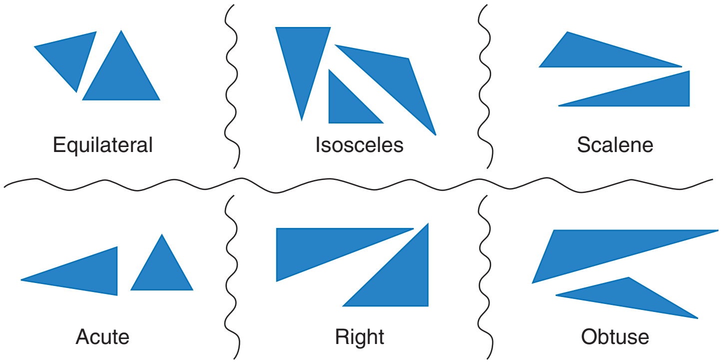
Figure 13.2

Examples of two-dimensional shapes across classifications.

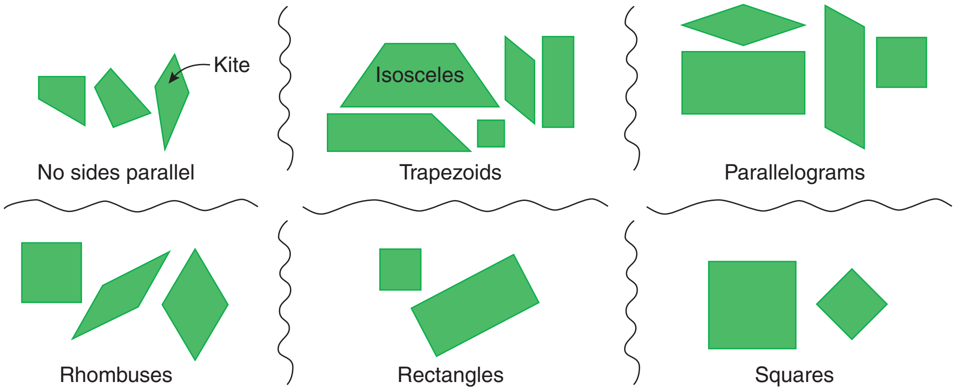
(a) Closed Shapes



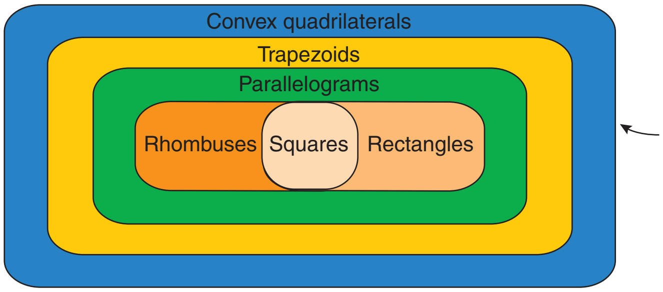
(b) Triangles



(c) Quadrilaterals



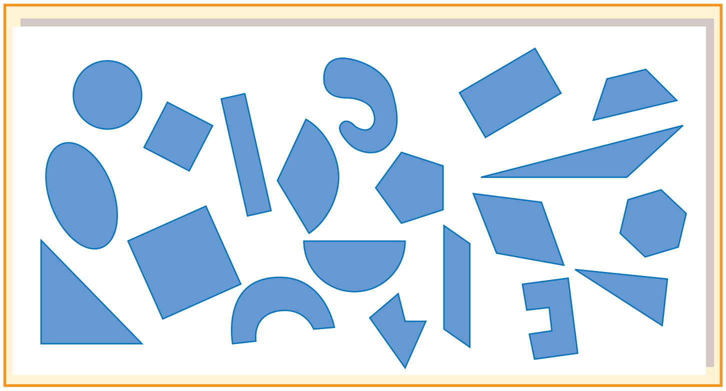
(d) Nesting of Categories of Quadrilaterals



Recognizing these subrelationships is an indication of being at level 2 thinking.

Figure 13.3

An assortment of shapes for sorting.



Categories of Three-Dimensional Shapes

Important and interesting relationships exist in three dimensions, and these relationships are important in measurements such as surface area and volume. Table 13.3 describes the classifications of solids. Or, you can download **Categories of 2- and 3-Dimensional Shapes**.

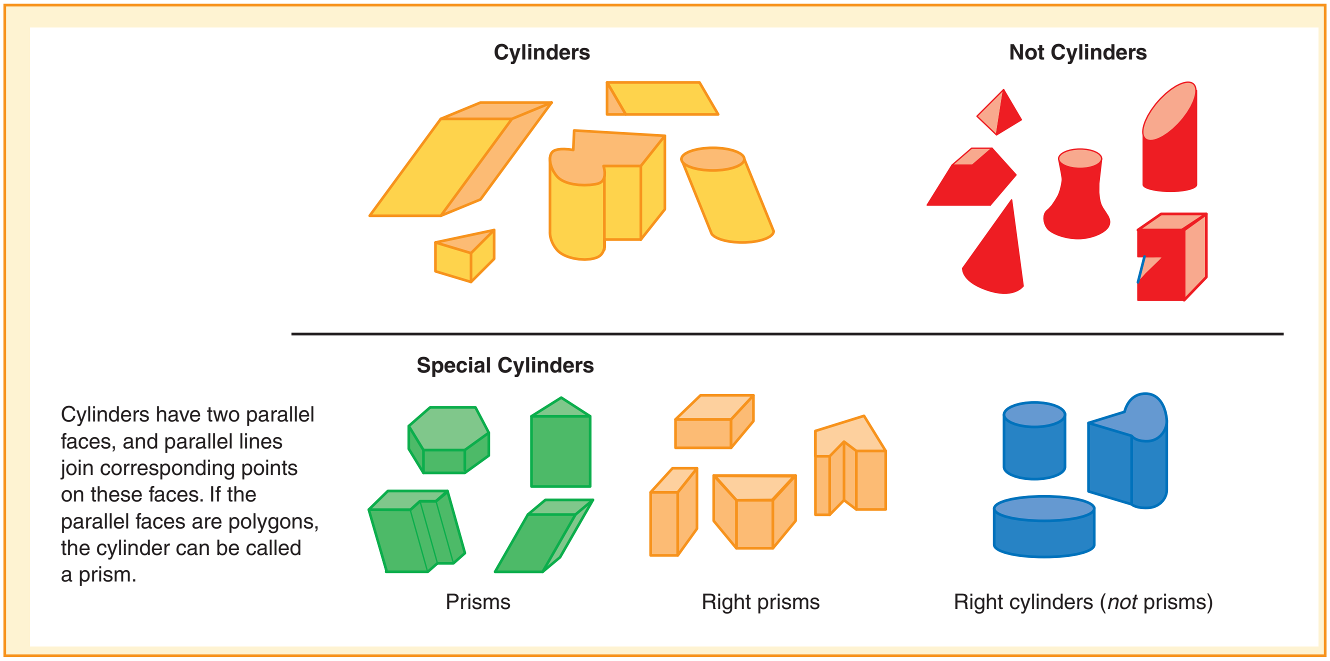
Three-dimensional shapes also have subcategories (Figures 13.4 and 13.5). For example, prisms are special cases of a cylinder with a polygon for a base (Zwillinger, 2011). Some textbooks may limit the definition of cylinders to just circular cylinders. Under that definition, the prism is not a special case of a cylinder. This points to the fact that definitions are conventions, and not all conventions are universally agreed on.

Table 13.3. Categories of three-dimensional shapes.

Shape	Description
Sorted by Edges and Vertices	
Sphere and "egglike" shapes	Shapes with no *edges* and no *vertices* (corners). Shapes with *edges* but no *vertices* (e.g., a flying saucer). Shapes with *vertices* but no *edges* (e.g., a football).
Sorted by Faces and Surfaces	
Polyhedron	Shapes made of all faces (a *face* is a flat surface of a solid). If all surfaces are faces, all the edges will be straight lines. Some combination of faces and rounded surfaces (cylinders are examples, but this is not a definition of a cylinder). Shapes with curved surfaces. Shapes with and without edges and with and without vertices. Faces can be parallel. Parallel faces lie in places that never intersect.
Cylinders	
Cylinder	Two congruent, parallel faces called *bases.* Lines joining corresponding points on the two bases are always parallel. These parallel lines are called *elements* of the cylinder.
Right cylinder	A cylinder with elements perpendicular to the bases. A cylinder that is not a right cylinder is an *oblique cylinder.*
Prism	A cylinder with polygons for bases. All prisms are special cases of cylinders.
Rectangular prism	A cylinder with rectangles for bases.
Cube	A square prism with square sides.
Cones	
Cone	A solid with exactly one face and a vertex that is not on the face. Straight lines (elements) can be drawn from any point on the edge of the base to the vertex. The base may be any shape at all. The vertex need not be directly over the base.
Circular cone	A cone with a circular base.
Pyramid	A cone with a polygon for a base. All faces joining the vertex are triangles. Pyramids are named by the shape of the base: *triangular* pyramid, *square* pyramid, *octagonal* pyramid, and so on. All pyramids are special cases of cones.

Figure 13.4

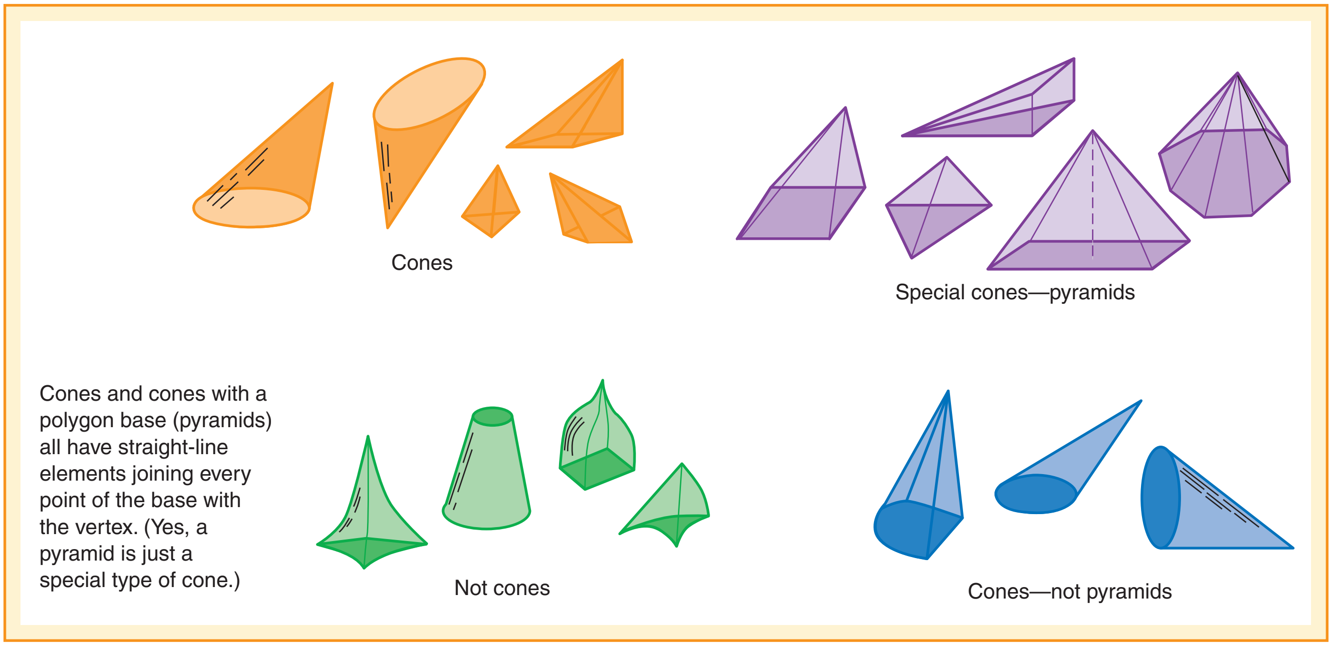
Cylinders and prisms.



Cylinders have two parallel faces, and parallel lines join corresponding points on these faces. If the parallel faces are polygons, the cylinder can be called a prism.

Figure 13.5

Cones and pyramids.



Cones and cones with a polygon base (pyramids) all have straight-line elements joining every point of the base with the vertex. (Yes, a pyramid is just a special type of cone.)

You can adapt Activity 13.3 to three-dimensional shapes (place three-dimensional shapes in an opaque bag). For three-dimensional shapes, use a collection of solids with a lot of variation (curved surfaces, etc.). Power Solids and other collections of three-dimensional shapes are available through various catalogs, or use real objects such as cans, containers, and balls.

Formative Assessment Note

Activity 13.3 can be used as a *diagnostic interview*. You hold the hidden shape and see what questions the students ask. The ways students describe two- or three-dimensional shapes are good evidence of their level of geometric thinking. Level 0 thinkers may ask, "Is it a parallelogram?" (because it looks like a rectangle), while level 1 thinkers may ask the same question but see the shape as a group of shapes. When you respond yes, look to see if the students include squares and rectangles. If they do not, point at a rectangle and ask, "Is this a parallelogram?" With level 1 thinkers, you may also hear these questions: "Is it convex? Is it obtuse? Does it have line symmetry? Rotational symmetry?" These questions focus on the properties of shapes and indicate that the students are at level 1 thinking.

Applying Definitions and Categories

Using definitions and categories helps students focus more deeply on the properties that make the shape what it is (and what it is not). In this section we explore the specific classifications within categories of polygons.

Exploring Properties of Triangles

Determining types of triangles is introduced in grade 4 with the concept of right triangles. It is then emphasized in fifth grade as students classify two-dimensional figures and focus on categories and subcategories of shapes, and extended in seventh grade, where the focus is on properties based on the measures of the sides and angles (CCSSO, 2010). Figure 13.2(b) lists the categories of triangles. Activity 13.4 explores all the ways to classify triangles.

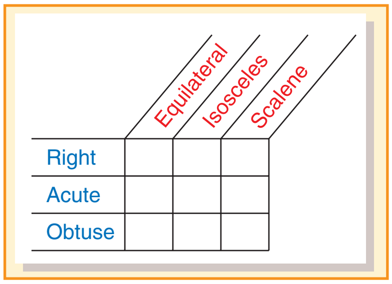
Activity 13.4

CCSS-M: 5.G.B.3; 5.G.B.4; 7.G.A.2

Triangle Sort

Make copies of the Assorted Triangles. Note the examples of right, acute, and obtuse triangles; examples of equilateral, isosceles, and scalene triangles; and triangles that represent every possible combination of these categories. Have students cut them out. Once the groups have been determined, provide appropriate terminology. For ELLs and other students who may struggle with the vocabulary, it is important to focus on the specialized meanings of the terms (e.g., contrasting *acute pain* and *acute angle*) as well as on root words (*equi-* meaning "equal" and *-lateral* meaning "side"). Ask students to sort the entire collection into three groups so that no triangle belongs to two groups. When this is done and descriptions of the groups have been written, students should find a second criterion for creating three different groups. Students with disabilities may need a hint to look only at angle sizes or only at the issue of congruent sides, but delay giving these hints if you can.

Activity 13.4 can also be explored with a piece of string. Ask students to create a triangle with specific conditions, like two congruent sides and one obtuse angle. As a follow-up activity, challenge students to sketch a triangle in each of the nine cells of the **Triangle Sort Chart**.



	Equilateral	Isosceles	Scalene
Right			
Acute			
Obtuse			

Stop and Reflect

Of the nine cells in the chart, two of them are impossible to fill. Can you tell which ones and why?

An important aspect of looking at types of triangles in seventh grade is to consider when given conditions result in a unique triangle, more than one triangle, or no triangle. For example, an equilateral triangle with sides of 4 cm is unique, a triangle with two sides the same and one angle the same can be made in many ways, and a triangle with two obtuse angles is impossible to make.

Activity 13.10, “Can You Build It?,” can also be adapted to explore the conditions of triangles. Adapt the list of properties to read, “A triangle with,” and then add the question, “How many ways?”

Can you make a triangle with one 90-degree angle and two sides of 5 cm and 8 cm? How many ways?

Can you make a triangle with two angles of 45 degrees? How many ways?

Can you make a triangle with sides of 4 cm, 8 cm, and 13 cm? How many ways?

Notice that these examples address both sides and angles (or a combination). You can also identify a single problem for investigation, pressing students to prove (informally, through the use of tools) whether the conditions make the triangle unique or not, as in these next two examples.

ARE THESE TRIANGLES UNIQUE?

A right triangle with legs of 3 units and 4 units

An isosceles triangle with an 80-degree angle and a side of 5 units

Students need many experiences with such activities (with triangles and other polygons). Thinking about how the conditions of a triangle define it is the foundation for high school geometry, in which, for example, students prove why angle-side-angle (ASA) makes a triangle unique. In doing transformations and scale drawings in middle school, students are better able to recognize what is true about congruent and similar shapes.

Teaching Tip

Once right triangles are introduced, the term *legs* is used for the two sides adjacent to the right angle, and the term *hypotenuse* is used to name the third side. Encourage students to use this language when discussing right triangles.

Middle-school students should explore the relationship of angles within a triangle (adding to 180 degrees). They should also use facts about supplementary, complementary, vertical, and adjacent angles to solve for an unknown angle in a figure (CCSSO, 2010). Activity 13.5 explores interior angles.

Activity 13.5

CCSS-M: 5.G.B.3; 7.G.A.2; 8.G.A.5

Angle Sums in a Triangle

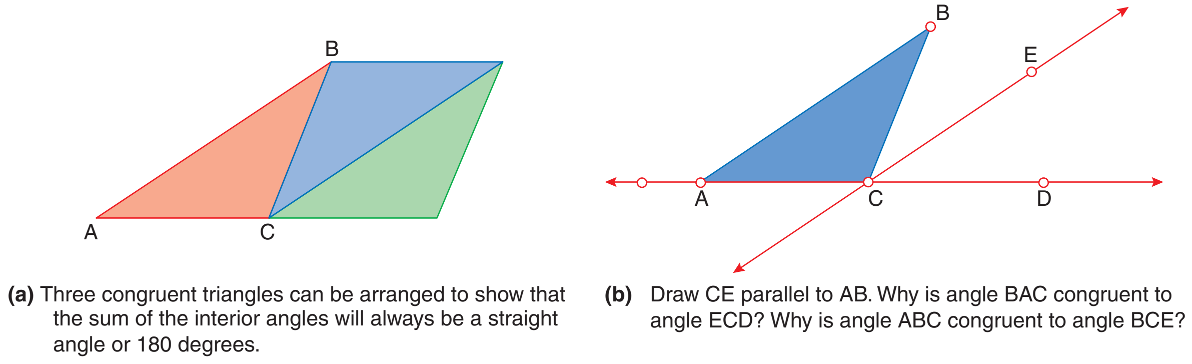
Distribute three copies each of the **Three Congruent Triangles** Activity Page to pairs or small groups of students. Ask students to stack the three sheets of paper and cut out each triangle (creating three equivalent triangles for each of the triangles, a total of nine triangles). Tell students to label each triangle with angles a, b, and c, making sure the corresponding angles are labeled with the same letter. Ask students to use a straight edge or a line drawn on another paper to explore what they notice about the angles within one set of congruent triangles. If students do not discover that when they put angles a, b, and c together they add up to 180°, then offer the following steps:

1. Place one triangle on a line and the second directly next to it in the same orientation.
2. Place the third triangle in the space between the triangles, as shown in Figure 13.6(a). Ask, “Will this relationship be true for any kind of triangle?”
3. Have students make two more triangles (in triplicate) that are different from each other to test.

Summarize by asking students to make a conjecture about the sum of the angles in a triangle.

Figure 13.6

Interior and exterior angles of triangles can be explored with various tools.



(a) Three congruent triangles can be arranged to show that the sum of the interior angles will always be a straight angle or 180 degrees.

(b) Draw CE parallel to AB. Why is angle BAC congruent to angle ECD? Why is angle ABC congruent to angle BCE?

Technology can be an important tool in exploring these relationships. Using a dynamic geometry program, the three triangles in Figure 13.6(a) can be drawn by starting with one triangle, translating it to the right the length of AC, and then rotating the same triangle about the midpoint of side BC. When the vertices of the original triangle are dragged, the other triangles will change accordingly and remain congruent. Although this exploration demonstrates to students that the angle sum is always a straight angle, it does not show them why. This requires examining exterior angles, as illustrated in Figure 13.6(b). Students can look at relationships among the interior and exterior angles, making and testing conjectures, which builds important foundations for the proofs they will do in high school.

Activity 13.6 is another example of a way to informally investigate important properties of triangles.

Activity 13.6

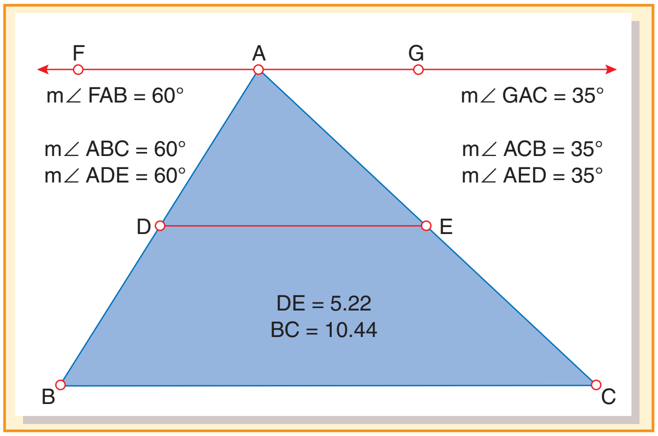
CCSS-M: 7.G.A.2; 8.G.A.5

Triangle Midsegments

Using a dynamic geometry program, draw a triangle, and label the vertices A, B, and C. Draw the segment joining the midpoints of AB and AC, and label this segment DE (Figure 13.7). Measure the lengths of DE and BC. Also measure angles ADE and ABC. Drag points A, B, and C. What conjectures can you make about the relationships between segment DE (the midsegment of triangle ABC) and segment BC (the base of triangle ABC)? For using this activity in a full lesson, go to **Expanded Lesson: Triangle Midsegments**.

Figure 13.7

The midsegment of a triangle is always parallel to the base and half as long.



The midsegment is half the length of the base and parallel to it. Students can explore why this is so by drawing a line through point A parallel to BC. List all pairs of angles that they know are congruent. Why are they congruent? Note that triangle ABC is similar to triangle ADE. Why is it similar? Encourage students to make logical arguments for why the things they observe to be true are in fact true for any triangle.

Exploring Properties of Quadrilaterals

Like triangles, quadrilaterals have subcategories with names (see Figure 13.2(c)) or refer to **Categories of 2- and 3-Dimensional Shapes** Activity Page. The activities described previously can be used to explore quadrilaterals as well as triangles. An important and difficult concept within quadrilaterals is how they relate to one another, for example, which is a subcategory of which? This concept is a focus of fifth grade in the *Common Core State Standards* expectations but continues to be important in seventh grade, as students explore the conditions that make a shape unique, possible, or impossible. The minimal defining list in Activity 13.2 begins to address these relationships. The next activity uses examples and nonexamples to explore the subcategories.

Activity 13.7

CCSS-M: 4.G.A.2; 5.G.B.3, 5.G.B.4; 7.G.A.2

Mystery Definition

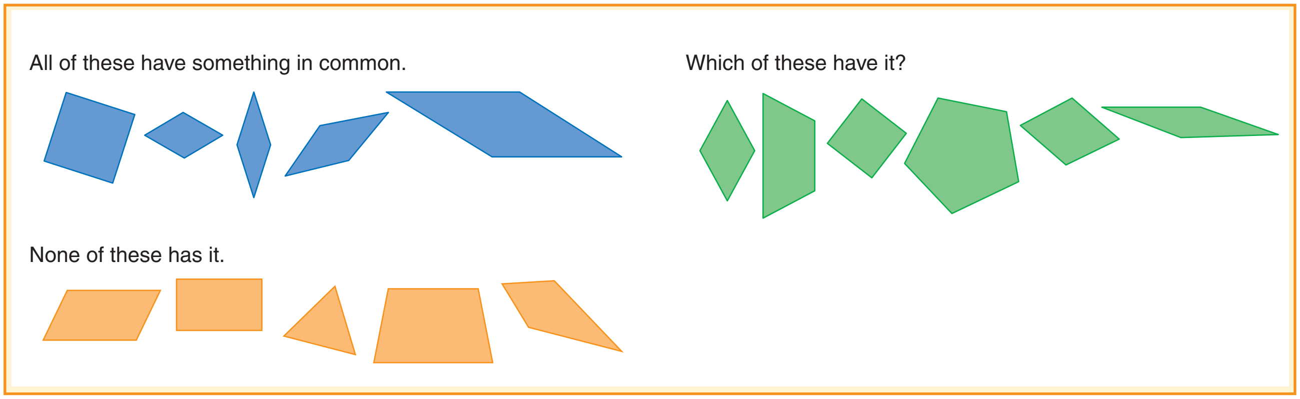
Project or distribute the **Mystery Definition** Activity Page (see Figure 13.8). With your first collection, be certain that you have allowed for all possible variations. In Figure 13.8, for example, a square is included in the set of rhombi. Similarly, choose nonexamples to —or mixed— be as close to the positive examples as is necessary to help with an accurate definition. The third—or mixed—set should include shapes with which students are most likely to be confused. For each shape in the third set, students should justify their choices in a class discussion. This activity can also be used with three-dimensional figures. Note that the use of nonexamples is particularly important for students with disabilities.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Figure 13.8

Solve the mystery. What unique features do the shapes have in common?



Teaching Tip

The word *definition* is not used in the CCSS for grades 6–8, but informal definitions lay the foundation for learning precise definitions in high school geometry.

Standards for Mathematical Practice

6 Attend to precision.

Through activities like "Mystery Definition" with two- and three-dimensional shapes, students develop informal definitions based on their own analysis of relevant properties. Developing such definitions should be a meaning-making experience, not a memorizing task.

Figure 13.2(a)–(c) illustrates various categories of two-dimensional shapes, any of which can be used for "Mystery Definition." After students' definitions have been discussed and compared, you can compare their ideas with the common definition for each shape and determine whether their definitions captured all of the necessary aspects of the shape and stated them precisely.

Quadrilaterals are an especially rich source of investigations. In addition to Activity 13.7, "Mystery Definition," Activities 13.1 and 13.2, "Property Lists for Quadrilaterals" and "Minimal Defining Lists," focus on having students generate definitions and compare them with the common definition for each shape. In defining shapes, it is important for students to explore which of these shapes are subcategories of other shapes (e.g., squares are a subset of rectangles). This is a fifth-grade expectation in the *Common Core State Standards*, but it is particularly difficult for students (and adults) and likely will need to be revisited through middle school. Figure 13.2(d) illustrates the nesting of categories of quadrilaterals.

As new relationships come up in student presentations and related discussions of defining properties, you can introduce proper terminology. For example, if two diagonals intersect such that they form 90-degree angles, then they are *perpendicular*. Other terms, such as *parallel*, *congruent*, *bisect*, and *midpoint*, can be clarified as you help students write their descriptions. This is also a good time to introduce symbols, such as ≈ for *congruent* and ∥ for *parallel*.

Recall that at van Hiele level 2, the focus of thought shifts to properties for categories of shapes. At this level, it is essential to encourage conjecture and explore informal deductive arguments. Middle-school students should begin to understand and use simple proofs. Let's revisit Activity 13.2, "Minimal Defining Lists," which is a level 2 activity. The parallelogram, rhombus, rectangle, and square all have at least four minimal defining lists. One of the less obvious, but very interesting, lists uses the properties of diagonals. For example, a quadrilateral with diagonals that bisect each other and are perpendicular (intersect at right angles) is a rhombus. Activity 13.8 explores the properties of diagonals among classes of quadrilaterals.

Teaching Tip

In number and algebra, students make conjectures about properties of operations (see Chapter 12); in geometry, students make conjectures about properties of shapes. When a student makes an observation or a statement about a geometric concept, write it on the board. Ask, "Is it always true? How can we prove it?"

Activity 13.8

CCSS-M: 5.G.B.3; 5.G.B.4; 6.RP.A.1; 7.G.A.2

Diagonals of Quadrilaterals

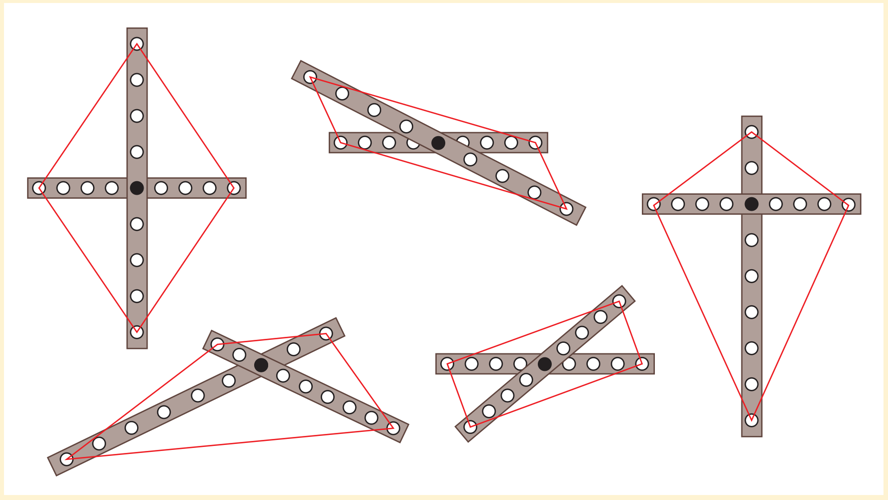
For this activity, students need three **Diagonal Strips** made of card stock. Punch the nine holes as marked. Use a brass fastener to join two strips. A quadrilateral is formed by joining the four holes (Figure 13.9). Provide students with the list of possible relationships for angles, lengths, and proportional comparisons of parts (ratios). Ask, "What quadrilaterals can these diagonals form?"

- Diagonals that are the same length
- Diagonals that are perpendicular
- Diagonals that bisect each other
- Diagonals that are perpendicular and of the same length

Students use the strips to determine the properties of diagonals that will produce different quadrilaterals. Have students make drawings on **1-Centimeter Dot Paper** to test the various hypotheses and record their findings on the **Properties of Quadrilateral Diagonals** Activity Page. See the **Expanded Lesson: Diagonals of Quadrilaterals** for this activity for the full description of this instructional experience. For ELLs, provide a list of the quadrilaterals with pictures next to the names of the various shapes (or refer them to a word wall or journal entry with each option). Ask, "What quadrilaterals can these diagonals form?"

Figure 13.9

Diagonals of quadrilaterals made with diagonal strips and brass fasteners.



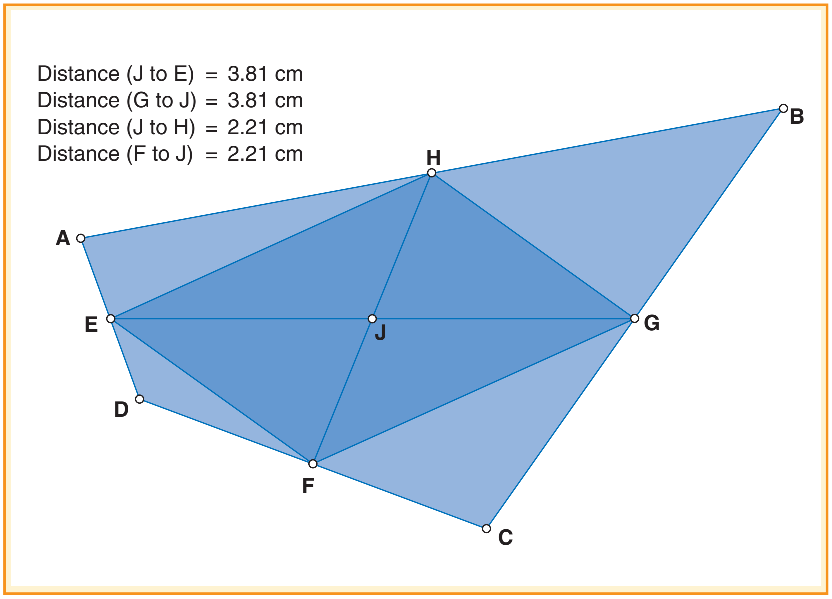
Every type of quadrilateral can be uniquely described in terms of its diagonals using only the conditions of length, proportional comparison of parts, and whether they are perpendicular. This activity can also be explored using dynamic geometry software (e.g., *The Geometer's Sketchpad* or *GeoGebra*), with which objects can be moved and manipulated easily. Lines can be drawn and designated as perpendicular, or a point can be placed as the midpoint of a segment. One of the most significant ideas is that when a geometric object is created with a particular relationship to another, that relationship is maintained no matter how either object is moved or changed. For example, in Figure 13.10, the midpoints of a freely drawn quadrilateral ABCD have been joined. The diagonals of the resulting quadrilateral (EFGH) are also drawn and measured.

Teaching Tip

Vocabulary support can also occur in the *After* phase of the lesson, not just in the *Before* phase. This allows vocabulary to grow out of students' meaningful experiences.

Figure 13.10

A construction made with dynamic geometry software illustrating an interesting property of quadrilaterals.



Standards for Mathematical Practice

5 Use appropriate tools strategically.

No matter how the points A, B, C, and D are dragged around the screen, even inverting the quadrilateral, the other lines will maintain the same relationships (joining midpoints and diagonals), and the measurements will be instantly updated. Without technology, if a quadrilateral is drawn, only one shape is observed, but when students can use dynamic geometry software, the shape can be stretched and altered in endless ways, and students can explore countless examples and find and test hypotheses of their own.

Notice that the "Minimal Defining List" and "Diagonals of Quadrilaterals" activities focus on the relationships among properties within a shape. Students are engaged in the general process of deciding the following question: "If we specify only this list of properties, will that guarantee this particular shape?" These tasks provide a rich opportunity to discuss what constitutes a definition and to develop foundations for informal proofs.

Exploring Polygons

Although more attention is given to triangles and quadrilaterals, middle-school students should also explore various polygons, examining their properties and the conditions that make such shapes unique or possible. In addition, investigating polygons is a good connection to generalizations and algebraic thinking (see Chapter 12 for examples of geometric growing patterns).

Activity 13.9

CCSS-M: 5.G.B.4; 7.G.A.2

True or False?

Prepare statements such as the following: "If it is a __________, then it is also a __________. All are __________. Some are __________." A few examples are suggested here, the first set focuses more on grade 5, and the second set focuses more on grade 7. Many more possibilities exist, and students can generate ones for their peers to test.

SET 1:

- If it is a square, then it is a rhombus.
- All squares are rectangles.
- Some parallelograms are rectangles.
- All parallelograms have congruent diagonals.
- If it has exactly two lines of symmetry, it must be a quadrilateral.
- If it is a cylinder, then it is a prism.
- All pyramids have square bases.

SET 2:

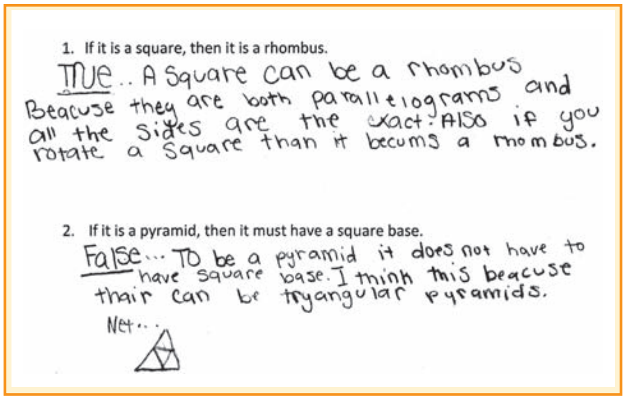
- If it has one obtuse angle, it cannot be a right triangle.
- If it has one acute angle, it cannot be a right triangle.
- If two triangles have all the same angles, they are congruent.
- If two triangles have all sides the same length, they are congruent.
- If two isosceles right triangles have one side the same, they are congruent.
- If a triangle has three acute angles, it is an equilateral triangle.

Select several of these for investigation. Students determine whether the statements are true or false and in the *After* phase of the lesson present an argument to support their decision. Once this format is understood, let students challenge their classmates by making their own statement(s). Extend to three-dimensional shapes. Each list should have a mix of true and false statements. Students' lists can be used in subsequent lessons, with a focus on informal ways to prove whether a statement is true or false. See Figure 13.11 for examples of student justifications.

The next task engages students in exploring patterns and developing informal proof. It is excellent for eighth-graders because it connects to both function and geometry expectations.

Figure 13.11

True or False? A student presents an argument to support her decision.



1. If it is a square, then it is a rhombus.

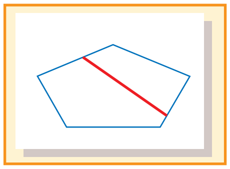
True.. A Square can be a rhombus Beacuse they are both parallelograms and all the Sides are the exact. Also if you rotate a Square than it becums a rhombus.

2. If it is a pyramid, then it must have a square base.

False... To be a pyramid it does not have to have square base. I think this beacuse thair can be tryangular pyramids.

Draw a line through a polygon to form two new polygons. Is there a relationship between the number of sides of the original polygon and the number of sides of the two new polygons?

For example, the accompanying pentagon has been partitioned into a quadrilateral and a pentagon. The number of sides of the original shape was five, and the number of sides of the two new shapes (combined) is nine. Will a pentagon always form two shapes with a sum of nine sides? Is there a relationship between the starting number of sides and the ending number of sides? Does the pattern for pentagons extend to other shapes?



Make a conjecture, and prepare a justification for why you think it is true.

Source: Adapted from Sconyers, J. M. (1995). Proof and the middle school mathematics student. *Mathematics Teaching in the Middle School, 1*(7), 516–518.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Teaching Tip

To avoid rubber band "incidents," require that students place a thumb over the peg where they first place the rubber band. This way, if the rubber band comes off as they stretch it, it will stay on their thumb.

The number of resulting sides depends on where the slice is made (from a vertex or from a side). With the exception of triangles, there are three possibilities. For each case, a pattern emerges across polygons.

Seventh-graders learn to draw geometric shapes with given conditions (CCSSO, 2010). Geoboards are excellent tools for building shapes with various properties or conditions. Rather than draw and erase sketches with a geoboard, you simply move the rubber band.

Activity 13.10

CCSS-M: 6.G.A.3; 7.G.A.2

Can You Build It?

Post "clues" of unique shapes and ask students to build or draw the shapes on the geoboard or **Coordinate Grid**.

Some possible clues include:

A. A shape with just one square corner and four sides

B. A shape with two square corners (or three, four, five, or six square corners)

C. A shape with one line of symmetry (or two lines of symmetry)

D. A shape with two pairs of parallel lines

E. A shape with two pairs of parallel lines and no right angles

If using the coordinate axis, you have more options for the challenges, such as:

F. A square with an area of 16 that is located in quadrants I and IV

And, if using the coordinate axis, the challenges can be distributed so that every student or every pair has one challenge, then they create a list of coordinates that fits that clue, and put their names on the page of clues and return that list to you. You redistribute the student-generated lists of coordinates to other groups. Each group looks at their coordinate lists and decides which clue fits, records the letter of the clue and returns it to the original group to see if they were correct.

Composing and Decomposing Shapes

Students must see shapes as composed of other shapes. This work begins as early as first grade and is an important component in middle school, where students are expected to decompose shapes in order to measure surface area and volume (addressed in Chapter 14). In addition, in grade 7, students compose (draw) shapes in order to study the properties of the shape: "Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle" (CCSSO, 2010, p. 50).

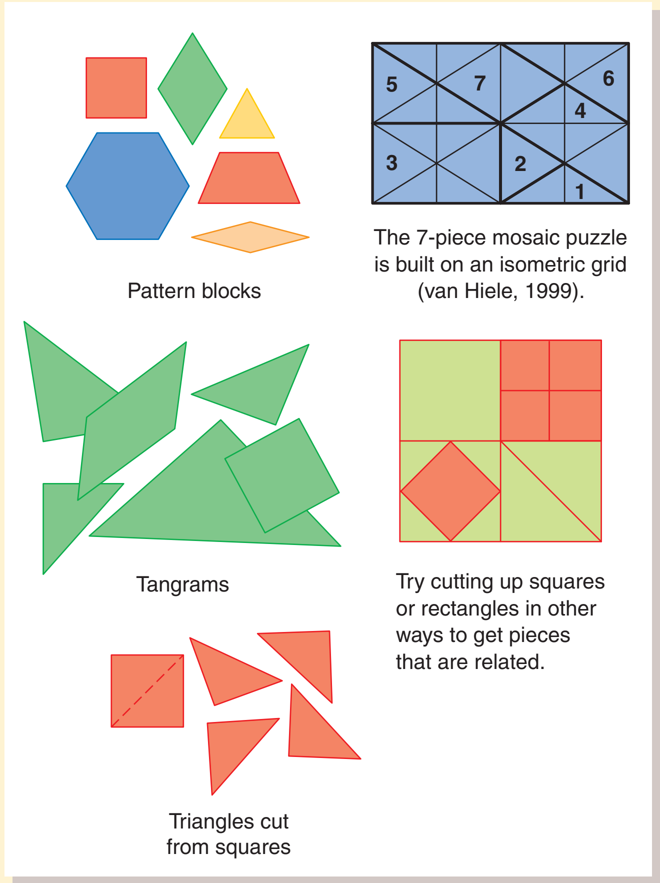
Teaching Tip

Provide free exploration time with these manipulatives before starting a task—it helps students notice how the shapes are related and minimizes off-task behaviors with the pieces during the lesson.

Students need to explore freely how shapes fit together to form larger shapes (compose) and how larger shapes can be taken apart into smaller shapes (decompose). Pattern blocks and **Tangram Pieces** are good tools for composing and decomposing shapes. Less known, but very interesting, is the **Mosaic Puzzle** (van Hiele, 1999), which contains five different angles (lending to discussions of types of angle measures and angle comparisons). Figure 13.12 illustrates the different tools and provides an example of a template that you can make.

Figure 13.12

Tools for exploring composing and decomposing shapes.



Pattern blocks

The 7-piece mosaic puzzle is built on an isometric grid (van Hiele, 1999).

Tangrams

Try cutting up squares or rectangles in other ways to get pieces that are related.

Triangles cut from squares

technology note

Tangram geometry activities and puzzles are readily available on the Internet. For example, “Logicville” posts lessons for geometry and fractions, as well as dozens of tangram puzzles that can be composed with an applet. The National Library of Virtual Manipulatives (NLVM) also offers a tangram applet with a set of 14 puzzle figures. Sharing these websites with families through newsletters can encourage students to solve such puzzles at home.

The geoboard is an excellent tool for composing and decomposing two-dimensional shapes, and middle-school students find it highly engaging. The geoboard, for example, would be effective for exploring the splitting-a-polygon task described previously. There are many possible activities to develop facility with composing and decomposing shapes.

Activity 13.10, "Can You Build It?" is a good example. Geoboard activities can also be done on dot or grid paper. Allow students to choose the tool (geoboard, grid paper, dot paper, or blank paper) that best supports their thinking for the given problem.

technology note

There are excellent electronic versions of the geoboard. One of them, found at the National Library of Virtual Manipulatives (http://nlvm.usu.edu), includes an option to calculate perimeter and area of a shape. The University of Illinois Office for Mathematics, Science, and Technology Education (http://mste.illinois.edu/users/pavel/java/geoboard) has a virtual geoboard with a clickable option that shows the lengths of the sides of the figure.

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Commercial materials, such as Geoshapes and Polydron, permit the creative construction of geometric solids. They include plastic polygons that can be snapped together to make three-dimensional models. With the Zome System, sticks and connectors are used to form three-dimensional skeletal shapes. Students are able to "see" slices of a shape by holding a card at various angles, which demonstrates what two-dimensional shape would be formed if the card fit exactly inside the three-dimensional shape in the designated position. Homemade constructions can be highly engaging for middle-school students. Here are three excellent options for skeletal models:

- *Plastic coffee stirrers with pipe cleaners.* Plastic stirrers can be easily cut into different lengths. To connect the corners, cut the pipe cleaners into 2-inch lengths. These are inserted into the ends of the stirrers.
- *Plastic bendable drinking straws.* With scissors, cut the straws lengthwise from the top down to the flexible joint. The slit ends can then be inserted into the uncut bottom ends of other straws, making a strong but flexible joint. Three or more straws are joined in this fashion to form two-dimensional polygons. To make three-dimensional shapes, use wire twist ties to join polygons side to side.
- *Rolled newspaper rods.* Fantastic large three-dimensional shapes can be built with newspaper and masking tape. Roll three large sheets of newspaper on the diagonal to form a rod. The more tightly the paper is rolled, the less likely the rod is to bend. Secure the roll at the center with a piece of tape. The ends of the rod are thin and flexible for about 6 inches, where there is less paper. Connect rods by bunching and taping (use a lot of tape!) the thin parts together. Additional rods can be joined after two or three are already taped (Figure 13.13).

The newspaper rod method is exciting because the structures quickly become large. Let students work in groups of four or five. They will soon discover what makes a structure rigid and will acquire ideas of balance and form. (They can also create poster-size, two-dimensional slices to insert inside these shapes, as discussed in the "Visualization" section of this chapter.)

Teaching Tip

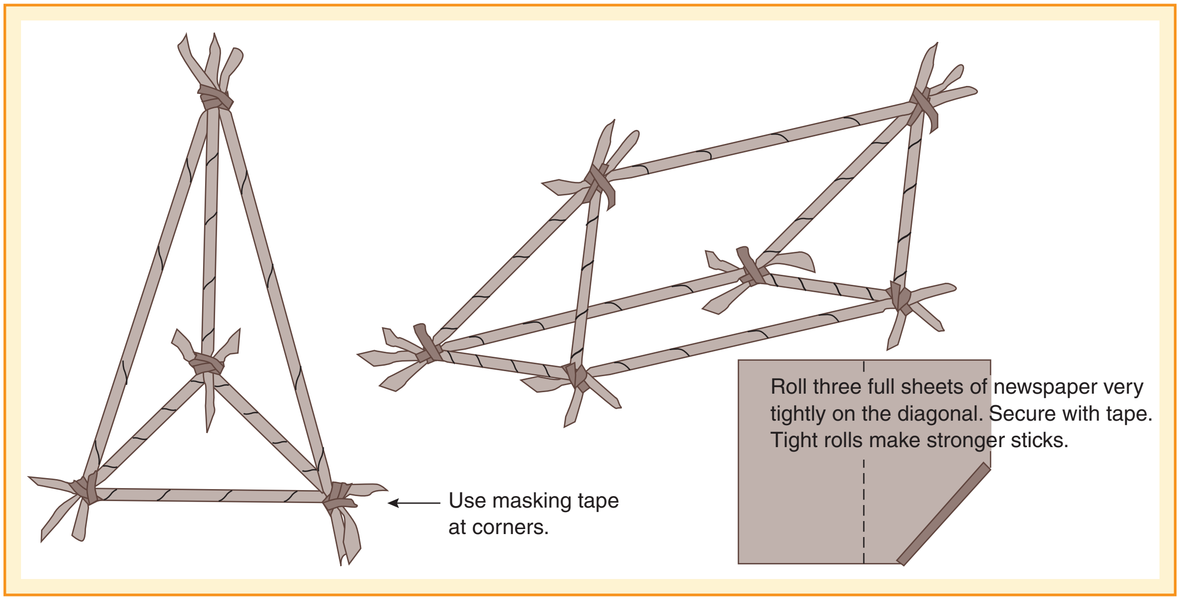
Asking students to create such homemade three-dimensional shapes can be an excellent homework assignment, especially if they have a model created in class to use as a reminder of how to build it.

Teaching Tip

One way to acknowledge the importance of the Pythagorean theorem is through learning about Pythagoras. Read about Pythagoras in *Mathematical Scandals* by Theoni Pappas and other books, or find interesting facts online.

Figure 13.13

Large, skeletal, three-dimensional shapes can be made from newspapers and used to look at the shapes of slices through them.



Pythagorean Theorem

The *Pythagorean theorem*, explored in eighth grade, is one of the most important mathematical relationships and warrants in-depth conceptual investigation. In geometric terms, this relationship states that if a square is constructed on each side of a right triangle, the areas of the two smaller squares will together equal the area of the square on the longest side, the hypotenuse.

Activity 13.11

CCSS-M: 8.G.B.6; 8.G.B.7

The Pythagorean Relationship

Have students draw a right triangle on 0.5-Centimeter Square Grid Paper. Assign each student a different triangle by specifying the lengths of the two legs. Students are to draw a square on each leg and on the hypotenuse and find the areas of all three squares. For the square on the hypotenuse, the exact area can be found by making each of the sides the diagonal of a rectangle (see Figure 13.14). Have students record their own group's data on the Table of the Areas Activity Page and then have them pair with another group to combine their data. Ask students to look for a relationship among the squares of a particular triangle.

The two large, congruent squares in Figure 13.15 together show a proof of the Pythagorean theorem (Nelson, 2001). Note that both squares contain four triangles that are the same but arranged differently. If the areas of the squares and the triangles are added and set equal, the Pythagorean relationship can be found by subtracting out the common areas in both squares. An algebraic recording of the thinking process is shown below the drawings. Instead of using *a*, *b*, and *c*, students can explore the relationship with examples. Pythagorean triples work well for this (e.g., 3-4-5, 6-8-10, or 5-12-13).

Standards for Mathematical Practice

4 Model with mathematics.

The applet at Illuminations entitled, "Proof without Words: Pythagorean Theorem," is a dynamic proof that is worth sharing with your students. And, explore this interesting video (https://www.youtube.com/watch?v=gjSAE8_FahM) titled, "Pythagorean Theorem—Proof without Words," that explores this concept using paper cutouts of triangles and a square.

Figure 13.14

The Pythagorean relationship. The areas of all squares can be calculated when they are drawn on a grid.

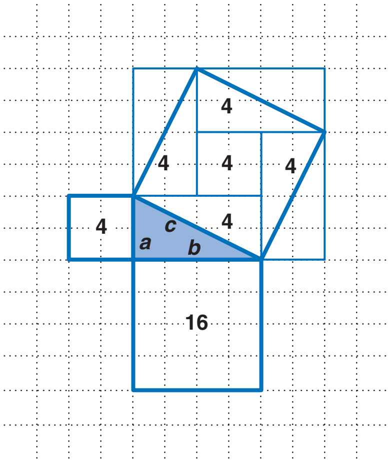
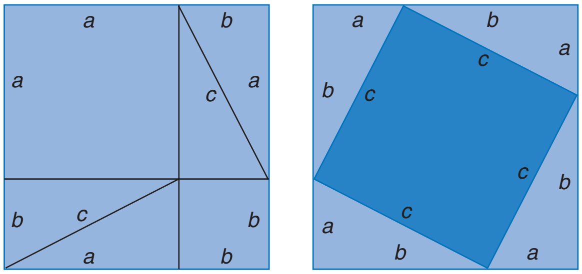


Figure 13.15

The two squares together form a "proof without words." Can you supply the words?



The area of the large square: $(a + b)^2 = a^2 + 2ab + b^2$.

The same area is also c^2 plus 4 times the area of one triangle:

$$c^2 + 4\left(\tfrac{1}{2}ab\right) = c^2 + 2ab$$

$$c^2 + 2ab = a^2 + 2ab + b^2$$

$$c^2 = a^2 + b^2$$

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Eighth-graders should be familiar with Pythagorean triples. Any set of three whole numbers that satisfy the Pythagorean theorem is called a *Pythagorean triple*. Pythagorean triples occur often in geometry tasks and are often "disguised" as multiples of commonly recognized triples.

Standards for Mathematical Practice

7 Look for and make use of structure.

Activity 13.12

CCSS-M: 8.G.B.6; 8.G.B.7

Finding Pythagorean Triples: 3-4-5 in Disguise

Begin with the most common Pythagorean triple, 3-4-5. Ask, "Will triangles that are similar to the 3-4-5 triangle also be Pythagorean triples?" Give students a ruler, grid paper, and a calculator (or have them explore the options with dynamic geometric software). Ask students to find at least three triples that form triangles similar to the 3-4-5 triangle. (Note: There are infinitely many, so once students notice a pattern, stop the exploration and discuss strategies for how to recognize the 3-4-5 in disguise.)

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

Recognizing Pythagorean triples saves the step of having to use the Pythagorean theorem to find the length of a missing side. Pose contextual and mathematical tasks with Pythagorean triples, such as these:

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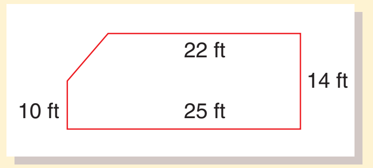
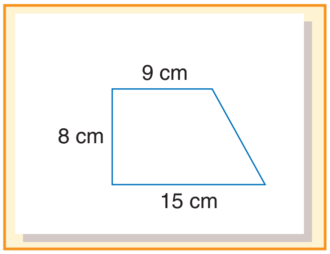
1. Banners are being hung around the classroom to celebrate the upcoming championship game. Each banner is a meter wide. How many banners need to be purchased to go end to end around the perimeter of the room?



2. Find the perimeter of the trapezoid.



These tasks both have a Pythagorean triple relationship that leads to finding the missing side. But these types of tasks should be mixed with tasks that use numbers that are not Pythagorean triples.

Transformations

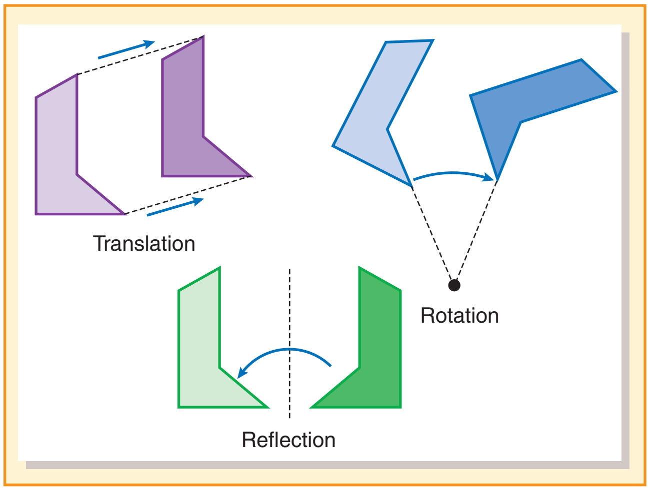
Transformations are changes in the position or size of a shape and are a major focus of eighth grade in the *Common Core State Standards*. Movements that do not change the size or shape of the object moved are called *rigid motions*. *Translations*, *reflections*, and *rotations* are rigid transformations that result in congruent shapes (Figure 13.16). *Dilations* preserve shape, but not size, and therefore result in similar shapes. Students need to know how to recognize and construct each transformation, and particular information is required for each transformation if it is to be performed.

Teaching Tip

Build meaning for the words *translation*, *reflection*, and *rotation* by connecting each one to ideas associated with the word in everyday language. For example, the earth *rotates* around the sun.

Figure 13.16

Translation, reflection, and rotation.



Translation

A translation requires a *direction* and a *distance*. In a translation, every point on the preimage moves in the same direction for the same distance to form the image. In middle school, this may be a move of "up 2 and over 3" on the coordinate axis for each of the points in the figure.

Reflection

A reflection requires a *line of reflection*. A reflection is a transformation in which an object is flipped across a line of reflection. The line of symmetry can be the x-axis or the y-axis, or any other line. If a shape is reflected over the y-axis, for example, the x-values of the preimage are the opposite of the x-values of the image, and the y-values in both images are the same.

Rotation

A rotation requires a *center of rotation (point)* and a *degree of rotation*. The point can be any point on the coordinate axis, though in middle school, rotation around the origin is most common. A figure can be rotated up to 360 degrees.

Beginning work with transformations can involve sketches on paper, without the use of a coordinate axis, as in Activity 13.13.

Activity 13.13

CCSS-M: 8.G.A.1a, b, c

Motion Man

Make copies of **Motion Man** Activity Page and the **Mirror Image Motion Man** Activity Page, back-to-back, so that they match up when held to the light (see Figure 13.17). Give all students a two-sided Motion Man.

Have students work with a partner. Assign one partner to show the preimage and one to show the image. Give students the required criteria for each type of transformation and ask them to show the preimage and image, showing the motion from one to the other (you may also want them to trace each type of transformation on paper). Practice by having everyone start with his or her Motion Man in the same orientation. As you announce one of the moves, students translate, reflect, or rotate Motion Man accordingly.

When you feel that students have an idea of the general meanings of each type of rigid transformation, they are ready to do more specific transformations. For example, for the translation, students need to know distance and direction. Write or say, “Translate Motion Man 4 inches right and 2 inches down.” A reflection requires a line. Write or say, “Draw a vertical or horizontal line and reflect Motion Man over that line.” A rotation requires a center and a degree measure. Write or say, “The center of Motion Man is the center of rotation; rotate him 90 degrees (clockwise).” For all students, ELLs in particular, it is important that these activities include explicit practice with the terms and that visuals are posted for reference.

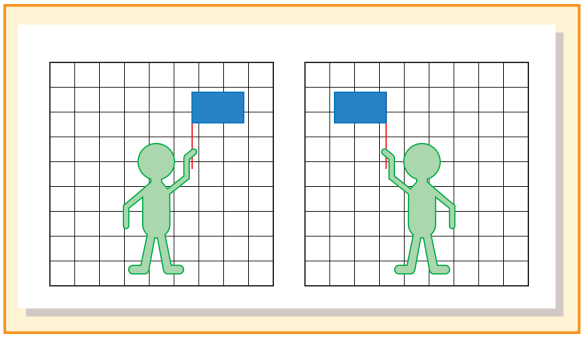
As a follow-up, or formative assessment, display two Motion Men side by side in any orientation. The task is to decide what motion or combination of motions will get the man on the left to match the man on the right. Students use their own man to work out a solution. Test the solutions that students offer.

Stop and Reflect

In the previous activity, instructions were given for one way to do each transformation. What slight shifts in the directions can adapt this task to provide more and different experiences for students to explore translations, reflections, and rotations in meaningful ways?

Figure 13.17

Motion Man can be used to begin an exploration of transformations.



Tessellations

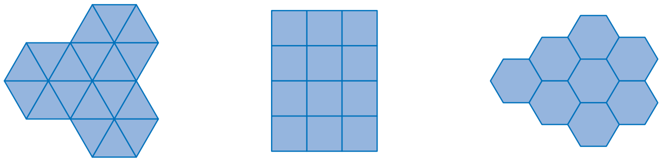
Tessellations are a motivating and artistic application of transformations. A *tessellation* is a tiling of the plane in which one or more shapes appear in a repeating pattern with no gaps or overlaps (Figure 13.18(a)). Tessellations are based on the circle, if the angle measures add up to 360 degrees, the shapes will fit together at a vertex with no overlaps or gaps. A *regular tessellation* is made of a single polygon. Therefore, only certain polygons can be used for regular tessellations.

Which regular polygons can be used to form regular tessellations? A regular triangle (equilateral) has angles of 60 degrees, so six triangles can form a tessellation. Likewise, four squares can form a tessellation, and so can three regular hexagons. A *semi-regular tessellation* is made of two or more different regular polygons. These tessellations are defined by the series of shapes

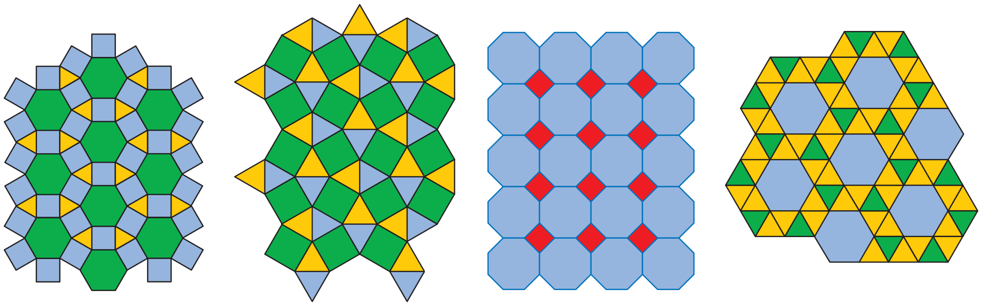
Figure 13.18

Regular and semi-regular tessellations with polygons.

(a) Regular Tessellations

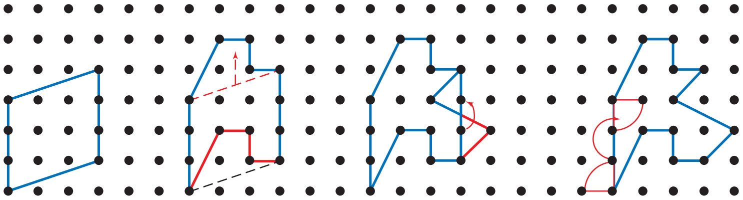


(b) Semi-Regular Tessellations



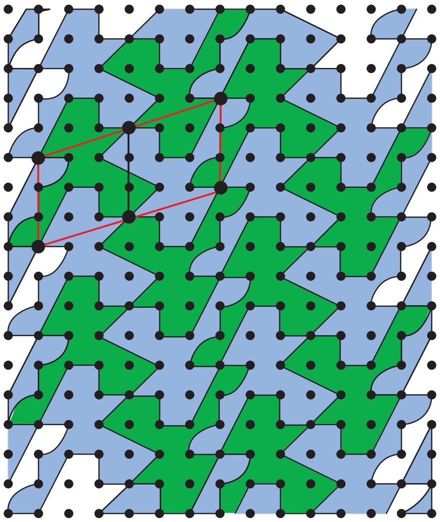
4-3-4-6 3-4-3-3-4 8-8-4 3-3-3-3-6

(c) Altering a Parallelogram to Create an Escher-Like Design



1. Start with a simple shape.
2. Draw the same curve on two opposite sides. This tile will stack up in columns.
3. Rotate a curve on the midpoint of one side.
4. Rotate a curve on the midpoint of the other side. Use this tile for tessellation (below).

A column of this tile will now match a like column that is rotated one complete turn. Find these rotated columns in the tessellation below.



meeting at a vertex. An excellent activity for middle-school students is to explore which polygons can form a semi-regular tessellation and design their own illustration of that tessellation (Figure 13.18(b)). All they need is one set of regular polygons, such as triangles, squares, pentagons, hexagons, and octogons. All polygons must have the same length sides. Students can then trace them on blank paper to make designs. For example, a curve drawn along one side might be *translated* (slid) to the opposite side. Or, **1-Centimeter Dot Paper** can be used to alter a polygon (see Figure 13.18(c)). If the altered line is rotated to an adjacent side, the shape will also tessellate, and the objects will look as if they have turned. Once a tile has been designed, it can be traced over and over again to create beautiful designs.

The Dutch artist M. C. Escher is famous for his tessellations, in which the tiles are very intricate and shaped like birds, horses, or lizards. Escher took a simple shape such as a triangle, parallelogram, or hexagon and altered the sides applying transformations.

technology note

There are also online tools for designing tessellations. Here are three:

- NCTM's Illuminations "Tessellation Creator"
- NRICH (students)—Semi Regular Tessellations
- Shodor inter*activate* "Tesselate!"

Symmetry

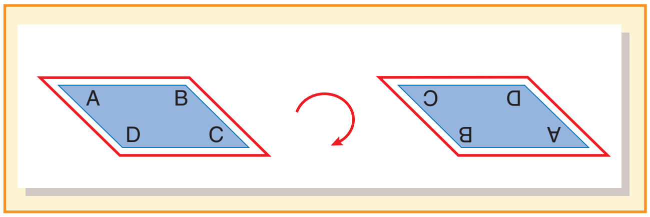
Symmetry is established through transformations. If a shape can be folded on a line so that the two halves match exactly, then it is said to have *line symmetry* (or *mirror symmetry*). Notice that the fold line is actually a *line of reflection*—the portion of the shape on one side of the line is reflected onto the other side.

A shape has *rotational symmetry* (also referred to as *point symmetry*) if it can be *rotated* about a point and land in a position exactly matching the one in which it began. A square has rotational symmetry, as does an equilateral triangle.

A good way to understand rotational symmetry is to take a shape with rotational symmetry, such as a square, and trace around it on a piece of paper. Call this tracing the shape's "footprint." The degrees refer to the smallest angle of rotation required before the shape matches itself or fits into its footprint. The parallelogram in Figure 13.19 has *180-degree rotational symmetry*. A square has *90-degree rotational symmetry*.

Figure 13.19

A parallelogram is rotated 180 degrees.



Composition of Transformations

One transformation can be followed by another. For example, a figure can be reflected over a line and then rotated about a point. A combination of two or more transformations is called a *composition*.

Activity 13.14

CCSS-M: 8.G.A.1a, b, c; 8.G.A.2

Motion Man-Double Move

To introduce composition of transformations, use two copies of the Motion Man Activity Page and a coordinate axis. Place the two Motion Men on a coordinate axis and challenge students to figure out how he got there. At first, students may be confused when they cannot get Motion Man into the new position with one transformation. Encourage them to work together with several Motion Men to see how they might move Motion Man to his new position. Oftentimes, there are numerous ways to get him to the new position, so this is a great opportunity for students to explain their thinking and justify their answers.

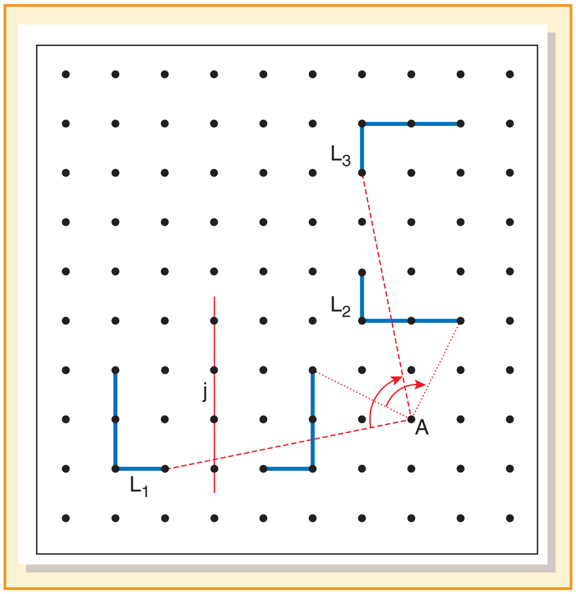
Have students experiment with compositions of transformations by using a simple shape on **1-Centimeter Dot Paper**, as a step toward using coordinates on the coordinate axis. For example, have students draw an L shape on a dot grid and label it L_1 (Figure 13.20). Reflect it over line j, and then rotate the image $\frac{1}{4}$ turn clockwise about point A. Call this image L_2. L_2 is the image of a composition of a reflection followed by a rotation. Notice that if L_1 is rotated $\frac{1}{4}$ turn clockwise about the same point (the result of which we will call L_3), there is a relationship between L_2 and L_3. Continue to explore different combinations of transformations (translations, reflections, and rotations). Also, note that compositions do not have to involve two different types of transformations, a shape could be reflected two different ways. Finally, take advantage of applets that engage students in exploring compositions of transformations, such as the ones on Shodor inter*activate*, NCTM's illuminations, and The Math Forum@NCTM.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Figure 13.20

Transformations on dot paper.



Congruence

Congruent shapes are defined in terms of transformations. Two shapes are considered congruent if you can apply rigid transformations from one shape to the other. The *Common Core State Standards* for eighth grade state that students should "understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them" (CCSSO, 2010, p. 55).

Having explored foundational experiences like Motion Man and transformations on dot paper, students are ready to explore transformations and compositions of transformations on the coordinate axis. A focus on congruence helps connect these two related ideas, as in Activity 13.15.

Activity 13.15

CCSS-M: 8.G.A.1; 8.G.A.2

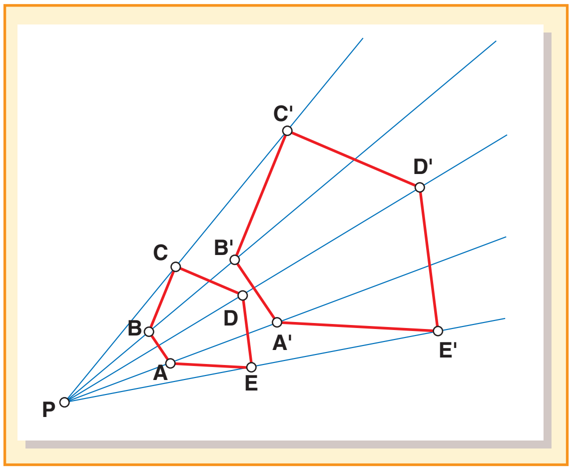
Are They Congruent?

Draw various triangles on a Coordinate Grid, some of which are congruent and some not. Ask students to find a match of two congruent triangles and prove they are congruent by stating the transformations that they applied in order to get one shape to cover exactly the one they selected as a match.

The coordinate axis is addressed in detail in a later section, "Location," in which transformations will continue to be explored.

Figure 13.21

Begin with figure ABCDE and place point P anywhere. Draw lines from P through each vertex. Place point A′ twice as far from P as A is from P (scale factor of 2). Do similarly for the other points. ABCDE is similar to A′B′C′D′E′.



Similarity

Two figures are *similar* if all of their corresponding angles are congruent and the corresponding sides are proportional. As noted in Chapter 11, proportional reasoning activities are good connections to geometry, such as Activity 11.6, "Look-Alike Rectangles," and Activity 11.7, "Scale Drawings."

A *dilation* is a nonrigid (can change size) transformation that produces similar two-dimensional figures. Figure 13.21 shows how a given figure can be dilated to make larger or smaller figures. In order to dilate a preimage, you must know the *scale factor*. A dilation can make the image smaller or larger, depending on whether the scale factor is less than or greater than 1.

If different groups of students use the same scale factor to dilate the same figure, they will find that the resulting figures are all congruent, even with each group using different dilation points. Dynamic geometry software makes the results of this exercise quite dramatic. The software allows the scale factors to be set at any value. Once a dilation is made, the dilation point can be dragged around the screen, and the size and shape of the image clearly remain unchanged.

Location

Location activities begin early in school, when young children describe objects as *under*, *near*, *far*, *between*, *left*, and *right*. The coordinate axis is introduced in grade 5 with a focus on quadrant I and is extended to all four quadrants in grade 6, when students connect and apply their understanding of rational numbers and draw polygons on the coordinate axis (CCSSO, 2010). This knowledge is needed for scale drawings and constructions in grade 7 (described previously) and is used extensively in grade 8 for graphing lines on the coordinate axis and performing transformations. Finally, the coordinate plane is used to explore the distance (length) of vertical and horizontal lines in grade 6 and of any lines (by using the Pythagorean theorem) in grade 8. It is clear that understanding and being able to use the coordinate plane is critical in middle-school mathematics.

These next three quick activities are designed simply to get students familiar with plotting points on a **coordinate axis**. Any or all of these can be used in sixth grade for full-class instruction, or as intervention or review for students who will need to be accurate in graphing on the coordinate plane.

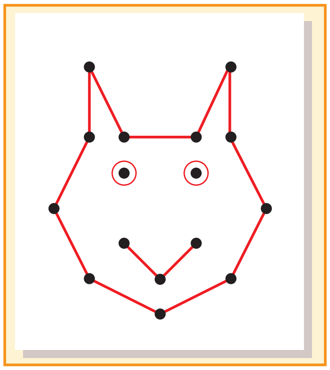
Activity 13.16

CCSS-M: 5.G.A.2; 6.G.A.3

Dot to Dot

Ask students to create a dot-to-dot picture with fewer than 20 coordinates.

Each student (or each pair) prepare(s) two products—one is the labeled graph, and the second is the list of coordinates (separated by which ones are to be connected to each other). Collect the second item (lists of coordinates), mix them up, and redistribute the lists to other students. Students create the graph of the new dot-to-dot picture. As a final step, they get the labeled graph from the original group to see if they completed the diagram accurately. Differences are studied to see where an error occurred.



Activity 13.17

CCSS-M: 5.G.A.2

Four in a Row

Adapt this classic game by requiring students to name the coordinates of the position they wish to mark with their color. If they name the coordinates backward or forget the negative value, then mark the point they named, not the point they meant to name. This can be done with the whole class (to start) or with partners.

Activity 13.18

CCSS-M: 5.G.A.2

Simon Says

Give students starting coordinates—for example, begin at ($^{-}3$, 4). Give a series of instructions, such as "Simon says move 3 units to the right. Simon says move 2 units up. Move 5 units down. Where are you?" Have students compare their final locations and figure out what might have caused different results. (Remember, if Simon does not say it—don't move!)

Transformations on the Coordinate Plane

Transformations were the focus of the previous section, in which the experiences prepared students to explore transformation on the coordinate plane. Experiences on the coordinate plane can begin with a single type of transformation, as in Activity 13.19, in which the focus is on translations.

Activity 13.19

CCSS-M: 6.G.A.3; 8.G.A.1; 8.G.A.3

Coordinate Slides

Ask students to plot and connect five or six points on **1-Centimeter Grid Paper** to form a small shape (Figure 13.22). You may want to begin with all coordinates in quadrant I, with *x* values and *y* values less than 12, so there is room on the graph to draw the translated shapes. Next, ask students to add 6 to each of the *x*-values of their shape, leaving the *y*-values the same. This new figure should be congruent to the original and translated to the right. Then, ask students to create a third figure by adding 9 to each *y*-value of the original coordinates.

Ask students to conjecture and test what could be done to the coordinates to move the figure along a diagonal line up and to the right. Figure 13.22 shows a translation created by adding 6 to the *x*-values and 9 to the *y*-values, thus translating the figure and maintaining congruence.

After this first experience with positive values, explore shapes that are positioned in the other quadrants. Explore moves that go down and to the left, as well. Also include fractional values. As they explore, ask:

> "What does adding (or subtracting) a number from the first coordinates cause?
>
> What if the number is added or subtracted from the second coordinates? From both coordinates?"

Finally, have students draw lines connecting corresponding points in the original figure with one of those where both coordinates were changed. What do they notice? (The lines are parallel and the same length.)

In “Coordinate Slides,” the figure did not twist, turn, flip over, or change size or shape. The shape “slid” along a path that matched the lines between the corresponding points. Reflections and rotations can also be explored on a coordinate grid. For reflections, begin with using the *x*- or *y*-axis as the line of reflection, as in the following activity.

Activity 13.20

CCSS-M: 8.G.A.1; 8.G.A.3

Coordinate Reflections

Ask students to draw a five-sided shape in the first quadrant of a **Coordinate Grid**. Label the figure ABCDE and call it Figure 1 (Figure 13.23). Use the *y*-axis as a line of symmetry, and draw the reflection of the shape in quadrant II. Label the reflected points A′B′C′D′E′ and call it Figure 2. Now, use the *x*-axis as the line of reflection and create Figure 3 (in quadrant III) and Figure 4 (in quadrant IV). Label the points of these figures with double and triple primes A″ and A‴, and so on. Write in the coordinates for each vertex of all four shapes. Explore the following:

- How is Figure 3 related to Figure 4? How else could you have gotten Figure 3? How else could you have found Figure 4?
- How are the coordinates of Figure 1 related to its image in the *y*-axis, Figure 2? What can you say about the coordinates of Figure 4?
- Make a conjecture about the coordinates of a shape reflected over the *y*-axis and a different conjecture about the coordinates of a shape reflected over the *x*-axis.
- Draw lines from the vertices of Figure 1 to the corresponding vertices of Figure 2. What can you say about these lines? How is the *y*-axis related to each of these lines?

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Figure 13.22

Example translations on a coordinate axis.

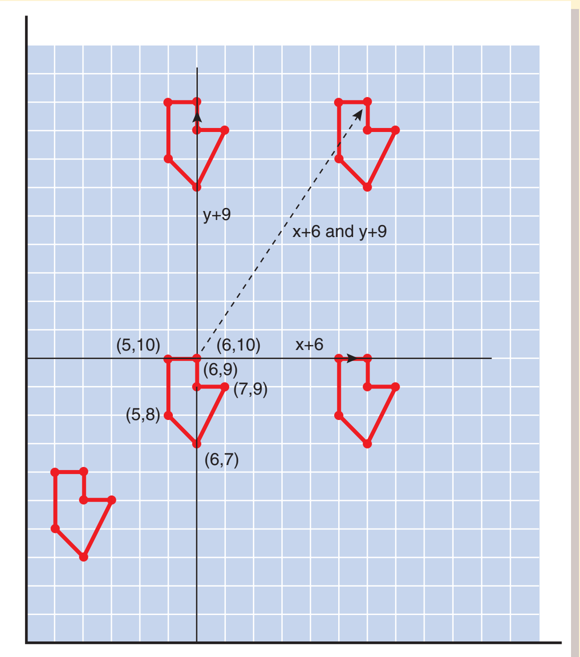
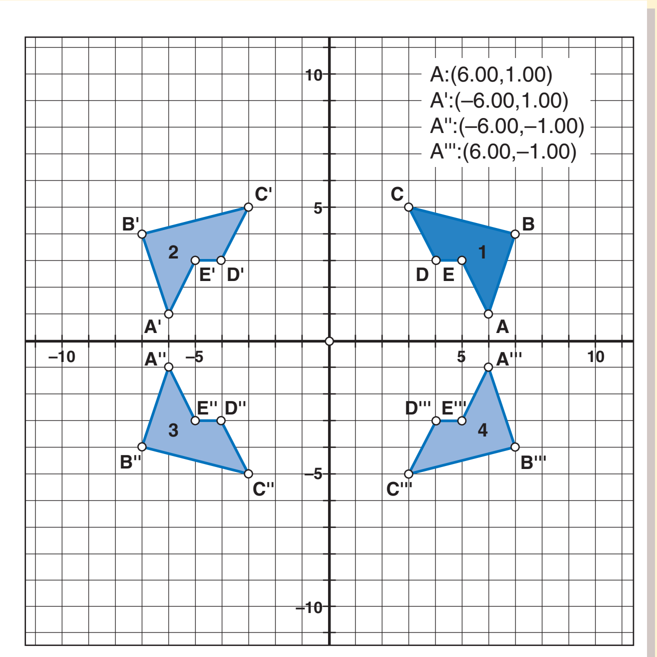


Figure 13.23

Figure 1 (ABCDE in quadrant I) is reflected over the *y*-axis. Then both figures are reflected over the *x*-axis.



Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Students need opportunities to determine which transformations have occurred to get from a preimage to an image. The next activity challenges students to use pentominoes (shapes made from five squares, each square touching at least one other square by sharing a full side (see Figure 13.24) to explore transformations on the coordinate plane. If students are not familiar with pentominoes, an engaging opening activity is to have students see how many different pentominoes they can find.

Activity 13.21

CCSS-M: 8.G.A.1

Pentomino Rotations and Reflections

Have students cut out a set of 12 **Pentominoes** and distribute **2-Centimeter Grid Paper** (Blackline Master 5). Mark one side of each pentomino piece to help remember whether it has been flipped over. Ask students, "How many different positions could each pentomino have on a coordinate plane using vertical and horizontal reflections and rotations of 90°, 180°, and 270°?" A position is considered "different" if the new position could not have been created using a translation from the original position (in other words, a reflection or a rotation is required to get from the preimage to the image). Also, explain that a new position needs to have vertices on whole number coordinates (otherwise, too many possibilities exist). To clarify the task, you may want to provide these two examples:

- The cross-shaped piece has no new positions because when it is reflected vertically or horizontally, or rotated 90°, 180°, or 270°, the shape is in the same position as it started (so it could have been created with a translation).
- The strip of five squares has one new position (if it starts vertical, it can be rotated 90° to be horizontal).

Ask students to explore one of the other 10 pentominoes, determining what new positions are possible and recording what type of transformation led to any new position. Additionally, you can ask them to record the coordinates in the preimage and the image. Some pentomino pieces have more than three new positions—challenge students to seek as many positions as possible for the shape they selected. Students can continue with other pentominoes, or they can trade work with groups who explored other pentominoes to critique whether the group's list of new positions were described correctly and if any additional ones could be added.

These activities have explored transformations that are rigid motions. But to understand what is required to have a rigid motion means understanding what types of transformations distort a shape. In the next activity, students see that multiplying a constant times the coordinates is a transformation that is *not* a rigid motion.

Activity 13.22

CCSS-M: 6.G.A.3; 7.G.A.1; 8.G.A.1; 8.G.A.3

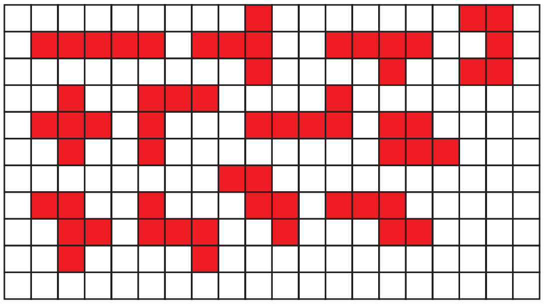
Polygon Dilations

Distribute a **Coordinate Grid** and ask students to create a four-sided shape in the first quadrant. Next, ask them to record the coordinates for their quadrilateral. With their list, they are to create a new set of coordinates by multiplying each of the original coordinates by 2. Ask, "What do you think the new polygon will look like? How will its perimeter change? Its area?" Now, have students multiply each of the original coordinates by $\frac{1}{2}$ and plot that shape. Ask the same questions and have students record their predictions. Have students plot the new shapes. Discuss how their predictions compared to what they see.

Finally, ask students to draw a line from the origin to a vertex of the largest shape. Repeat for one or two additional vertices, and ask for observations. (An example is shown in Figure 13.25.)

Figure 13.24

There are 12 different pentomino shapes. A distinct piece cannot become one of the other pentominoes through a transformation.

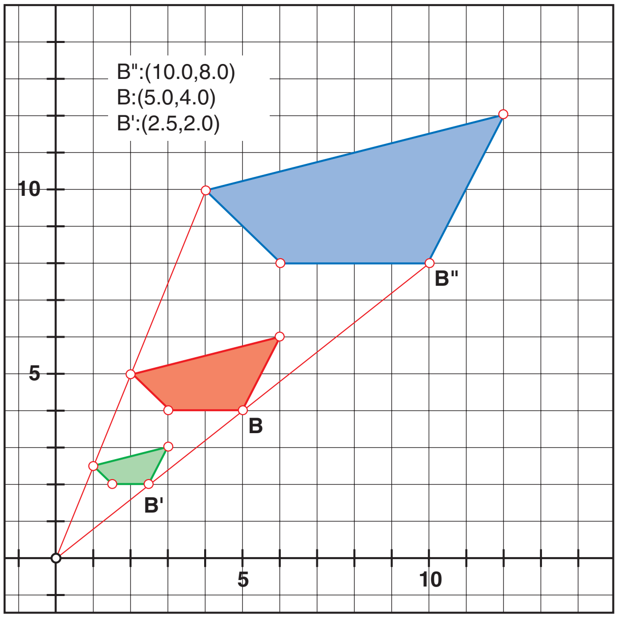


There are 12 pentominoes.

Finding all possible shapes made with five squares—or six squares (called “hexominoes”) or six equilateral triangles and so on—is a good exercise in spatial problem solving.

Figure 13.25

Dilation of a trapezoid (original is dark gray) when the scale factors are 0.5 and 2.0.



Stop and Reflect

How do the lengths of sides and the areas of the shapes compare when the coordinates are multiplied by 2? What if they are multiplied by 3 or by $\frac{1}{2}$?

Teaching Tip

Invite students to bring a picture of a favorite TV, book, video, or comic strip character. They can use coordinate points to create a modified version of the picture and dilate it to form a larger or smaller character.

Your students may enjoy exploring dilations a bit further, including the connection to scale drawings. Any diagram (e.g., a sailboat) can undergo a dilation, resulting in a larger or smaller diagram of the same shape.

Distortions are engaging because of the interesting ways they change the shapes, and they communicate the impact of the scale of a coordinate on the outcome of the transformation. If students start with a drawing of a simple face, boat, or some other shape drawn with straight lines connecting vertices, they will create an interesting effect by multiplying just the first coordinates, just the second coordinates, or using a different factor for each. When only the second coordinate is multiplied, the vertical dimensions alone are dilated, so the figure is proportionately stretched (or shrunk) vertically, as illustrated in Figure 13.26. Students can explore this process to distort shapes in various ways. Imagine being able to control transformations, not just in the plane but also for three-dimensional figures. The process is identical to computer animation techniques.

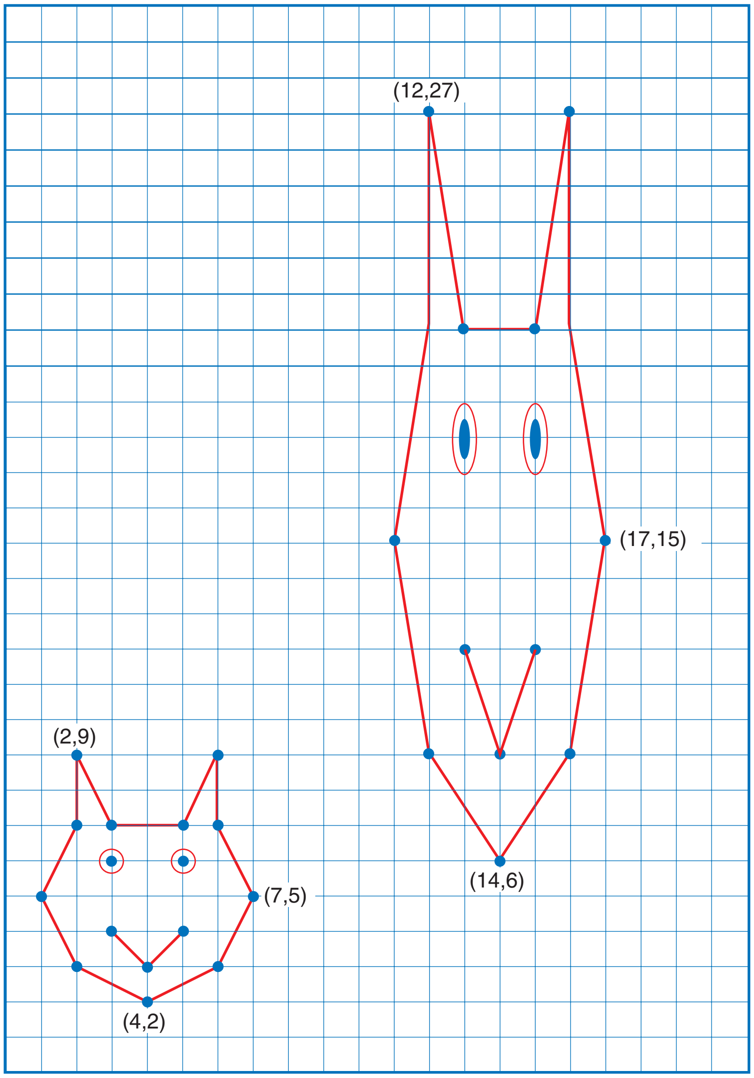
While exploring transformations, challenge students with questions such as the following, which deepen their understanding of transformations:

- How should the coordinates be changed to cause a reflection if the line of reflection is not the *y*-axis but is parallel to it?

- Can you discover a single rule for coordinates that would cause a reflection across one of the axes followed by a rotation of a quarter turn? Is that rule the same for the reverse order—a quarter turn followed by a reflection?
- If two successive translations are made with coordinates and you know what numbers were added or subtracted, what number should be added or subtracted to get the figure there in only one move?
- What do you think will happen if different factors are used for different coordinates in a dilation?

Figure 13.26

A distortion that is dilated differently for x- and y-values. This distortion was (x + 10, 3y).



Measuring Distance on the Coordinate Plane

Measuring in the coordinate plane begins in grade 6 with students measuring vertical and horizontal lines and moves in grade 8 to student exploration of distance—eventually leading to the distance formula.

How far apart are these points?

$(^-3, 5)$ and $(4, 5)$ $(10, ^-5)$ and $(10, 25)$

$(^-25, ^-10)$ and $(^-25, ^-35)$ $(-4\frac{3}{4}, -3)$ and $(1\frac{1}{4}, -3)$

Students can explore a variety of vertical lines by drawing them and conjecturing on how to find the distance without drawing it and counting units. Then they can see whether these conjectures also hold for horizontal lines.

Activity 13.23

CCSS-M: 6.G.A.4

Is It a Square Deal?

Distribute Coordinate Grid paper or use graphing software. Give students a set of four points and ask them to determine if the points make a square or not. For example, give the following points:

$(-2, 3), (3, 3), (3, -2), (-2, -2)$

$(-3, -5), (-3, 14), (-12, -5), (-12, 14)$

$(-3, 5), (-3, 14), (-12, 14), (-12, 5)$

$\left(\frac{1}{2}, \frac{5}{2}\right), \left(\frac{1}{2}, \frac{-1}{2}\right), \left(\frac{7}{2}, \frac{-1}{2}\right), \left(\frac{7}{2}, \frac{5}{2}\right)$

If it is a square deal, ask students to find the perimeter and area. If it is not a square deal, then explain why it is not a square.

After students have explored this set, ask them to make their own example and nonexample. You can use their examples to trade with other groups and see if they were given a 'square deal' or not. In discussion, ask students to look across the examples and explain how you can decide if the coordinates make a square without graphing to figure it out.

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

If you can find a vertical distance and a horizontal distance, you can make a triangle and use the Pythagorean theorem to find any length. The following activity uses the coordinate grid and the Pythagorean relationship to develop a formula for the distance between two points.

Activity 13.24

CCSS-M: 8.G.B.8

Developing the Distance Formula

Begin by asking students to draw a line between two points in the first quadrant that are not on the same horizontal or vertical line (on Coordinate Grid paper or using dynamic geometry software). Ask, "What is the length of this line?" Give time for students to give ideas of an estimate as well as how they might find the actual length. Next, ask students to draw a right triangle by using the line as the hypotenuse (the vertex at the right angle will share one coordinate with each end point). Tell students to apply the Pythagorean theorem to find the distance. Ask students to do two to four more examples and look for patterns across their examples. Next, have them look through all of their calculations and see how the coordinates of the two end points were used. Challenge students to use the same type of calculations to get the distance between two new points without drawing any pictures.

Standards for Mathematical Practice

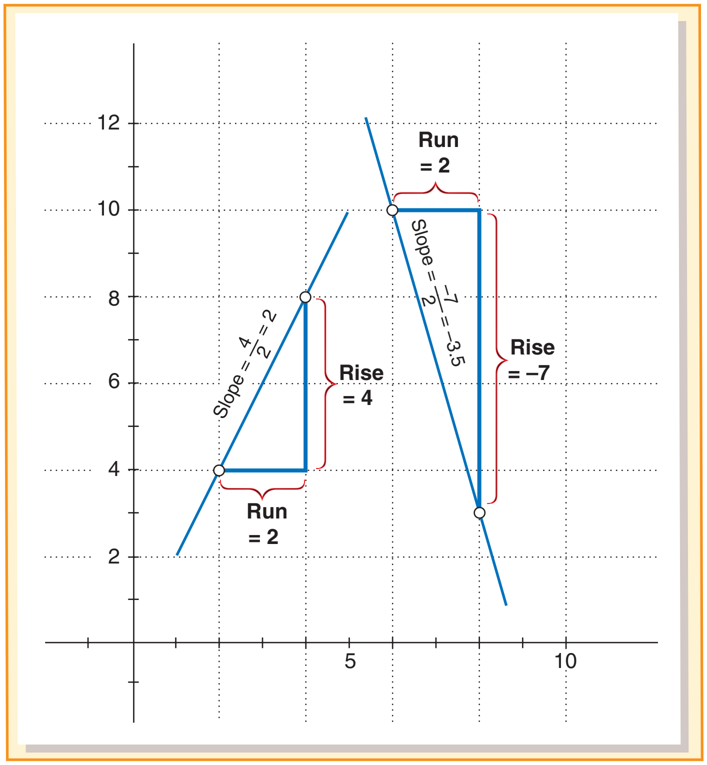
8 Look for and express regularity in repeated reasoning.

Eighth-graders do not need to construct proofs independently but should be able to follow the rationale if shown proofs. By using the Pythagorean theorem to find the length of one line (or the distance between the end points), you provide students with the opportunity to make an important connection between two big mathematical ideas.

Notice that the distance formula is closely connected to slope (discussed in Chapter 12). Measuring slope requires a reference line (the x-axis). The *rise* is the vertical change from the left point to the right point—positive if up, negative if down. The *run* is the horizontal distance from the left point to the right point. Slope is then defined as the ratio of rise to run, or the ratio of the vertical change to the horizontal change. Figure 13.27 illustrates slope and the close connection to the distance formula.

Figure 13.27

Measuring distance on the coordinate plane is closely connected to slope.



Visualization

Visualization might be called "geometry done with the mind's eye." It involves being able to create images of shapes and then turn them around mentally, thinking about how they look from different viewpoints—predicting the results of various transformations. It includes the mental coordination of two and three dimensions—in sixth grade, for example, by determining the net for a three-dimensional shape. Any activity that requires students to think about, manipulate, or transform a shape mentally or to represent a shape as it is seen visually will contribute to the development of their visualization skills.

Nets

A flat shape that can be folded up to make a solid figure is called the *net* of that solid. The following activity suggests several challenges involving nets. Sixth-graders construct nets and describe the faces of three-dimensional shapes made up of rectangles and triangles, and they use the nets to calculate the surface area.

Activity 13.25

CCSS-M: 6.G.A.4

Net Challenges

The following tasks can be done as a series of activities, or any one of them can be selected to explore nets.

- Examine a set of pentominoes and determine which ones are "box makers." Test conjectures by cutting out the pentominoes and trying to use them to make a box. For each that is a box maker, see in how many different places a sixth square can be attached to create a top for the box (a net for a cube). Are there other nets for a cube that do not begin with a pentomino?
- Begin with a solid, such as a rectangular prism or square pyramid. Sketch as many nets as possible for this shape. Add to the collection some arrangements of the sides of the solid that are not nets. Challenge a friend to decide which are nets of the shape and which are not.
- Use a Polydron or a three-dimensional Geoshape to create a flat figure that you think will fold up into a solid. Test the result. If the number and/or type of flat shapes is specified, the task can be made more or less difficult. Can you make the net of a solid with 12 regular pentagons or 8 equilateral triangles? (These can be made into a *dodecahedron* and an *octahedron*, respectively, two of the five completely regular polyhedra, also known as the five *Platonic solids*.)

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

technology note

Students can use "Dynamic Paper" from NCTM Illuminations to create nets of three-dimensional figures with specified dimensions. This multipurpose tool creates custom graph paper, number grids, nets, number lines, shapes, spinners, and tessellations that can be exported in .jpeg and .pdf formats.

Perspective Drawings

One of the main goals of visualization in the geometry strand of the *Common Core State Standards* is to be able to identify and draw two-dimensional images of three-dimensional figures and to build three-dimensional figures from two-dimensional images. Activities aimed at this goal often involve drawings of small "buildings" made of 1-inch cubes.

Activity 13.26

CCSS-M: 6.G.A.4; 7.G.A.1

Building Views

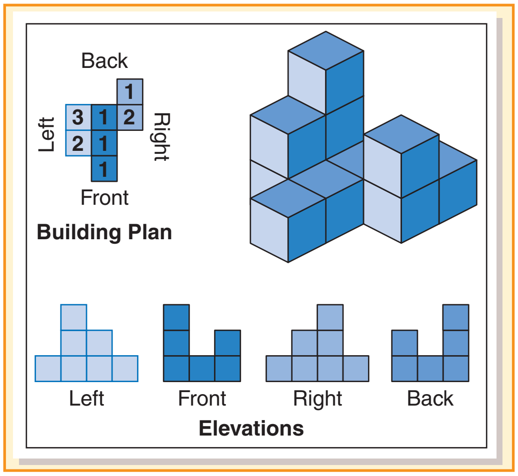
For this activity, students will need **1-Centimeter Grid Paper** for drawing a building plan and 1-inch blocks for constructing a building.

Version 1: Students begin with a building made of the blocks and draw the left, right, front, and back views (these are called *elevations*). In Figure 13.28, the building plan shows a top view of the building and the number of blocks in each position. After students build a building from a plan like this, they draw the elevations (views) of the front, right, left, and back, as shown in the figure.

Version 2: Students are given right and front elevations. Ask students to build the corresponding building. To record their solution, they draw a building plan (top elevation with numbers).

Figure 13.28

“Building Views” tasks. Students are given one representation and build another.



Notice that the front and back elevations are symmetric, as are the left and right elevations. That is why only one of each is given in the second part of the activity. To expand “Building Views” into a more challenging activity, students can prepare three-dimensional drawings (isometric) of the block buildings or match three-dimensional drawings with buildings using **2-Centimeter Isometric Grid Paper** (Blackline Master 9) or **1-Centimeter Isometric Dot Paper** (Blackline Master 10). Isometric grids are a form of *axonometric* drawing in which the scale is preserved in all dimensions (height, depth, width). The next activity provides a glimpse of this form of visualization activity.

Activity 13.27

CCSS-M:
6.G.A.4; 7.G.A.1

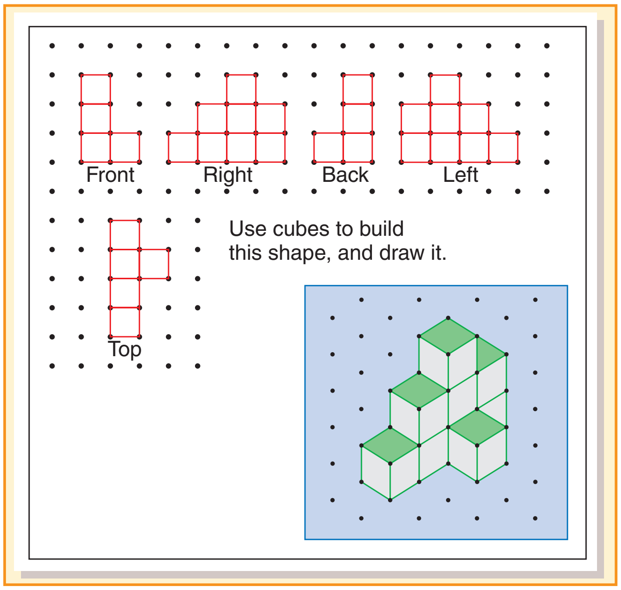
Three-Dimensional Drawings

Version 1: Students begin with being given an isometric three-dimensional drawing on **2-Centimeter Isometric Grid Paper** or **1-Centimeter Isometric Dot Paper**. The assumption is that there are no hidden blocks. From the drawing, the students build the actual building with their blocks. To record the result, they draw a building plan (top view) indicating the number of blocks in each position.

Version 2: Students are given the four elevation views and a building plan (top view) (Figure 13.29). They build the building accordingly and draw two or more of the elevation views. There are four possible views: the front left and right and the back left and right. For students who struggle, have them build the building on a sheet of paper with the words *front*, *back*, *left*, and *right* written on the edges to keep them from confusing the different views.

Figure 13.29

Develop visual perception with elevations and building plan views.



technology note

An amazing computer tool for drawing two- and three-dimensional views of block buildings is the NCTM Illuminations "Isometric Drawing Tool." This applet uses mouse clicks to draw either whole cubes, any single face of a cube, or just lines. The drawings, however, are actually "buildings" and can be viewed as three-dimensional objects. They can be rotated in space to be seen from any vantage point. Watch this video clip (http://www.youtube.com/watch?v=CtmXu2yt5SE), which shows components of two lessons on imagery in a seventh-grade classroom. One explores the building activities just described and the lessons end with slicing solids, which is the focus of the next section.

Two-Dimensional Slices of Three-Dimensional Shapes

Another connection between two- and three-dimensional shapes is found by slicing solids in different ways. This is a standard for grade 7 (CCSSO, 2010). When a solid is sliced into two parts, a two-dimensional figure is formed on the slice faces. Slices of solids made of clay sliced with a potter's wire can be explored.

Activity 13.28

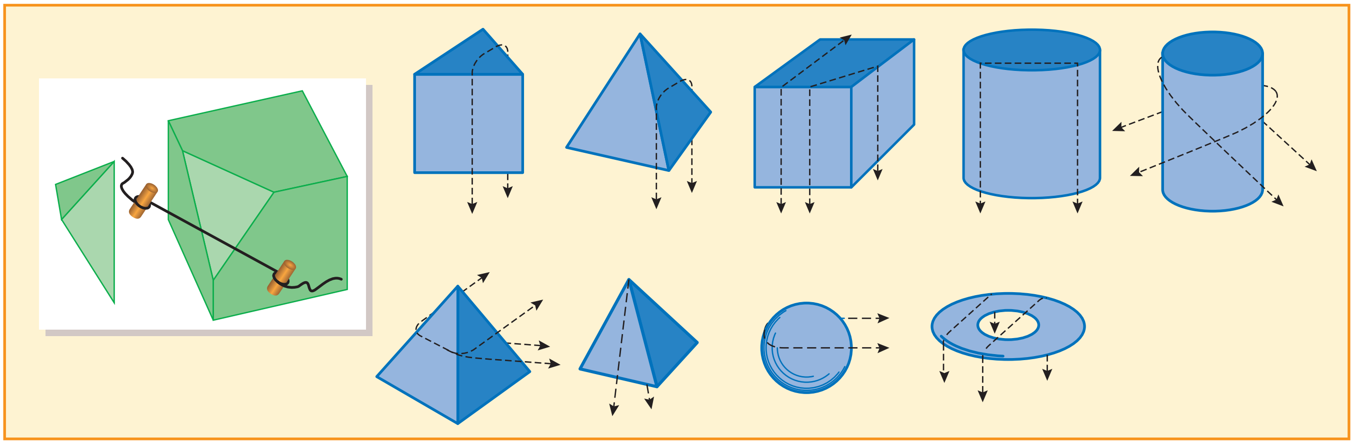
CCSS-M: 7.G.A.3

Slicing 3-D Shapes

Distribute something moldable to students, such as dough or clay, and something to slice with (string or plastic knife) (see Figure 13.30). Ask students to make a rectangular solid as best as they can. Ask students what shape the remaining surface will be if they 'cut off' a corner. After they make their predictions, have them slice off a corner. Challenge students to find other 2-D shapes by slicing in additional ways. Then, ask students to make the other 3-D shapes, such as the ones in Figure 13.30, first predicting the shapes that will be made for each cut, and then actually slicing the shapes. For ELLs, provide labeled illustrations of each 3-D shape, as well as for the names of the 2-D shapes they may be able to create through their slicing.

Figure 13.30

Predict the shape of the slice face, then cut with a thin, sturdy string or wire.



Another engaging method (beyond dough or clay) is to fill a plastic solid (such as one of the Power Solids) partially with water. The surface of the water simulates a slice and models the face of the solid as if it had been cut at that location. By tilting the shape in different ways, every possible "slice" can be observed. Ask students to see if they can find a particular plane shape by slicing the three-dimensional shape they have. For example, ask, "Can you slice the rectangular solid to have a trapezoid face? A triangle face? A square face?"

These explorations of two- and three-dimensional shapes illustrate not only real-life connections (between two-dimensional representations and our three-dimensional world), but also the importance of exploration, conjecture, and proof in learning about shapes.

Literature Connections

Cubes, Cones, Cylinders and Spheres *Hoban, 2000*

This wordless book contains dramatic photographs of three-dimensional geometric shapes in the environment. Using digital cameras to create Hoban-like books invites students to seek and identify three-dimensional shapes in the world around them. For example, students can use their phones or classroom iPads to photograph their own three-dimensional shapes and then create nets that represent those shapes. See also Tana Hoban wordless books with two-dimensional shapes (e.g., Shapes, Shapes, Shapes) for exploring polygons.

The Greedy Triangle *Burns, 1995*

This delightful book is the story of a triangle that is very busy being a sail or a roof or fitting into the crook of the arm of someone standing with hand on hip. Soon he becomes bored and travels to the local shape-shifter with a request for one more side and one more angle. Now a quadrilateral, there are new things to try as he fits into different four-sided figures in the environment. This story can be used to launch into lessons about areas of triangles and quadrilaterals (grade 6), scale drawings of geometric figures (grade 7), or to explore congruence and similarity through transformations (grade 8).

14

Exploring Measurement Concepts

BIG IDEAS

1 Measurement involves a comparison of an attribute of an item or situation with a unit that has the same attribute. Lengths are compared with units of length, areas with units of area, time with units of time, and so on.

2 Estimation of measures and development of benchmarks for frequently used units of measure help students increase their familiarity with units, preventing errors and aiding in the meaningful use of measurement.

3 Area and volume formulas provide a method of measuring these attributes with only measures of length.

4 Area, perimeter, surface area, and volume are related. For example, as the shapes of regions or of three-dimensional objects change while their areas or volumes are maintained, their perimeters and surface areas are affected.

Measurement is the process of describing a continuous quantity with a numerical value. It is one of the most useful mathematics content strands. From gigabytes that measure amounts of information, to font size on computers, to miles per gallon, to recipes for a meal, people are surrounded daily with measurement concepts that apply to a variety of real-world contexts and applications. The focus of measurement in middle school [or CCSS-M] is on area, surface area, and volume:

> *Grade 6:* Investigate formulas for the area of triangles and quadrilaterals, use nets to explore the surface area of three-dimensional shapes, and find the volume of right rectangular prisms.
>
> *Grade 7:* Discover and apply circumference and area of circles; measure angles, including solving problems involving

supplementary, complementary, vertical, and adjacent angles; and solve problems involving area, surface area, and volume.

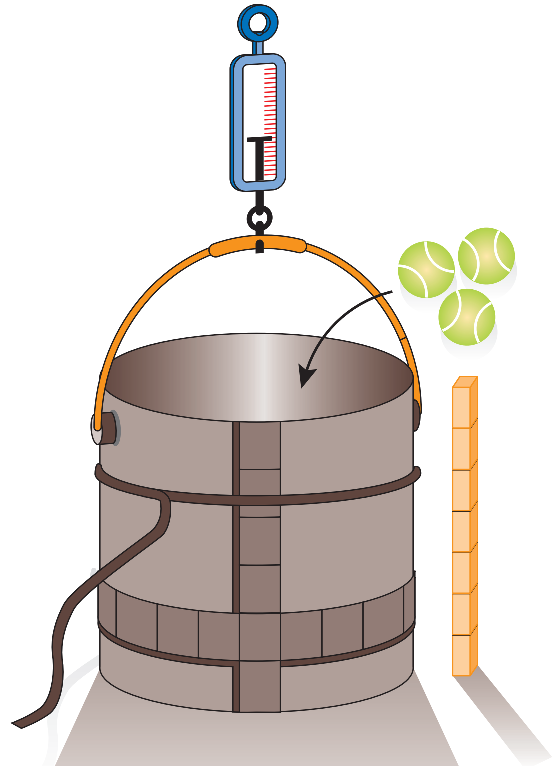
Grade 8: Extend their knowledge of volume of three-dimensional shapes, with a focus on cylinders, cones, and spheres.

Foundations of Measuring

Suppose that you asked your students to measure an empty bucket, as in Figure 14.1. The first thing they would need to know is *what* about the bucket is to be measured. They might measure the height (depth), diameter (distance across), or circumference (distance around). All of these are length measures. The surface area of the side could be determined. A bucket also has volume (or capacity) and weight. Each aspect that can be measured is an *attribute* of the bucket.

Figure 14.1

Measuring different attributes of a bucket.



Attribute: Weight
Units: objects that stretch the spring in the scale
How many units will pull the spring as far as the bucket will?

Attribute: Volume/Capacity
Units: cubes, balls, cups of water
How many units will fill the bucket?

Attribute: Length
Units: rods, toothpicks, straws, string
How many units are as tall as the bucket?
How much string is needed to go around the bucket?

Attribute: Area
Units: index cards, squares of paper, tiles
How many cards will cover the surface of the bucket?

Once students determine the attribute to be measured, they then choose *how* they will measure it. This involves determining an appropriate unit that has the same attribute being measured. Length is measured with units that have length, volume with units that have volume, and so on.

Meaning of Measuring

Technically, a *measurement* is a number that indicates a comparison between an attribute of the object (or situation or event) being measured and the same attribute of a given unit of measure. We commonly use small units of measure to determine a numeric relationship (the measurement) between what is measured and the unit. For example, to measure a length, the comparison can be done by lining up copies of the unit directly against the length being measured. For most of the attributes measured in schools, we can say that *measuring* means that the attribute being measured is being "filled" or "covered" or "matched" with a unit of measure that has the same attribute.

Process of Measuring

In brief, to measure something takes three steps:

1. Decide on the attribute to be measured.
2. Select a unit that has that attribute.
3. Compare the units—by filling, covering, matching, or using some other method—with the attribute of the object being measured. The number of units required to match the object is the measure.

The skill of measuring with a unit must be explicitly linked to the concept of measuring as a process of using measuring units and measuring instruments to compare attributes, as outlined in Table 14.1. This may seem

Table 14.1. Plan for measurement instruction.

	Goal	Type of Activity	Notes
Step 1	Students will understand the attribute to be measured.	Make comparisons based on the attribute—for example, longer or shorter, heavier or lighter. Use direct comparisons whenever possible.	When it is clear that the attribute is understood, there is no further need for comparison activities.
Step 2	Students will understand how filling, covering, matching, or making other comparisons of an attribute with measuring units produces a number called a *measure*.	Use physical models of measuring units to fill, cover, match, or make the desired comparison of the attribute with the unit.	In most instances, it is appropriate to begin with informal units. Progress to the direct use of standard units when appropriate, and certainly before using formulas or measuring tools.
Step 3	Students will use common measuring tools with understanding and flexibility.	Make measuring instruments, and compare them with the individual unit models to see how the measurement tool is performing the same function. Make direct comparisons between the student-made tools and the standard tools.	Standard measuring instruments such as rulers, scales, and protractors make the filling, covering, or matching process more concrete and comprehensible.

simple, but middle-school students struggle to match appropriate units with length, area, and volume measures.

Teaching Tip

Use precise language when helping students make comparisons. Avoid using *bigger than,* and instead use language such as *longer than* or *holds more than*.

Making Comparisons

Sometimes, with a measure such as length, a direct comparison can be made in which one object is lined up and matched with another. But often, an indirect method using a third object (e.g., a ruler) is necessary. For example, if students compare the volume of one box with the volume of another, they must devise an indirect way to compare the sizes of the two boxes (i.e., using a measuring instrument).

Using Physical Units to Measure

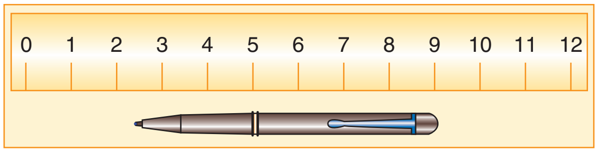
All measurement is actually a comparison. The object being measured is being compared with the unit. A 15-inch book is measured by placing a unit of 1 inch down and counting how many units will match the length of the book. This process, called *iteration*, was also discussed with fractions—for example, how many fourths are in $3\frac{1}{2}$? Although iteration is learned early in elementary school, it is worth revisiting in middle school because this understanding will support students' thinking as they develop formulas for measuring the area of triangles and quadrilaterals.

Using Measuring Instruments

In the 2003 National Assessment of Educational Progress (NAEP) examination, only 56 percent of eighth graders could give the correct measure of an object not aligned with the end of a ruler, as in Figure 14.2 (Kloosterman, Rutledge, & Kenney, 2009). These results point to the need to teach explicitly how to use measuring devices. Students also experienced difficulty when the increments on a measuring device were not one unit.

Figure 14.2

How long is this stylus?



If students use unit models with which they are familiar to construct simple measuring instruments, it is more likely that they will understand how an instrument measures. A ruler is a good example. If students line up individual physical units along a strip of card stock and mark them off, they can see that it is the *spaces* on rulers, not the hash marks or numbers, that are important. It is essential that students discuss how measurement with the iteration of individual units compares with measurement with an instrument. Without this comparison and discussion, students may not understand that these two methods are essentially the same.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Formative Assessment Note

Use an **Observation Checklist** to see whether students are using measuring tools accurately and meaningfully. Rulers and protractors are often poorly understood. Or, in a *diagnostic interview*, ask students to measure using a unit that they iterate and measure using a measuring tool. Second, ask them to react to the technique used by another student. ("Monique measured the width of her locker. She lined up the 10-cm mark on one side. The measure of the other side of the locker ended up between the 42- and 43-cm marks. Monique is confused about how wide her locker is. Without measuring the locker again, how would you help Monique?")

Choosing Appropriate Units

Should the room be measured in square feet or square inches? Should the concrete blocks be weighed in grams or kilograms? The answers to questions such as these involve more than simply knowing how big units are, though that is certainly required. Another consideration involves the need for precision. If you were measuring your wall in order to cut a piece of molding to fit, you would need to measure it very precisely. The smallest unit would be an inch or a centimeter, and you would also use fractional parts. But, if you were determining how many 8-foot molding strips to buy, the nearest foot would probably be sufficient.

Standards for Mathematical Practice

6 Attend to precision.

Activity 14.1

CCSS-M: 5.MD.C.3; 7.G.B.6

Guess the Unit

Use any or all of the five sets of Measurement Cards to pose situations that ask students to determine the unit of measure. You can give each set to one small group of students, and then trade sets, or you can pose any of the situations as quick warm-up activities. Additionally, look for examples of measurements of all types on the internet, in newspapers, on signs, or in other everyday situations. Present the context and measures, but without units. For example, a story in a national newspaper described Abe Lincoln's Manchester, Vermont, home as 8,000-square-feet (almost 750 square meters). Show students text that reads "Abe Lincoln had a home built in Vermont that was 750 [smudge]." Ask students to predict what units of measure were used. Then share the actual article. Have students discuss their choices. Similar estimation can be done for surface area or for angle measures. For students with disabilities, you may want to provide the possible units so they can sort the real-world measures into groups. For ELLs, having a list with measurement names with illustrations will ensure they are able participate.

Measurement Systems and Units

The United States is now the only nation that has not converted to the metric system, actually titled "International System of Units" or SI. Because of its international importance, U.S. students must know SI (metric system) for such things as product design, manufacturing, marketing, and labeling in order to participate in the global marketplace (as a consumer or an employee) (National Institute of Standards and Technology [NIST], 2015; NCTM, 2015). Results of the 2004 NAEP revealed that only 37 percent of eighth-graders knew how many milliliters were in a liter (Perie, Moran, & Lutkus, 2005). Customary units continue to be important in the United States for various careers (e.g., carpentry) and contexts, so U.S. students need to develop familiarity with multiple systems of measure (NCTM, 2015). Interestingly, U.S. students do better with metric units than with customary units (Preston & Thompson, 2004). Familiarity with the units in both systems helps students consider reasonableness of their measurements. Having everyday objects as points of reference can help: A doorway is a bit more than 2 m high, and a doorknob is about 1 m from the floor. A bag of flour is a good reference for 5 pounds. A paper clip weighs about a gram and is about 1 cm wide. A pineapple or liter of water weigh about 1 kilogram.

As students grasp the structure of decimal notation, develop the metric system with all seven places: three prefixes for smaller units (*deci*-, *centi*-, *milli*-), one for the unit itself (e.g., meters), and three for larger units (*deka*-, *hector*-, *kilo*-). Avoid mechanical rules, such as this one: "To change centimeters to meters, move the decimal point two places to the left." Instead, use measurement conversions as an application of the proportional reasoning that is central to middle school.

Estimation and Approximation

Measurement estimation is the process of using mental and visual information to measure or make comparisons without measuring instruments. We use this practical skill almost every day: Do I have enough sugar to make cookies? Is this ribbon long enough to wrap this gift? Is this suitcase over the weight limit? Here are several reasons for including estimation in measurement activities:

- Estimation helps students focus on the attribute being measured and the measuring process. Think about how you would estimate the area of the cover of this book in square centimeters. To do so, you would have to think about what area is and how the units might be placed on the book cover.
- Estimation provides an intrinsic motivation for measurement activities. It is interesting to see how close to the actual measurement you can come in your estimate.
- When standard units are used, estimation helps develop familiarity with the units. If you estimate the height of the door in meters before measuring, you must think about the size of a meter.
- The use of a benchmark to make an estimate promotes multiplicative reasoning. The width of the building is about one-fourth of the length of a football field—perhaps 25 yards.

Approximation is different from estimation—it does not have to do with predicting the result, it has to do with the actual result. All measurements are approximations. If you measure to the nearest inch, the object is only approximately 6 inches. If you decide to measure to the nearest eighth of an inch, you may measure the same object at $6\frac{3}{8}$ inches. Still, it could be measured to the nearest sixteenth of an inch, or even to a higher level of precision. Emphasize that all measures are approximate, and that smaller units lead to a greater degree of *precision*. A length measure can never be more than one-half unit in error. Because there is mathematically no "smallest unit," there is always some measurement error. As stated in the CCSS-M standards, students need to be able to "express numerical answers with a degree of precision appropriate for the problem context" (p. 7).

Standards for Mathematical Practice

6 Attend to precision.

Strategies for Estimating Measures

Begin measurement activities with estimates. Students need explicit guidance on effective measurement estimation. Three important ways to support estimating measures are using benchmarks, chunking, and iterating:

1. *Develop and use benchmarks or referents for important units.* Students who have acquired mental benchmarks for measurements *and* have practiced using them in class activities are much better estimators than students who have not learned to use benchmarks (Joram, 2003). Students must pay attention to the size of the unit to estimate well (Towers & Hunter, 2010). Referents should be things that are easily envisioned by the student.
2. *Use "chunking" or subdivisions.* Chunking is the process of grouping parts of what is to be measured. For example, the weight of a stack of books is easier to estimate if some estimate is given for the weight of an "average" book. Subdivisions are like chunking. For example, if a wall length to be estimated has no useful chunks (parts of the wall), it can be mentally subdivided into halves, fourths or even eighths until a more manageable length is found. This partial area estimate can then be used to estimate the entire wall. Length, area, surface area, and volume all lend themselves to being chunked or subdivided in order to estimate.
3. *Iterate a unit mentally or physically.* For length, area, and volume, it is sometimes easy to mark off single units visually. You might use your hands or make marks or folds to keep track as you go. If you know, for example, that your stride is about $\frac{3}{4}$ meters long, you can walk off a length and then multiply to get an estimate.

Standards for Mathematical Practice

7 Look for and make use of structure.

Tips for Teaching Measurement Estimation

Each of the previous strategies should be explicitly taught and discussed with students; then students should have more opportunities for selecting from among the strategies a method that works best in a particular situation. Consider these additional teaching tips:

1. *Explicitly teach each strategy and practice each approach.* After learning each, students are then able to choose from the three options, based on their preference and the numbers in the situation.
2. *Create lists of visible benchmarks for students to use.* Record and post on a class chart for reference.
3. *Discuss how different students made their estimates.* This will confirm that there is no single right way to estimate while reminding students of other useful approaches.
4. *Accept a range of estimates.* Think in relative terms about what is a good estimate. Within 10 percent for length is quite good. Even 30 percent off may be reasonable for weights or volumes.
5. *Encourage children to give a range of estimates (e.g., the door is between 7 and 8 feet tall) that they believe includes the actual measure.* Rather than just give a single value, stating a range helps students focus on reasonable minimum and maximum values.

Teaching Tip

Do not promote a "winning" estimate. It discourages estimation and promotes only seeking the exact answer.

6. *Make measurement estimation an ongoing activity.* Post a weekly measurement to be estimated. Students can record their estimates and discuss them for five minutes at the end of the week. Invite students to select measurements to estimate, with a student or team of students assigned this task each week.
7. *Be precise with your language, and do not use the word "measure" interchangeably with the word "estimate"* (Towers & Hunter, 2010). Randomly substituting one word for the other will cause uncertainty and possibly confuse students.

Measurement Estimation Activities

Estimation is an excellent way to apply the measurement formulas students are learning in middle school! Estimation activities need not be elaborate. Any measurement activity can have an "estimate first" component but also include tasks in which estimates are all that are needed. Here are some measurement estimation activities that can be done briefly and repeatedly.

Activity 14.2

CCSS-M: 6.G.A.1; 6.G.A.2; 7.G.B.4; 8.G.C.9

Estimation Exploration

Select a single object, such as a box, a painting on the wall of the school, a jar, or a desk. Each day, select a different attribute or dimension to estimate. For the jar or desk, for example, students can estimate volume and surface area. Focus on the shapes and types of measures for your grade. Invite students to identify objects to be measured.

Activity 14.3

CCSS-M: 6.G.A.1; 6.G.A.2; 7.G.B.4; 8.G.C.9

Estimation Scavenger Hunt

Conduct estimation scavenger hunts. Give teams a list of measurements, and have them find things that are close to having those measurements. Focus on the shapes and types of measures for your grade. See the Estimation Scavenger Hunt Activity Page for ideas. Examples include:

- **Something with a length of 5 mm**
- **Something that weighs between 1 kg and 2 kg**
- **Object with an angle of 45 degrees or 135 degrees**
- **A container that holds about 200 milliliters**
- **Object with a surface area of 80 square inches**

Let students suggest how to judge results in terms of accuracy. Students with special needs may benefit from having a point of reference, for example, an illustration of 1 square inch or 1 milliliter. ELLs are likely to have a stronger measurement sense of metric, and native U.S. students of U.S. measures; heterogeneous grouping allows each students strengths to support the group's efforts to find objects of the appropriate size.

Activity 14.4

CCSS-M: 6.G.A.1; 6.G.A.2; 7.G.B.4; 8.G.C.9

E–M–E Sequences

Use estimate–measure–estimate (E–M–E) sequences to help students practice with benchmarks. The **Estimate–Measure–Estimate** Activity Page can be used in its entirety, or you can cut and use the ones that fit your grade level. Focus on the shapes and types of measures for your grade. Select pairs of objects to estimate that are somehow related or close in measure but not the same. Have students estimate the measure of the first and check by measuring, then have them estimate the second. Here are some examples of pairs:

- Width of a window, width of a wall
- Area of a book cover, area of a desk surface
- Surface area of a jewelry box, surface area of a cereal box
- Volume of a coffee mug, volume of a pitcher

Formative Assessment Note

Estimation tasks are a good way to assess students' understanding of both measurement and units. Use a *checklist* while students estimate measures of real objects and distances inside and outside the classroom. Prompt students to explain how they arrived at their estimates to get a more complete picture of their measurement knowledge. Asking only for a numeric estimate can mask a lack of understanding and will not give you the information you need to provide appropriate remediation.

Angles

"Understand concepts [or attributes] of angles and measure angles" is one of the *Common Core State Standards* beginning in grade 4 (CCSSO, 2010, p. 28) and developing in middle school as seventh-graders use "facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure" (CCSSO, 2010, p. 50). Working with angle measures continues to be emphasized in grade 8 and is critical to high school geometry.

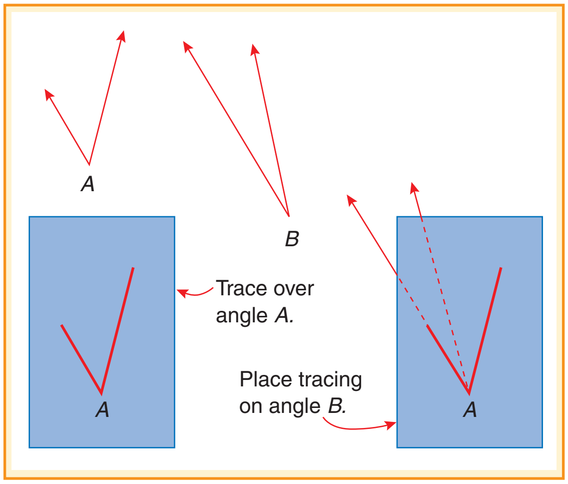
Angle measurement can be a challenge for two reasons: The attribute of angle size is often misunderstood, and protractors are often used without students understanding how they work. Angle units are based on an angle formed by rays extending from the center of a circle. A "one-degree angle" is an angle whose turn is $\frac{1}{360}$ of a circle, as illustrated in the semicircles in the **Degrees and Wedges** reference page. In this section, and the others in this chapter, we follow an instructional sequence that starts with comparison, then considers informal units, and finishes with the use of measurement tools.

Comparison Activities

The attribute of angle size might be called the spread of the angle's rays. Angles are composed of two infinitely long rays with a common vertex. They differ in size only by how widely or narrowly the two rays are spread apart. An angle with n turns of $\frac{1}{360}$ is an angle of measure *n* degrees. To help students conceptualize the attribute of the spread of the rays, two **angles** can be directly compared by tracing one and placing it over the other (Figure 14.3). Be sure to have students compare angles with rays of different lengths. A wide angle with short sides may seem smaller than a narrow angle with long sides. This is a common misconception among students (Munier, Devichi, & Merle, 2008). As soon as students are able to differentiate between a large angle and a small angle, regardless of the length of the rays, they are ready to measure angles.

Figure 14.3

Which angle is larger?



Tools for Angular Measure

A unit for measuring an angle must be an angle. Nothing else has the same attribute of spread that we want to measure.

Activity 14.5

CCSS-M: 4.MD.C.5; 7.G.B.5

A Unit Angle

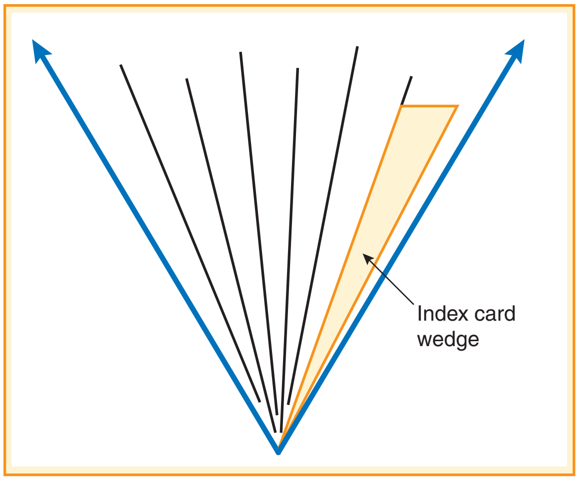
Give each student an index card. Have students draw a narrow angle on the card with a straightedge and then cut it out (or use the Wedges Activity Page). The resulting wedge can then be used as a unit of angular measure by counting the number of wedges that will fit in a given angle (Figure 14.4). Distribute copies of the Angles Activity Page and ask students to label each angle, a–f. Then use their "unit" angle to measure each angle. Because the students made different unit angles, the results will differ and can be discussed and compared in terms of unit size. Next, have students actually measure their wedge to determine the number of degrees. ELLs will need to know the specialized meaning of the word degree in measuring angles. Ask students to write equations to represent the number of degrees in their angle measures. For example, if $\angle A$ measured 2 wedges/units, and their wedge was 20 degrees, they write 20 degrees + 20 degrees or 2 × 20 degrees. This is an informal way to begin exploring supplementary and complementary angles, and writing equations with angles.

Activity 14.5 illustrates that measuring an angle is like measuring length or area; unit angles are used to fill or cover the spread of an angle, just as unit lengths fill or cover a length.

Middle-school students need an approximate mental image of angle size for common benchmark angle measures, such as 30, 45, 60, 90, 180, 270, and 360 degrees. Activity 14.6 develops a tool that can be used to help students develop a familiarity with these angle measures.

Figure 14.4

When a small wedge is cut from an index card and used as a unit angle, this angle measures about $7\frac{1}{2}$ wedges. The accuracy of measurement with nonstandard angles is less important than the idea of how an angle is used to measure the size of another angle.



Activity 14.6

CCSS-M: 7.G.A.2

Wax Paper Protractor Projects

The first part of this activity involves making the tool, incorporating concepts of circles and angles.

Make the protractor: Give each student a piece of wax paper that has been cut into a circle. Ask, "How many degrees in a full circle?" Ask students to fold the wax paper carefully in half and in half again, forming fourths. Ask, "How many degrees in the angles in the center?" [90 degrees.] Next, ask students to refold the circle in fourths, and then to fold that fourth into thirds, crease, and open. Ask, "How many pieces?" [12 pieces.] What is the angle measure of each twelfth (sector)?" [30 degrees.] (See Figure 14.5, where the homemade protractor is used to measure angles in a polygon.)

Use the protractor: Point to or hold up various objects with vertices, and ask students to use their protractor to estimate the angle measure of those objects. Discuss the reasonableness of their estimates. Then have students use their wax paper protractor to measure. For example, challenge students to find five objects, each of which has a different angle measure and at least two of which have an angle measure greater than 90 degrees. Students can also explore the sum of angles for polygons, as illustrated in Figure 14.5.

One reason to explore angle measures is that they can provide important information about polygons, which is particularly relevant in grades 7 and 8 as students study special triangles and quadrilaterals (CCSSO, 2010). The next activity is targeted at building this connection.

Teaching Tip

Be sure to include measurements of 180 degrees and greater so that students realize there are measures beyond 180 degrees.

Activity 14.7

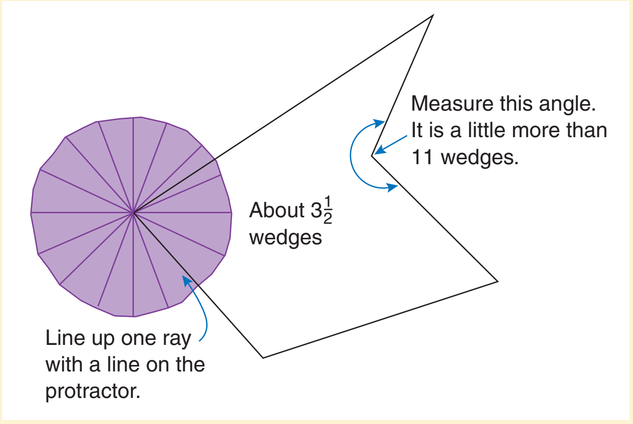
CCSS-M: 7.G.A.2; 7.G.B.5; 8.G.A.5

Angle Sums of Polygons

Ask students what they think the sum of the angles in a quadrilateral (or triangle or other polygon) might be. Collect their conjectures. Explain that students are going to investigate this question and report back what they learn. To do this, they must measure as many different kinds of quadrilaterals as they can. Students may choose to draw the quadrilaterals, to build them on a geoboard (see Blackline Master 27, **Geoboard Recording Sheets**) or create them by using short pieces of uncooked spaghetti as sides and taping them down. Discuss with students the types of shapes they might include, including concave and convex shapes (see Figure 14.5). For students with special needs, you may want to prepare a page of quadrilaterals with different shapes, rather than having them generate the shapes. Students use their wax paper protractor (or angle ruler) to measure the angles.

Figure 14.5

Measuring angles of a polygon with a wax paper protractor.

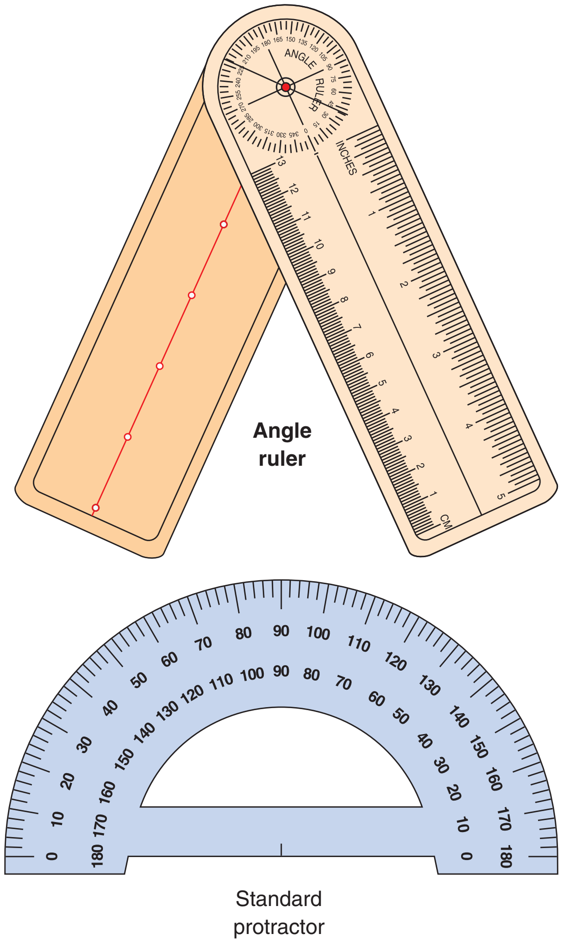


Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

Figure 14.6

Tools used to measure angles.



The marks on a plastic protractor are the rays of degrees. A degree is just a very small angle.

Such investigations build familiarity with angle units in general and important angle benchmarks, and they build the foundation for noticing important relationships among angles such as supplementary, complementary, vertical, and adjacent angles.

Another homemade protractor can be created by adapting **Circular Fraction Pieces**. Cut out two identical ones on two different colors of cardstock. Cut one radius for both circles. They slide together and can be rotated to show different angle measures. The fraction pieces are readily adaptable, for example fourths become 90 degrees, eighths become 45 degrees, sixths become 60 degrees, and so on. Or, use the **Rational Number Wheel**, which is based on 100 percent or 1 whole. It is an excellent reasoning task for students to be able to translate between angles (based on 360 degrees) and fractions or decimals (based on 100 percent or 1 whole). For example, students might recognize that 30 degrees is one-third of 90 degrees (one-fourth of 360 degrees), so one-third of one-fourth, or one-twelfth. Or they might think that 30 degrees is one-third of 90 degrees, which is the same as one-third of 25 percent, so approximately 8 percent.

Using Protractors and Angle Rulers

The two tools commonly used for measuring angles are the protractor and the angle ruler (Figure 14.6). The protractor is one of the most poorly understood measuring instruments. Part of the difficulty arises because the units (degrees) are so small. In addition, the numbers on most protractors run clockwise and counterclockwise along the edge, making the scale hard to interpret without a strong conceptual foundation. Angle rulers are a good choice for measuring angles, but they also require experience in order to understand how to set the tool on the angle to be measured and how to read the scale.

Standards for Mathematical Practice

5 Use appropriate tools strategically.

When introducing these tools, have students check their measures with one of their homemade tools, engage students in making an estimate, or do both. The best way to minimize errors in reading the protractor or angle ruler is to have students monitor their measuring, asking themselves, “Is this angle measure reasonable?” When students ask this question, they will not record 200 degrees when the angle is less than 180 degrees and will recognize that the value they should be reading on the protractor is 160 degrees.

Using Angle Measures to Determine Angle Measures

In middle school, students explore how other measures can be used to determine the value of a missing angle measure. This includes knowing that the sum of the angles in a triangle is 180 degrees, that the sum of the angles in a quadrilateral is 360 degrees, that the sum of two adjacent angles that form a right angle must be 90 degrees (*complementary angles*), and that the sum of two adjacent angles that form a line must be 180 degrees (*supplementary angles*). To explore the sum of angles within a triangle, see Activity 13.5, “Angle Sums in a Triangle,” or adapt Activity 14.5, “A Unit Angle.” Activity 14.8 engages students in discovering relationships among angles of intersecting lines, but it can also be applied to polygons or other shapes.

Activity 14.8

CCSS-M: 7.G.B.5

Angle Relationships

To explore supplementary, complementary, vertical and adjacent angles, you can invite students to discover the relationship, employing their algebraic reasoning skills, by actually measuring angles across examples and then look for patterns. For example, you can follow these steps:

- Ask pairs of students to draw two intersecting lines.
- Measure the four angles created by the intersection.
- Record equations, noting relationships they notice within their set.
- Compare to another groups findings (other groups will have different angle measures but should have noticed the same relationships).
- Discuss as a whole class if these relationship are always true and teach the related vocabulary of supplementary, vertical and adjacent angles.

For the benefit of ELLs, and all students, practice saying the words and record the new vocabulary with illustrations.

Once students have an understanding of the relationships among angles, challenge them to solve problems involving missing angles. It is important for them to realize that they cannot always use one of these strategies to find the unknown angle; sometimes they will have to actually use a measuring tool. There is one such situation in the following activity.

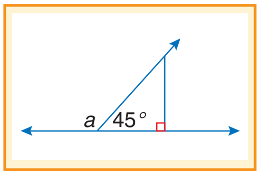
Activity 14.9

CCSS-M: 7.G.B.5

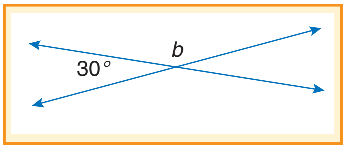
In Search of Missing Angles

Begin by distributing the **In Search of My Missing Angle** or give students any problem with a triangle in which some angle is missing. Two examples from the activity page are illustrated here:

What is the measure of angle *a*?



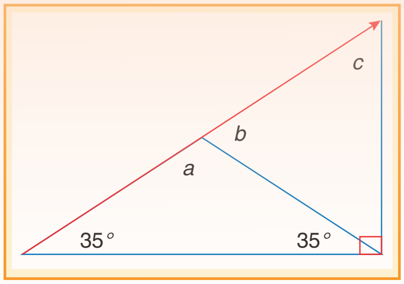
What is the measure of angle *b*?



Ask students to talk (not solve), sharing how they can find the missing angle first, then apply their strategy. Encourage students to use the language of supplementary, complementary, vertical, and adjacent angles, as appropriate.

Formative Assessment Note

Journal entries are a good way to evaluate what relationships students notice between angles. Give students a diagram with some angles labeled with measures and some angles not labeled.



In their journals, have them explain the properties or relationships they used to figure out each missing angle measure ("I knew angle *a* was 110 degrees because . . ."). Writing an explanation such as this one reinforces the concept as well as the vocabulary.

Area

Area is the two-dimensional space inside a region. As with other attributes, students must first understand the attribute of area before measuring. Data from the 2003 NAEP suggest that fourth- and eighth-grade students have an incomplete understanding of area (Blume, Galindo, & Walcott, 2007). Estimating area and measuring area begin in third grade, as they connect to learning multiplication, and continue in grades 4 and 5 with finding the area of rectangles through tiling. In grade 6, students explore area of a wide range of polygons and learn the formulas for many of these polygons. In grade 7, students explore area of circles. Across middle school, students apply concepts of area to surface area and to real-world situations (CCSSO, 2010). Real-world situations could include measuring and constructing a container. Using inches (fractions) or centimeters (decimals) is a great way to reinforce fluency with rational numbers.

Comparison Activities

Comparing area measures is a bigger conceptual challenge than comparing length measures because areas come in different shapes. Comparison activities with areas should help students distinguish between size (or area) and shape, length, and other dimensions. For example, project or distribute rectangles that have different dimensions but are similar in their area, such as the ones on the **Rectangle Comparison** Activity Page. A long, skinny rectangle may have less area than a triangle with shorter sides. Many students, and even adults, do not understand that rearranging an area into different shapes does not affect the amount of area (though the perimeter can change).

Direct comparison of two areas is frequently impossible except when the shapes involved have some common dimension or property. For example, two rectangles with the same width can be compared directly, as can any two circles. Comparison of these special shapes, however, fails to deal with the attribute of area. Instead, activities in which one area is rearranged (conservation of area) are suggested.

Tangrams can be used for this purpose. The two small triangles can be used to make the parallelogram, the square, and the medium triangle. This permits a similar discussion about pieces having the same size (area) but different shapes (and is a great foundation for sixth-grade investigation of area formulas of triangles and quadrilaterals). To become familiar with

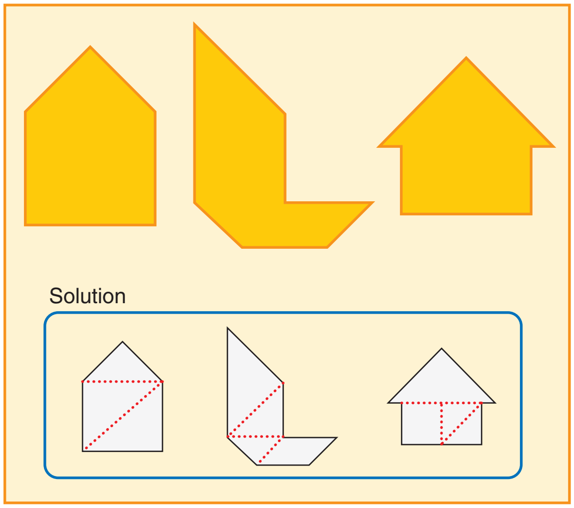
Tangrams and the relationship among the area of each piece, students can cover outlines of shapes made with Tangrams. The National Library of Virtual Manipulatives and PBS Kid's Cyberchase provide ways to do this virtually. Covering shapes can (and should) be repeated with different shapes that *vary in size*—that is, use more or fewer of the tangram pieces (see examples in Figure 14.7). Let students explain which shapes they think have a larger area and how they came to their conclusions.

Teaching Tip

Two good literature links for tangrams are the popular *Grandfather Tang's Story* (Tompert, 1997) and the middle-school-friendly *The Legend of the Tangram Prince* (Foster, 2007). The illustrations in the first book use one full set, and those in the second book use two full sets, so the areas of the various figures within each book are congruent.

Figure 14.7

Compare area of shapes made of tangram pieces.



Tools for Area Measure

Before learning formulas, students should have significant "cover-the-surface" experiences with area. Tools for covering include manipulatives, which tend to be squares (but don't have to be) and grid paper. Both are discussed here.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

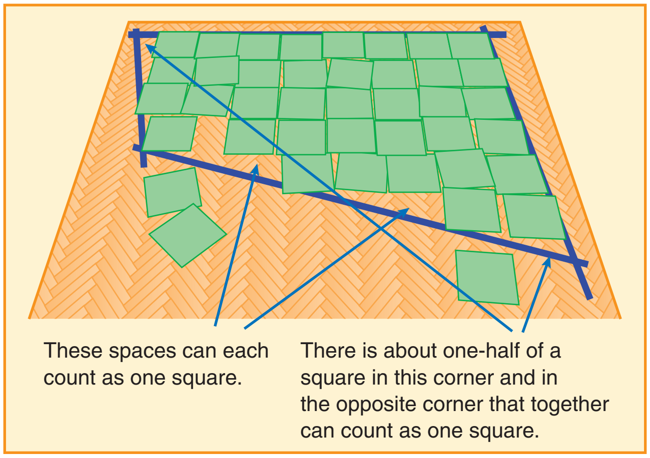
Tiles

Square tiles are the most commonly used tools for exploring area, though any object that has area as its main attribute can be used. Square tiles can be easily cut from card stock or construction paper so that each student will have a set that he or she can keep and use at home. Square manipulatives also come in other forms, for example, the 1-inch or 6-inch tiles found at a home improvement store or 5-foot square table cloths. The unit does not have to be a square. Surfaces can be covered with rectangles such as sticky notes, sheets of notebook paper, or sheets of newspaper.

In activities that involve covering surfaces, there are typically units that only partially fit in, or spaces that are too small for another full unit to fit in. Students should estimate about how many units will fill those gaps (Figure 14.8). Beginning in fifth grade, students begin using fractional units of measure, so covering a surface with partial pieces is a good way to start thinking about parts of units. The following activity is a good starting point to assess your students' thinking related to units of area.

Figure 14.8

Measuring the area of a large shape drawn with tape on the floor. Units are card stock squares of the same size.



Activity 14.10

CCSS-M: 4.MD.A.3; 5.NF.B.4B; 6.G.A.1

Cover and Compare

Draw three shapes of different sizes on a sheet of paper: two rectangles and one irregularly curved shape. Make sure it is not obvious which shape is largest or smallest. Ask students to predict which shape has the smallest and which has the largest area. Next, ask students to find the approximate area of each shape by using a comparing technique that they design. Ask the groups to brainstorm strategies for comparing before they get started. Alternatively, and particularly for students with special needs, you may want to create a list of options and have them discuss each option and how it would be used to help compare.

There are several approaches to this problem. One is to measure each shape with tiles and compare the approximate areas. Or students can select one shape as the basis for comparison and cut out the others, laying them over the first. Or students can cut out each shape and place it over grid paper. Students should explain their strategy and justify their choice of the shape with the largest and smallest area.

Your initial objective is to develop the idea that area is measured by covering or tiling, and that shape and size are two related but different attributes.

Grids

Grids of various types can be thought of as “area rulers.” A grid of squares does for area what a ruler does for length. It lays out the units for you. Square grids of various sizes (**0.5-Centimeter Grid Paper**, **1-Centimeter Grid Paper**, or **2-Centimeter Grid Paper**, Blackline Masters 7, 6, or 5) can be copied on transparency paper and placed over a region to serve as a clear grid so the square units inside can be measured and counted. Alternatively, an object can be traced or placed onto a piece of grid paper.

Perimeter and Area

Using tiles or outlining shapes on grid paper allows the simultaneous exploration of perimeter and area. Yet, teaching these concepts during a close timeframe is particularly challenging for students with disabilities (Parmar, Garrison, Clements, & Sarama, 2011). So, although these concepts were taught in elementary school, if middle-school students are struggling, teaching them separately and connecting to concrete situations and visuals may better support their learning. The next activities are designed to reinforce and strengthen students' understanding of the relationship between perimeter and area; after exploring each, explicitly ask students how perimeter and area are related.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Activity 14.11

CCSS-M: 5.NF.B.4B

Fixed Perimeters or Areas

In this two-part activity students explore the impact of a fixed perimeter and a fixed area.

Part 1: Explain to students that they are building a pen for a rabbit and that you have 24 feet of wire fence. Ask, “Will it matter what shape rectangle I use to make my pen?” Give students a piece of nonstretching string that is 24 cm long and a sheet of **1-Centimeter Grid Paper**. Ask students to explore how many different-size rectangles can be made with a perimeter of 24 cm, recording the perimeter and area for each new sketch of the pen. As a way to incorporate nonwhole rational numbers, change the total number of feet to 30, and encourage students to measure sides to the nearest one-half inch. Ask students to record all of the results on the **Fixed Perimeter Recording Sheet**.

Part 2: Using the same context, explain to students that when you bought your rabbit, you were told that the best space for outdoor exercise is 36 square feet. You need to buy the wire fence and are wondering how much to buy, but you haven't yet decided on a shape for the rectangle. Provide students with 1-cm grid paper. Ask them to design several options for you and for each one tell you the area and the perimeter (fence) needed. Ask students to record all of the results on the **Fixed Area Recording Sheet**. See also the **Expanded Lesson: Fixed Areas**, focused on Part 2 of this Activity.

Stop and Reflect

How are area and perimeter related? For "Fixed Perimeters," will the areas remain the same? Why or why not? For "Fixed Areas," will all of the perimeters be the same? If not, what can you say about the shapes with longer or shorter perimeters?

When students complete Activity 14.11, have them cut out all the figures. Label either two charts or two locations on the board "Perimeter" and "Area," and have teams come up and place their figures (left to right) from smallest perimeter (or area) to largest perimeter (or area) on the appropriate chart. Ask students to state what they observe, make conjectures, and see whether any conclusions can be drawn. Students may be surprised to find out that rectangles with the same areas do not necessarily have the same perimeters, and vice versa. And, of course, this fact is not restricted to rectangles.

Students will notice interesting relationships between perimeter and area. When the area is fixed, the shape with the smallest perimeter is "squarelike," as is the rectangle with the largest area. If you allow any shape whatsoever, the shape with the smallest perimeter for a fixed area is a circle. Also, students will notice that the "fatter" a shape, the smaller its perimeter; the "skinnier" a shape, the larger its perimeter. (This is true in three dimensions—replace perimeter with surface area and area with volume.)

Developing Formulas for Area

When students *develop* formulas, they gain conceptual understanding of the ideas and relationships involved, and they engage in "doing mathematics." Also, there is less chance that students will confuse formulas or forget them altogether. And, if they are forgotten, with a conceptual understanding of how the formulas are related, students can regenerate the formulas.

It is important for students to understand how shapes are related and how formulas for finding their areas are also related. In the *Common Core State Standards* for sixth grade, students are to "find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes" (CCSSO, 2010, p. 44). The formula for the area of a rectangle is one of the first that is developed and is usually written as $A = L \times W$, or stated as "area equals length times width." Thinking ahead to other area formulas, an equivalent but more unifying idea might be $A = b \times h$, or "area equals base times height."

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

The formula *base* × *height* can be generalized to all parallelograms (not just rectangles) and is useful in developing the area formulas for triangles and trapezoids. Furthermore, the same approach can be extended to three dimensions—volumes of cylinders are given in terms of the *area of the base* times the height.

Student Misconceptions

Students have various challenges with area formulas. The common error of confusing the formulas for area and perimeter has already been discussed. Student difficulties related to area may be due to an overemphasis on formulas without understanding the meaning of the formulas. Two major misconceptions related to area are:

Teaching Tip

Constantly ask students what the variables in the formula represent and why they are there. Look for students to connect back to the idea that area is the measure of the surface, and it is found by using the general formula base × height.

1. *Confusing linear and square units.* One of the major issues with area measurement is thinking about area as the length of two lines (length and width), rather than the measure of a surface (Putrawangsa, Lukito, Amin, & Wijers, 2013). A focus on the formula and the use of a ruler to measure the sides confuses the unit as well as the tool for measuring area (as the use of the ruler is indirect). This confusion can cause some students to believe that if there are no sides to measure (no length and width), the shape doesn't have an area (Zacharos, 2006).

   The tasks in Figure 14.9 cannot be solved with simple formulas; they require an understanding of concepts and how formulas work. "Length times width" is not a definition of area; instead, area is a measure of a two-dimensional surface enclosed by a boundary.

Teaching Tip

A good way to strengthen students' understanding of height is to have them identify the height for every base of a triangle or quadrilateral. (Be sure to vary the shapes so that the height falls inside some of the shapes and outside others.)

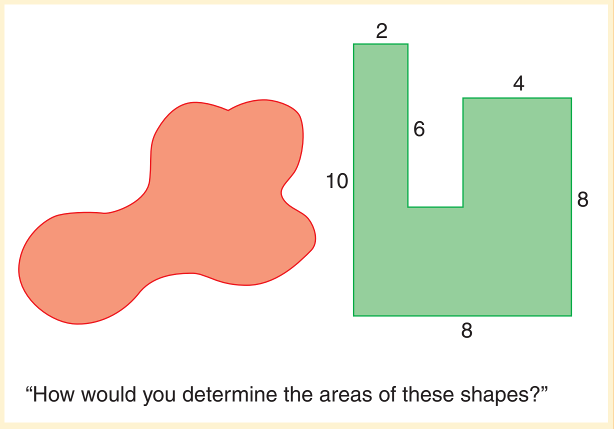
2. *Difficulty in conceptualizing the meaning of height and base.* The shapes in Figure 14.10 each have a slanted side and a height given. Students tend to confuse these two. Any side of a figure can be called a *base*. For each base that a figure has, there is a corresponding height. If the figure were to slide into a room on a selected base, the *height* would be the height of the shortest door it could pass through without tipping—that is, the perpendicular distance to the base. The confusion may be because students have a lot of early experiences with the length-times-width formula for rectangles, in which the height is exactly the same as the length of a side.

Rectangles

Research suggests that it is a significant leap for students to move from counting squares inside a rectangle to acquiring a conceptual development of a formula. Students often try to fill in empty rectangles with drawings of squares and then count the result one square at a time (Battista, 2003). Although area of rectangles (and the formula) have been taught in elementary school, it is important to assess whether middle-grade students have a strong understanding of the formula because it will be used to develop the other area formulas.

Figure 14.9

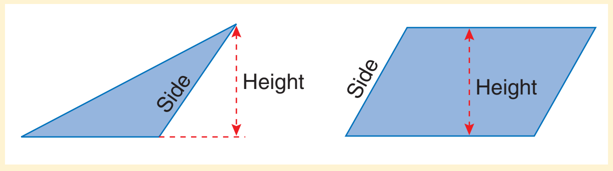
Understanding the attribute of area.



"How would you determine the areas of these shapes?"

Figure 14.10

Heights of two-dimensional figures are not always measured along an edge.



Do students connect area to multiplication (of whole numbers or fractions), in which rectangles were used as a visual representation? When we multiply a length times a width, we are not multiplying "squares times squares." Rather, the *length* of one side indicates how many squares will fit on that side. If this set of squares is taken as a unit, then the *length* of the other side (not a number of squares) will determine how many of these *rows of squares* can fit in the rectangle. Then the amount of square units covering the rectangle is the product of the length of a row and the number of rows: column × row = area.

Do students connect length to the idea of base? Explain to students that you like the idea of measuring one side to tell how many squares will fit in a row along that side. You would like them to call or think of this side as the *base* of the rectangle, even though some people call it the *length* or the *width*. Then the other side can be called the *height*. Which side is the base? It can be either side—and students should realize that they can always rotate a shape in order to use more convenient numbers and use any side as the base.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

From Rectangles to Other Parallelograms

Once students understand the base-times-height formula for rectangles, the next challenge is to determine the areas of parallelograms. Rather than providing a formula, use the following activity, which asks students to devise their own formula, building on what they know about rectangles.

Activity 14.12

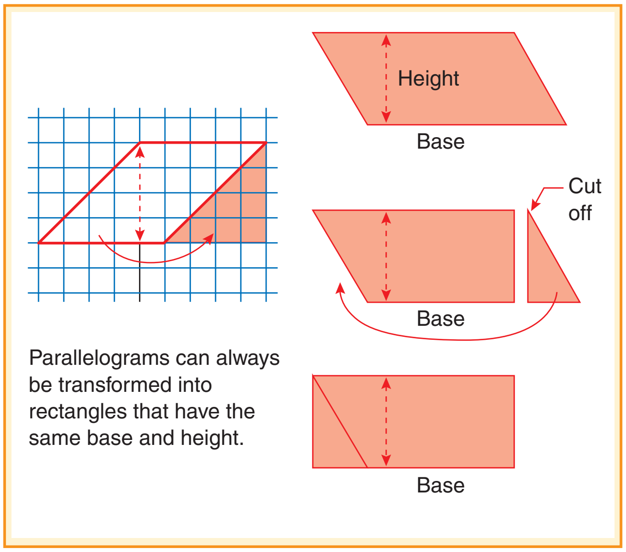
CCSS-M: 6.G.A.1

Area of a Parallelogram

Give students two or three parallelograms, drawn either on grid paper or, for a slightly greater challenge, on plain paper labeled with all dimensions—the lengths of all four sides and the height. Ask students to use what they have learned about the area of rectangles to determine the areas of these parallelograms. Students should find a method that will work for any parallelogram, even if it is not drawn on a grid.

Figure 14.11

Area of a parallelogram.



Parallelograms can always be transformed into rectangles that have the same base and height.

If students are stuck, ask them to examine ways in which the parallelogram is like a rectangle or how it can be changed into a rectangle. As shown in Figure 14.11, a parallelogram can always be transformed into a rectangle with the same base, the same height, and the same area. Thus, the formula for the area of a parallelogram is exactly the same as that for a rectangle: base times height.

From Parallelograms to Triangles

Knowing the relationship between area of rectangle and parallelogram, the area of a triangle can logically follow. Engage students in discovering the relationship between parallelograms and triangles, the focus of Activity 14.13.

Activity 14.13

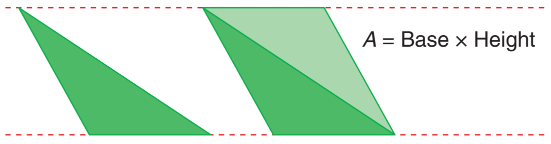
CCSS-M: 6.G.A.1

Area of a Triangle

Provide students with at least two triangles drawn on grid paper or distribute cut-out Triangles and have students use blank or grid paper (you may want to remove right triangles because they are an easier special case). Challenge the students to use what they have learned about the area of parallelograms to find the area of each of the triangles and to develop a method that will work for any triangle. They should be sure that their method works for all the triangles given to them, as well as at least one more that they draw. For students with disabilities or those that need more structure, ask, "Can you find a parallelogram that is related to your triangle?" Then suggest that they fold a piece of paper in half, draw a triangle on the folded paper, and cut it out, making two identical copies. Use the copies to fit the triangles together into a parallelogram. This provides a nice visual of how a triangle is related to a parallelogram.

Figure 14.12

Two triangles always make a parallelogram.



Two copies of any triangle will always form a parallelogram with the same base and height; therefore, the triangle has an area of half of the parallelogram, $A = \frac{1}{2}(\text{Base} \times \text{Height})$.

As shown in Figure 14.12, two congruent triangles can always be arranged to form a parallelogram with the same base and same height as those of the triangle. The area of the triangle will, therefore, be one-half that of the parallelogram.

Stop and Reflect

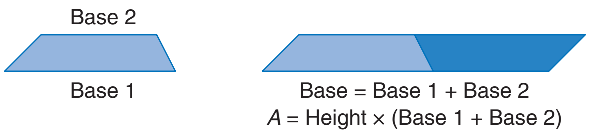
There are three possible parallelograms, one for each triangle side that serves as a base. Will the computed areas always be the same? How might you engage students in exploring the three different parallelograms?

From Parallelograms to Trapezoids

After developing formulas for parallelograms and triangles, students are ready to develop the formula for area of a trapezoid. There are at least 10 different methods of arriving at a formula for trapezoids, each related to the area of parallelograms or triangles. One method uses the same general approach that was used for triangles. Suggest that students work with two trapezoids that are identical, just as they did with triangles. Figure 14.13 shows how this method results in the formula. Not only are all of these formulas connected, but similar methods were used to develop them, as well.

Figure 14.13

Two trapezoids always form a parallelogram.



Two trapezoids always make a parallelogram with the same height and a base equal to the sum of the bases in the trapezoid. Therefore, $A = \frac{1}{2} \times \text{Height} \times (\text{Base 1} + \text{Base 2})$.

Here are a few suggestions, each leading to a different approach to finding the area of a trapezoid:

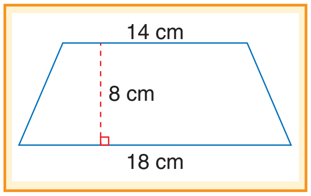
- Make a parallelogram inside the given trapezoid by using three of the sides.
- Make a parallelogram by using three sides that surround the trapezoid.

- Draw a diagonal, forming two triangles.
- Draw a line through the midpoints of the nonparallel sides. The length of that line is the average of the lengths of the two parallel sides.
- Draw a rectangle inside the trapezoid, leaving two triangles, and then put those two triangles together.

Or give students an example of a trapezoid with the measurements labeled and ask them to try the following:

Find the area of the trapezoid by using only the formulas for area of rectangles, area of parallelograms, and area of triangles.

After they have worked through several related problems, ask students what they notice across the situations that they might generalize to all trapezoids.



Stop and Reflect

Do you think that students should learn special formulas for the area of a square? Why or why not? Do you think students need formulas for the perimeters of squares and rectangles?

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

technology note

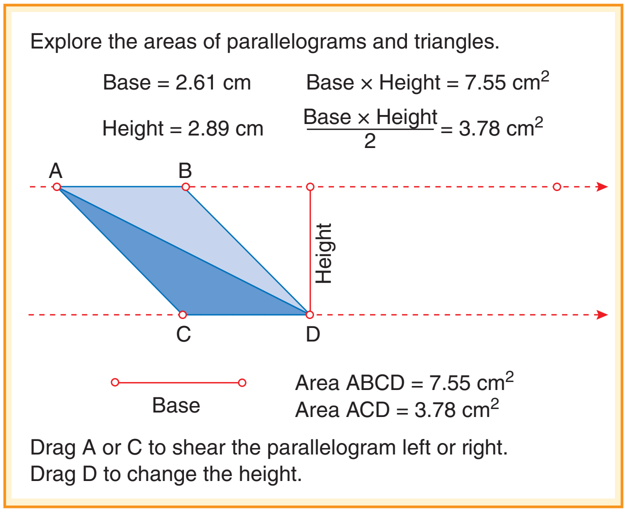
The relationship between the areas of rectangles, parallelograms, and triangles can be dramatically illustrated with a dynamic geometry program such as *GeoGebra* (free public domain software). Draw two congruent segments on two parallel lines (Figure 14.14). Then connect the end points of the segments to form a parallelogram and two triangles. The height is indicated by a segment perpendicular to the parallel lines. Either of the two line segments can be dragged left or right (*sheared*) to change the slant of the parallelogram and triangle without changing the base or height (and therefore maintaining the area).

Surface Area

After learning how to find area of two-dimensional shapes, students are ready to explore surface area of three-dimensional shapes composed of two-dimensional faces. According to the *Common Core State Standards*, sixth-graders are expected to find the surface area of shapes composed of faces of rectangles and triangles. In seventh grade, this requirement is extended to shapes composed of faces of triangles, quadrilaterals, and polygons, and to structures composed of cubes and right prisms (CCSSO, 2010). Nets are an effective strategy for introducing surface area. A great way to illustrate nets is to create several rectangular or triangular prisms made of card stock, including cubes with sides held together by small pieces of Velcro. In this way, students can think about the net of the figure as they break the figure into faces and calculate the surface area.

Figure 14.14

Dynamic geometry software shows that figures with the same base and height have the same area.



Explore the areas of parallelograms and triangles.

Base = 2.61 cm

Height = 2.89 cm

Base × Height = 7.55 cm^2

$\frac{\text{Base} \times \text{Height}}{2}$ = 3.78 cm^2

Area ABCD = 7.55 cm^2

Area ACD = 3.78 cm^2

Drag A or C to shear the parallelogram left or right.
Drag D to change the height.

Activity 14.14

CCSS-M: 6.G.A.4; 7.G.B.6

Making "To-Go" Boxes

This activity engages students in a context that involves nets. Begin by sharing the situation with the students: Joan's Dine & Dessert Shop orders to-go boxes in three sizes (dinner, pie, and cake), but the boxes are getting very expensive. Joan wants an estimate of how much cardboard is used for each size so that she can decide whether she should make her own boxes. Please share with her how much cardboard is used for these boxes (assume no overlap of cardboard—it is taped at the edges).

- Dinner box: 7 in. × 7 in. × 3 in.
- Pie box: 5 in. × 4 in. × 3 in.
- Cake box: 8 in. × 8 in. × 5 in.

Have grid and blank paper available to students for creating their net for each box. Have students compare their diagrams and surface areas. They may have picked different bases, and the illustrations may look different. This provides a great opportunity to talk about why these different nets for the same box are equivalent. For ELLs, the word *net* may be confusing because it may connect to the Internet or to nets used for fishing or basketball. Also, getting food in a "to-go" box may not be a practice in a student's country of origin.

technology note

See Shodor *Interactivate* "Surface Area and Volume" to explore how changing the dimensions of a polyhedron changes its surface area (and volume), or go to Annenberg Learner and search for "Interactives: Geometry 3D Shapes" for explorations of surface area through animations ("Surface Area & Volume" tab).

Circumference and Area of Circles

As students explore circumference and area of circles, they are introduced to π, one of the most important numbers in mathematics. Understanding the formulas for circumference and area, and exploring the π relationship can be very interesting for students, as the activities in this section highlight . See also the NCTM Illuminations applet "Circle Tool," which allows students to investigate the relationship between radius, diameter, circumference, and area of circles in a dynamic way.

Circumference

The circumference of every circle is about 3.14 times as long as the diameter. The exact ratio is an irrational number close to 3.14, represented by the Greek letter pi (π). So, $\pi = \frac{C}{D}$, the circumference divided by the diameter. In a slightly different form, $C = \pi D$. Half the diameter is the radius (r), so the same equation can be written $C = 2\pi r$.

There are many ways to explore the ratio of circumference and diameter of a circle. Activity 14.15 engages students in measuring to analyze the relationship.

Activity 14.15

CCSS-M: 7.G.B.4

Ratio of Diameter to Circumference

Place circular items such as jar lids, cans, wastebaskets, and even hula hoops at various stations around the room. Explain to students that they will be going to each station and carefully measuring the diameter and circumference of the shapes and recording their data in a table. Give each group of students a string and a ruler. Model how to measure each attribute: To measure the diameter, stretch the string across the circle, through the center. Since the diameter is the widest segment through a circle, students should look for where the string can be the longest as it is pulled across the circle; to measure the circumference wrap the string once around the object, and then measure the length of string needed to go around exactly once.

After students have been to every station, have them add a column to their table in which they record the ratio of circumference to diameter for each circle.

With graphing calculators, you can have each group enter its measures for diameter (x) and circumference (y) into the TABLE function of a graphing calculator, view the graph, and find the line of best fit.

Recall from Chapter 11 that graphs of equivalent ratios are always straight lines through the origin. If measured carefully, the ratio of circumference to diameter will be close to 3.14.

Another way to investigate the relationship is to cut a set of strings in a particular length (e.g., 8 cm). Use one string as a diameter, and use a compass to draw a circle around it. Then see how many of the remaining strings are needed to go around the circle. Repeat for different-size circles. Students are likely to be surprised that regardless of the size of a circle, they need only a little more than three strings to go around it.

What is most important in this activity is that students develop a clear understanding of π as the ratio of circumference to diameter (whole-to-part ratio) in any circle, regardless of size. If you know one measure, you can find the other. The quantity π is not some mystical number that appears in mathematics formulas; it is a naturally occurring and universal ratio.

Teaching Tip

Outliers (data that do not result in a ratio close to 3.14) can be examined to see if there was measurement error, focusing student thinking on data (outliers) and measurement (precision)!

Area of Circles

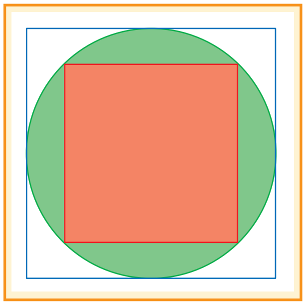
As with polygons, students should investigate the area formula for circles, rather than just be given the formula. Here are four ways to explore area of circles that build conceptual understanding for the formula:

1. **Cover a circle with tiles.** Cover a circle with square units and see how many cover a circle. Use square tiles (e.g., color tiles), or cut out squares that can be placed on the circle and glued. The advantage of cutouts is that students can cut some that are only partially inside the circle and place the extra pieces somewhere else in the circle. The key to this approach is that the students need to get a measure of the radius from the sides of the tiles. One group, for example, might have a circle that has a radius of 3 tiles, while another group explores a circle that has a radius of 5 tiles.
2. **Use radius squares.** Draw a circle and draw a radius. Set the radius as the unit. Use the radius unit to make squares—we will call them *radius squares*. Form a grid that is 2 × 2 radius squares, and draw a circle inside. By observation, you can see that the circle

inscribed in this grid has an area of less than 4 radius squares (because so much of the grid is outside the circle). Students need two identical copies of this picture. They use one set to cut out the radius squares to see how many it takes to cover the circle in the other picture exactly. Students can reach an estimate that the number of radius squares needed is about 3.1 or 3.2. This is not as accurate a technique, but it does reinforce the geometric idea that r^2 is a square with a side of r.

Figure 14.15

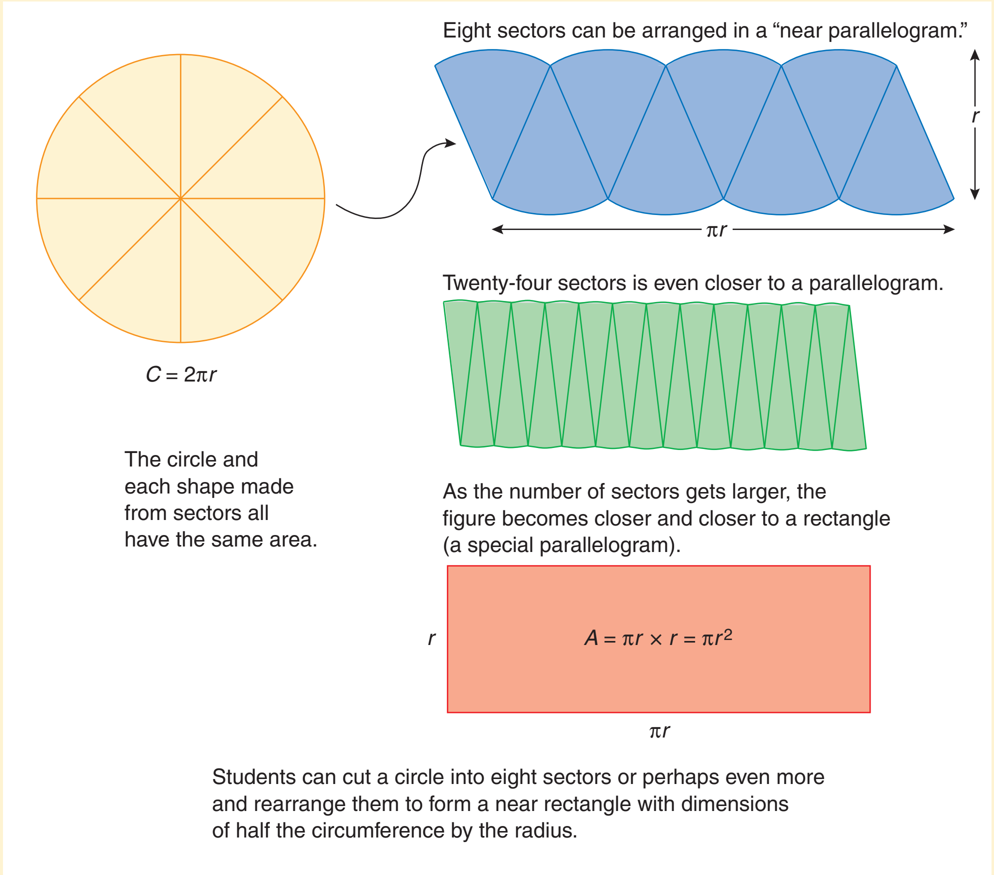
Using inscribed and superscribed squares to estimate the area of a circle.



3. **Draw Inscribed and Superscribed Squares.** This approach is something close to the method that Archimedes actually used to approximate π. In this approach, students draw a circle on grid paper with a given radius (or the circle can be drawn beforehand on a handout). They draw a square inside the circle and outside the circle (Figure 14.15). Find the areas of the two squares, and average them to find the area of the circle. Archimedes began this way and then started using polygons that got closer to the shape of the circle. See NCTM Illuminations for full lessons in which this approach is used.
4. **Cut to make a parallelogram.** Cut a circle apart into sectors and rearrange them to look more like a parallelogram. For example, students can cut from 3 to 12 sectors from a circle and build them into what looks like parallelogram. Recall that in the angle investigation, students made a wax paper circle. This same circle can be remade and cut to form the pieces to explore area of a circle. You may need to help them notice that the smaller the size of the sectors used, the closer the arrangement gets to a rectangle. Figure 14.16 presents a common development of the area formula $A = \pi r^2$.

Figure 14.16

Development of the formula for area of a circle.



Volume and Capacity

Volume and *capacity* are two terms for the measure of the "size" of a three-dimensional region. This topic begins in fifth grade. It continues to be emphasized throughout middle school as students solve real problems involving volume and develop the formula for volume of right rectangular prisms in grade 6 and for cylinders, cones, and spheres in grade 8 (CCSSO, 2010).

The term *capacity* is generally used to refer to the amount that a container will hold. Standard units of capacity include quarts, gallons, liters, and milliliters. The term *volume* can be used to refer to the capacity of a container but is also used for the size of solid objects. Standard units of volume are expressed in terms of length units, such as cubic inches or cubic centimeters.

Comparison Activities

One method of comparing capacities is to fill one container with something and then pour this amount into the comparison container. Students and adults can have difficulty accurately predicting which of two containers will hold more. The apparent volumes of solid objects are sometimes misleading, and a method of comparison is also difficult. To compare the volumes of solids, such as a ball and an apple, a displacement method must be used. You can have students predict which object has the smaller or greater volume and then place the object in a measuring cup or beaker filled with water to see how much the water rises.

Surface Area and Volume

Just as students should understand the relationship between perimeter and area (see Activity 14.11), they should understand the relationship between volume and surface area, as in Activity 14.16.

Activity 14.16

CCSS-M: 5.MD.C.5a; 5.MD.C.5b; 6.G.A.2; 7.G.B.6

Fixed Volume: Comparing Prisms

Give each pair of students a supply of centimeter cubes or wooden cubes. If you have ELLs, provide a visual of a rectangular solid, labeling all the key words they will need for the lesson (*length*, *width*, *height*, *surface area*, *cube*, *volume*, *side*). Ask students to use 64 cubes (or 36, if you prefer) to build different rectangular prisms and record in a table the surface area for each prism formed. Then ask students to describe any patterns they notice. In particular, what happens to the surface area as the prism becomes less like a tall, skinny box and more like a cube? (See Expanded Lesson: Fixed Volume for details.)

Another highly engaging way to compare surface area and volume is through the demonstration described in Activity 14.17.

Activity 14.17

CCSS-M: 6.G.A.4; 7.G.B.7

Which Silo Holds More?

Take two sheets of construction paper. Make a tube shape (cylinder) from one sheet by taping the two long edges together. Make a shorter, fatter cylinder from the other sheet by taping the short edges together. Ask, "If these were two silos, would they hold the same amount, or would one hold more than the other?" Survey students first to see how many select each option. To test the conjectures, use a filler, such as beans. Place the skinny cylinder inside the fat one. Fill the inside tube and then lift it up, allowing the filler to empty into the fat cylinder. Surprising to many, the volumes are different, even though the sizes of the papers holding the filler are the same. Ask students to investigate why this is true.

The goal of these activities is for students to realize that volume does not dictate surface area, but that there is a relationship between surface area and volume, just as there is between perimeter and area—namely, that cube-like prisms have less surface area than long, narrow prisms with the same volume.

Tools for Volume and Capacity Measures

Two types of units can be used to measure volume and capacity: solid units and containers. Solid units are objects like wooden cubes or old tennis balls that can be used to fill the container being measured. The other type of unit model is a small container that can be filled; the filler is then poured repeatedly into the container being measured. The following are a few examples of units that you might want to collect:

- Liquid medicine cups
- Plastic jars and containers of almost any size
- Wooden cubic blocks or blocks of any shape (as long as you have a lot of the same size)
- Styrofoam packing peanuts (which still produce conceptual measures of volume despite not packing perfectly)

Activity 14.18

CCSS-M: 5.MD.C.4; 5.MD.C.5; 6.G.A.2; 7.G.B.6

Box Comparison—Cubic Units

Provide students with a pair of small boxes that you have folded up from poster board (Figure 14.17). For units, use dimensions that match the cube blocks that you have. Students are given two boxes, one block, and an appropriate ruler. (If you use 2-cm cubes, make a ruler with the unit equal to 2 cm.) Ask students to decide which box has the greater volume or if the volumes of the two boxes are the same.

Continue to explore, giving them dimensions of other boxes, such as jewelry boxes. Here are some suggested box dimensions ($L \times W \times H$):

6 in. × 3 in. × 4 in.   5 in. × 4 in. × 4 in.   3 in. × 9 in. × 3 in.   6 in. × 6 in. × 2 in.

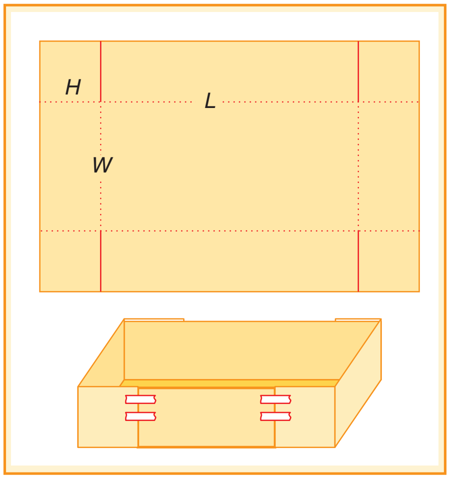
Students should use words, drawings, and numbers to explain their conclusions. Repeat with boxes with fractional values. For example, ask students to estimate and then determine which of the following shipping boxes has the greatest and least volume:

$$\frac{1}{4}\,ft \times 3\,ft \times 2\frac{1}{2}\,ft \quad 4\,ft \times \frac{3}{4}\,ft \times \frac{1}{2}\,ft$$

$$\frac{5}{12}\,ft \times 2\,ft \times \frac{3}{4}\,ft \quad 2\,ft \times \frac{1}{4}\,ft \times 3\,ft$$

Figure 14.17

Make small boxes by starting with a rectangle and drawing a square on each corner. Cut on the solid lines and fold the box up, wrapping the corner squares to the outside and taping them to the sides.



Instruments for measuring capacity are generally used for small amounts of liquids or pourable materials, such as rice or water. These tools are commonly found in kitchens and laboratories. Students should use measuring cups to explore recipes for foods, papier-mâché, or Oobleck for science experiments (google "making Oobleck" for recipes).

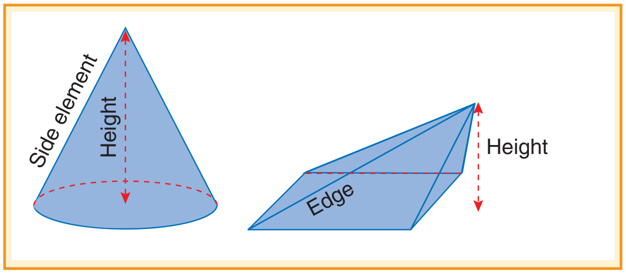
Developing Formulas for Volumes of Common Solid Shapes

The relationships in the formulas for volume are completely analogous to those for area. A common error that is repeated from two- to three-dimensional shapes occurs when students confuse the meaning of *height* and *base* in their use of formulas (see Figure 14.18). The base of the figure can be any flat surface of a figure.

As you read, notice the similarities between rectangles and prisms, between parallelograms and slanted (oblique) prisms, and between triangles and pyramids. Not only are the formulas related, but the processes for developing the formulas are similar.

Figure 14.18

Heights of three-dimensional figures are not always measured along an edge or surface.



Teaching Tip

A way to help students think about height (versus side length) is to picture an upright stick entering through a doorway: The *height* would be the height of the shortest door the stick could pass through without bending.

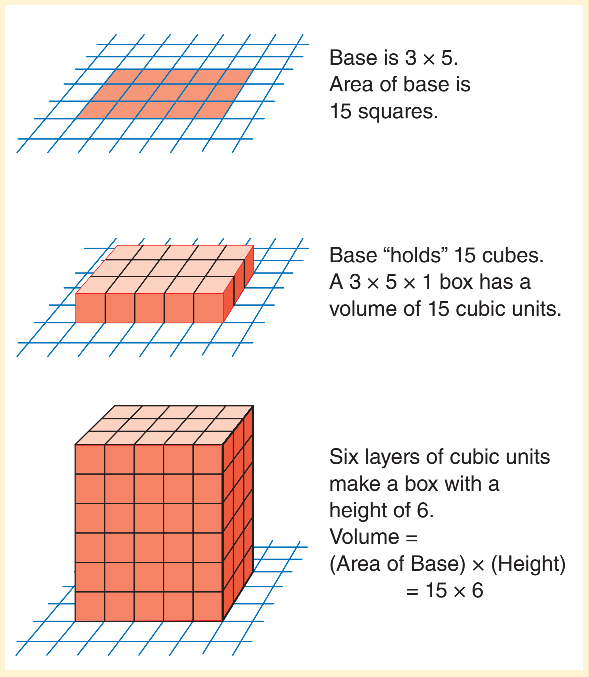
Volume of Cylinders

A *cylinder* is a solid with two congruent, parallel bases and sides with parallel elements that join corresponding points on the bases. There are several special classes of cylinders, including *prisms* (with polygons for bases), *right prisms*, *rectangular prisms*, and *cubes* (see Chapter 13) (Zwillinger, 2011). Interestingly, all of these solids have the same volume formula, and that one formula is analogous to the area formulas for rectangles and other parallelograms.

Review Activity 14.18, "Box Comparison—Cubic Units." The development of the volume formula from this box exploration is parallel to the development of the formula for the area of a rectangle (Figure 14.19). The *area* of the base (instead of *length* of the base for rectangles) determines how many *cubes* can be placed on the base to form a single unit—a layer of cubes. The *height* of the box then determines how many of these *layers* will fit in the box, just as the height of the rectangle determined how many *rows* of squares will fill the rectangle.

Figure 14.19

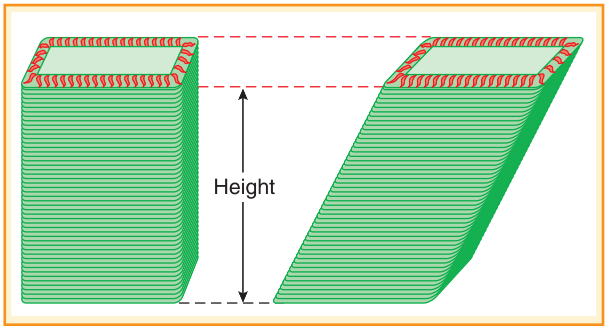
Volume of a right prism: area of the base × height.



Recall that a parallelogram can be thought of as a slanted rectangle, as was illustrated with dynamic geometry software (see Figure 14.14). Show students a stack of three or four decks of playing cards (or a stack of books). When stacked straight, they form a rectangular solid. The volume, as just discussed, is $V = A \times H$, with A equal to the area of one playing card. If the stack is now slanted to one side (Figure 14.20), what will the volume of this new figure be? Students should be able to argue that this figure has the same volume (and same volume formula) as the original stack.

Figure 14.20

Two prisms with the same base and same height have the same volume.



What if the cards in this activity were some other shape? If they were circular, the volume would still be the area of the base times the height; if they were triangular, still the same. The conclusion is that the volume of *any* cylinder is equal to (area of the base) × height.

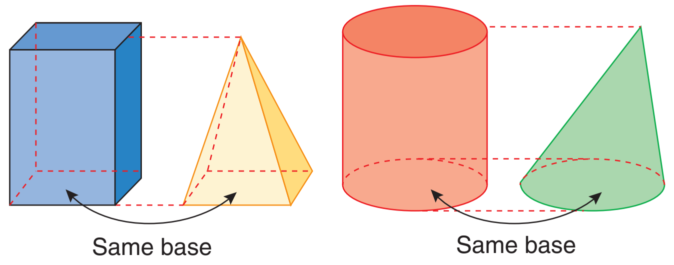
Volumes of Cones, Pyramids, and Spheres

Understanding formulas for volumes for these shapes is an eighth-grade expectation. Recall that when a parallelogram and a triangle have the same base and height, the areas are in a 2-to-1 relationship. Interestingly, the relationship between the volumes of a cylinder and a cone with the same base and height is 3 to 1. That is, *area* is to *two-dimensional* figures what *volume* is to *three-dimensional* figures. Furthermore, triangles are to parallelograms as cones are to cylinders.

To investigate the relationship, use plastic models of these related shapes (e.g., translucent Power Solids). Have students estimate the number of times the pyramid will fit into the prism. Then have them test their predictions by filling the pyramid with water or rice and emptying the filler into the prism. They will discover that exactly three pyramids will fill a prism with the same base and height (Figure 14.21). The volume of a cone or pyramid is exactly one-third the volume of the corresponding cylinder with the same base and height.

Figure 14.21

Comparing the volume of a prism with the volume of a pyramid, and comparing the volume of a cylinder with the volume of a cone.



The volume of a pyramid or cone is one-third the volume of a prism or cylinder with the same base and same height.

Using the same idea of area of the base times height, it is possible to explore the volume of a sphere ($\frac{2}{3}$ of the volume of a cylinder with the same height and base as identified through pouring water into the cylinder with the same height and base). The height of the matching cylinder is the sphere's radius doubled ($2r$). The volume of the cylinder is the area of the base (πr^2) × height ($2r$). So we find that the volume of the corresponding sphere is $\frac{2}{3}(2\pi r^3)$ or $\frac{4}{3}\pi r^3$. Beyond three-dimensional shapes and pouring water to visibly see this relationship, students can create solids using dough or clay. This has the added benefit of engaging students in measuring the dimensions accurately and applying the formulas.

Standards for Mathematical Practice

6 Attend to precision.

The connectedness of mathematical ideas can hardly be better illustrated than with the connections of all of these formulas to the single concept of *base times height*. A conceptual approach to the development of formulas helps students understand that they are meaningful and efficient ways to measure different attributes of the objects around us. After developing formulas in conceptual ways, students can derive formulas from what they already know. Mathematics does make sense!

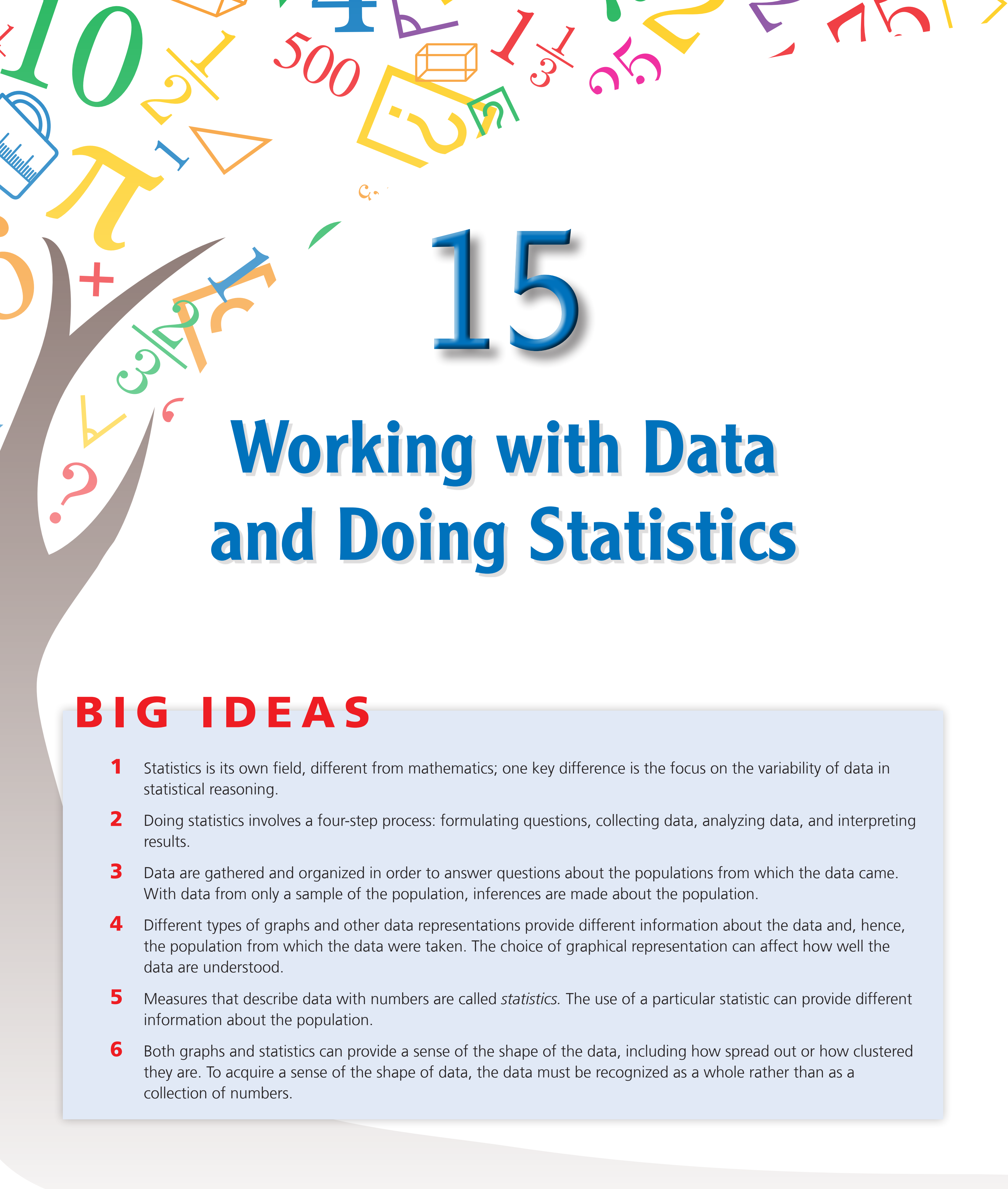
Literature Connections

Holes *Sachar, 1998*

In this clever chapter book, boys are in a prison camp where they are forced to dig holes each day that are 5 feet deep and 5 feet in diameter—the length of their shovels. One of the boys (XRay) has his own shovel that he claims is a fraction of an inch shorter. Students can explore the volume of the holes the boys dig with 5 feet shovels and compare that the volume of the hole that XRay digs if his shovel is just one-half an inch shorter.

The Librarian Who Measured the Earth *Lasky, 1994*

This biography about Eratosthenes offers interesting highlights of his life, focusing on how he questioned and eventually found the measurement for the circumference of the earth. The mathematics that Eratosthenes used can prompt an investigation into the relationship between central angles of a circle and its circumference. For example, students can measure the central angle and the arc length it creates and then estimate the circumference of the circle.



15

Working with Data and Doing Statistics

BIG IDEAS

1 Statistics is its own field, different from mathematics; one key difference is the focus on the variability of data in statistical reasoning.

2 Doing statistics involves a four-step process: formulating questions, collecting data, analyzing data, and interpreting results.

3 Data are gathered and organized in order to answer questions about the populations from which the data came. With data from only a sample of the population, inferences are made about the population.

4 Different types of graphs and other data representations provide different information about the data and, hence, the population from which the data were taken. The choice of graphical representation can affect how well the data are understood.

5 Measures that describe data with numbers are called *statistics.* The use of a particular statistic can provide different information about the population.

6 Both graphs and statistics can provide a sense of the shape of the data, including how spread out or how clustered they are. To acquire a sense of the shape of data, the data must be recognized as a whole rather than as a collection of numbers.

Graphs and statistics bombard the public in areas such as advertising, opinion polls, population trends, health risks, and the progress of students in schools. We hear that the average amount of rainfall this summer is more than it was last summer, or that the average American family consists of 3.19 people. We read in U.S. Census Bureau news that in June 2016 the median sales price of a new house was $306,700 and the mean sales price was $358,200 (census.gov, 2016). Knowing these statistics should raise an array of questions: How were these data gathered? What was the purpose? Why are the median and the mean for home sales more than $50,000 different, and which one is more helpful to me?

Statistical literacy is critical to understanding the world around us, essential for effective citizenship, and vital for developing the ability to question information presented in the media (Shaughnessy, 2007). Misuse of statistics occurs even in trustworthy sources like newspapers, where graphs are often designed to exaggerate a finding. In elementary school, students learn how data can be categorized and displayed in various graphical forms (grades K–1). They collect and organize sets of data, using frequency tables, bar graphs (horizontal and vertical), line plots (including fractional units), and pictographs to display and analyze data (grades 2–5) (CCSSO, 2010). Statistical thinking is also embedded in the first Mathematical Practice, which states, “Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends.” (CCSSO, 2010, p. 6). Based on recommendations in the *Common Core State Standards*, the study of statistics is a priority throughout middle school, as summarized here:

Grade 6: Recognize a statistical question; explore and understand measures that describe the distribution of data, including center, spread, and overall shape; display numerical data in plots and summarize numeric data sets in relation to their context.

Grade 7: Understand and use random sampling as a means of learning about a population; and extend understanding of single data distributions to informally compare two data distributions.

Grade 8: Investigate patterns in bivariate data through examining lines, scatter plots, and tables (note the strong connection to algebra).

What Does It Mean to Do Statistics?

Doing statistics is, in fact, a different process from doing mathematics. This is a notion that has received much attention in standards documents and research (Burrill & Elliott, 2006; Franklin et al., 2005; Shaughnessy, 2003). As Richard Scheaffer (2006), past president of the American Statistics Association, noted:

> Mathematics is about numbers and their operations, generalizations and abstractions; it is about spatial configurations and their measurement, transformations, and abstractions. . . . Statistics is also about numbers—but numbers in context: these are called data. Statistics is about variables and cases, distribution and variation, purposeful design or studies, and the role of randomness in the design of studies, and the interpretation of results. (Scheaffer, 2006, p. 310–311)

Statistical literacy is a life skill as necessary as being able to read. We interpret data every day and base important life decisions on that data. This section describes some of the big ideas and essential knowledge regarding statistics and explains a general process for doing statistics.

Is It Statistics or Is It Mathematics?

Statistics and mathematics are two different fields; however, questions designed to assess statistics understanding are often more focused on a computational skill than on statistical reasoning, as shown by the following examples (Scheaffer, 2006).

Read the following questions, and label each as "doing mathematics" or "doing statistics."

1. The average weight of 50 prize-winning tomatoes is 2.36 pounds. What is the combined weight, in pounds, of these 50 tomatoes? (This is a NAEP [National Assessment of Educational Progress] sample question.)
   a. 0.0472
   b. 11.8
   c. 52.36
   d. 59
   e. 118
2. Joe had three test scores of 78, 76, and 74, whereas Mary had scores of 72, 82, and 74. How did Joe's average (mean) compare with Mary's average (mean) score? (This is a TIMSS [Trends in International Mathematics and Science Study] eighth-grade released item.)
   a. Joe's was 1 point higher.
   b. Joe's was 1 point lower.
   c. The two averages were the same.
   d. Joe's was 2 points higher.
   e. Joe's was 2 points lower.
3. The following table gives the times each of three girls has recorded for seven trials of the 100-m dash this year. Only one girl may compete in the upcoming track meet. Which girl would you select for the meet and why?

	Race						
Runner	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Suzie	15.2	14.8	15.0	14.7	14.3	14.5	14.5
Tanisha	15.8	15.7	15.4	15.0	14.8	14.6	14.5
Dara	15.6	15.5	14.8	15.1	14.5	14.7	14.5

Stop and Reflect

Which of these involves statistical reasoning? All of them? None of them? What do you consider to be the aspects of a task that make it a statistics-focused task versus a mathematics-focused task?

Questions 1 and 2 test mathematical knowledge. The first requires knowing the formula for averages, but the thinking required is to work backwards through a formula, which is mathematical thinking, not statistical thinking. Similarly, in the second problem, one must know the formula for the mean, but the question is about the computational process of using the formula. In both of these, you might notice that the *context is irrelevant* to the problem. The final question is statistical in nature because the situation requires analysis—graphs or averages might be used to determine a solution. The mathematics here is basic; the focus is on statistics. Notice that the *context is central* to responding to the question, which is an indication that statistical reasoning is involved.

In statistics, the context is essential to analyzing and interpreting the data (Franklin & Garfield, 2006; Franklin et al., 2005; Langrall, Nisbet, Mooney, & Jansem, 2011; Scheaffer, 2006). Looking at the spread, or shape, of the data and considering the meaning of unusual data points (outliers) are determined based on the context.

Teaching Tip

A good way to decide whether the task you are selecting is statistical in nature (as opposed to mathematical) is to ask yourself, "Does the context matter in answering the question posed in this task?"

The Shape of Data

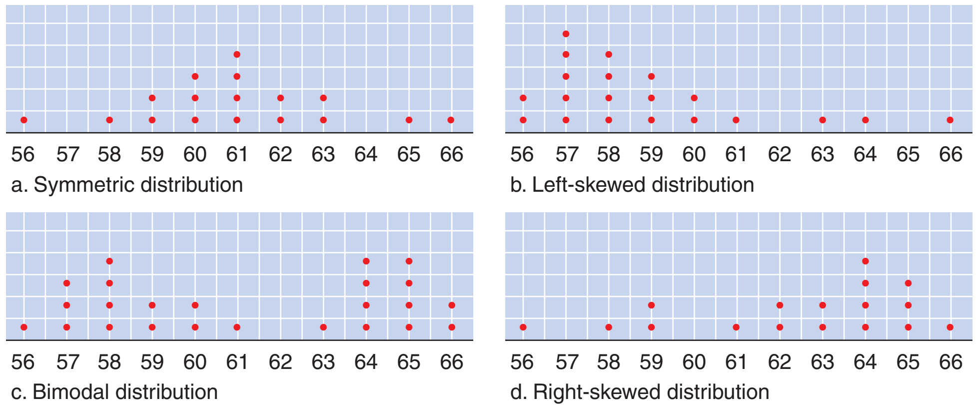
A big idea in data analysis is the shape of the data getting a sense of how the data are spread out or grouped, determining which characteristics of the data set as a whole can be described, and identifying what the data tells us globally for the population from which they are taken. In sixth grade, students must understand that the "set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape" (CCSSO, 2010, p. 45). Graphs, such as dot plots, can illustrate the distributions of data (Kader & Jacobbe, 2013). Figure 15.1 shows four different dot plots (line plots that use dots instead of X's), each showing a different shape of the data.

Different graphing techniques or types of graphs can provide a different snapshot of the shape of the data as a whole. For example, bar graphs and circle graphs (percentage graphs) each show how the data cluster in different categories. The circle graph focuses more on the relative values of this clustering, whereas the bar graph adds a dimension of quantity. The choice of which and how many categories to use in these graphs will result in different pictures of the shape of the data.

Part of understanding the shape of data is being aware of how spread out or clustered the data are. For numeric data, there are statistics that tell us how data are spread or dispersed. In middle school, measures of center include the median and the mean; measures of spread (variability) include range, interquartile range, and absolute deviation. In particular, students must be able to describe the overall shape, noting the variability based on the context of the data that were gathered. This focus on describing the data based on the context is the key to teaching statistics effectively. This is a very different approach from the one offered in many textbooks, in which students work primarily on skills (e.g., "find the mean" and "make

Figure 15.1

Dot plots showing different distributions (shape) of data.



a scatter plot") for the sake of the skills. Instead, these skills must be embedded in the collection and analysis of meaningful data with the purpose of having students be able to describe something about the population from which the data came.

Figure 15.2

The process of doing statistics.

1. Develop Statistical Questions
   - Use interesting, relevant questions for the students to explore
   - Engage students in generating questions that can be answered using statistics
   - Focus on questions that include variability
   - Consider what data will answer the question:
     - Include questions that generate numerical data
     - Include data that can be gathered within the classroom or school
2. Gather and Organize Data
   - Decide who will be asked the question(s) or what source(s) will be used for gathering data
   - Outline a plan for collecting the data
   - Choose the best way to record the data
   - Carry out the data collection plan
3. Choose a Data Analysis Plan
   - Decide which representation will best tell the story related to your question(s)
   - Prepare those data displays (by hand or using technology)
   - Select ways to interpret the data (combining or comparing)
   - Carry out the analysis plan
4. Interpret the Findings
   - Use representations from #3 to answer the question(s) from #1
   - Look at the shape of the data if in a graphical representation
   - Explore questions such as:
     - What does our data tell us about our class? And, what doesn't it tell us?
     - What might we infer?
     - What new questions might we have?
   - Look for factual information as well as inferences that go beyond the data

The Process of Doing Statistics

Just as learning to divide fractions involves much more than the procedure, statistics is much more than being able to create a graph or calculate the mean. As the discussion of the shape of data suggests, if students are to be engaged *meaningfully* in learning and doing statistics, they should be involved in the full process, from asking and defining questions to interpreting results. *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A Pre-K–12 Curriculum Framework* describes a four-step process: (1) develop a statistical question, (2) gather data, (3) analyze data, and (4) interpret findings. Figure 15.2 presents these steps, along with activities within each step, serving as an advanced organizer for this chapter.

Formulating Questions

Statistics begins with asking and answering questions about our world. Data collection should be for a purpose, to answer a question. Then the analysis of data actually adds information about some aspect of the world, just as political pollsters, advertising agencies, market researchers, census takers, wildlife managers, medical researchers, and hosts of others gather data to answer questions and make informed decisions.

Students should be given opportunities to generate their own questions, decide on appropriate data to help answer these questions, and determine methods of collecting the data (CCSSO, 2010). Whether the question is teacher-initiated or student-initiated, students should engage in conversations about how well defined the question is. For example, if students want to know how many shoes a typical sixth-grader owns, questions may arise as to whether they should count bedroom slippers and flip-flops. When students formulate the questions, the data they gather become more meaningful. How they organize the data and the techniques for analyzing them have a purpose. Often statistical questions will arise naturally in a discussion, or in other classes (e.g., social studies, science). The next two sections suggest many ideas.

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Classroom Questions

Middle school students are naturally interested in their peers! This may include their peers' interests/hobbies, families, their likes and dislikes, and so on. Begin with questions that can be answered by each class member contributing one piece of data. Here are some ideas:

- *Favorites.* TV shows, games, movies, ice cream, video games, sports teams, music

- *Numbers*. Number of pets or siblings, hours spent watching TV or sleeping, bedtime, time spent on the computer
- *Measures*. Height, arm span, area of foot, long-jump distance, shadow length, seconds to run around the track, minutes spent traveling to school

Teaching Tip

When a "favorite" is likely to have lots of possibilities, restrict the number of choices and have students select from a list.

Beyond One Classroom

The questions in the previous section are designed for students to contribute data about themselves. These questions can be expanded by asking, "How would this result compare with that of another class or of a nonstudent group?" Comparison questions are a good way to help students focus on the data they have collected and the variability within those data (Russell, 2006). In seventh grade, students begin to compare two populations and observe differences between them (CCSSO, 2010). Students might examine questions where they compare the responses of boys versus girls, adults or teachers versus students, or categories of full-time workers compared to college students. These situations involve issues of sampling and making generalizations and comparisons. In addition, students can ask questions beyond the classroom. Newspapers suggest all sorts of data-related questions. For example, how many full-page ads occur on different days of the week? What types of stories are on the front page? Which comics are really for kids, and which are not? Discussions about communities provide a good way to integrate social studies and mathematics.

Activity 15.1

CCSS-M: 6.SP.A.1; 7.SP.B.3

Who's in Our Village?

The picture book *If the World Were a Village: A Book About the World's People* provides an excellent opportunity to compare class data or school data with data from the wider population in the world. The book explores global wealth, culture, language, and other influences, providing statistics in the adapted case of the world being a village of 100 people. Read the book to the class (or at least several of the comparisons). Ask students if they think the data in the book for particular topics represents their class (or school). Gather data and compare. Then ask students what else they think might be interesting data about our world that could be added to the pages of this book. If you have students who are from other countries, or who have lived in other countries, they can provide valuable cultural insights as well as ideas for other statistical questions. (See Riskowski, Olbricht, & Wilson, 2010, for details on a project exploring concepts of statistics with 100 students.)

Science is about inquiry, and therefore planning statistics units are excellent times for interdisciplinary teaching. For example, consider this list of science-related statistical questions:

- How many plastic bottles or aluminum cans are placed in the school's recycling bins over a given week?
- How many days does it take for different types of bean, squash, and pea seeds to germinate when kept in moist paper towels?
- Which brand of bubble gum will give you the largest bubble?
- Do some liquids expand more than others when frozen?
- Does a plant grow faster when watered with water, soda, or milk?

As noted earlier, a distinguishing feature of statistics is that the context is front and center. Therefore, it is particularly important that the context be culturally meaningful.

Teaching Tip

Ask students to turn in a note card with three statistical questions they would like to investigate during the year. This provides an anonymous or confidential way in which students can share interests and curiosities that can then be connected to statistics.

Culturally meaningful contexts create a supportive classroom environment (McGlone, 2008). This can be as basic as asking about a favorite family meal or game, or it can include an exploration of family customs. For such questions to lead to a supportive classroom environment, you must share the results in a way that helps others in the class appreciate the unique features of the lives of their classmates.

Activity 15.2

CCSS-M: 6.SP.A.1; 6.SP.B.4; 6.SP.B.5

What Can We Learn about Our Community?

This activity plays out over several days. First, ask students to turn in a notecard with three statistical questions they would like to investigate during the year. This could be assigned as homework, with students' families helping to brainstorm ideas. Collect these ideas. For students with disabilities, consider providing categories from which they can write questions and/or examples of possible questions. When you have time to start the investigation, take one question from the set. As a class, refine the question to one that can be answered using statistics. Examples of questions include:

- How many different kinds of restaurants or stores are in our community (fast food restaurants versus "sit down" restaurants; Italian, Mexican, or American; convenience stores, grocery stores, clothing stores, variety stores)?
- How many responses are made by local firefighters each month? How many different types of responses are made by local firefighters each month (fire, medical, hazardous, public service)? (Data can usually be found on websites of local institutions.)
- How many state and local government officials are elected by voters?

Discuss ways to gather the data. Set up a plan and deadline for gathering the data. When students bring in the data, encourage students to select and use data displays (e.g., circle graph/pie chart, line plot, etc.). Invite students to share. Return to the question and ask, "What does our data tell us about ______?" Consider how the different data displays communicate the answer to this question. See **Expanded Lesson: Using Data to Answer a Question** for a full lesson of this activity.

Students may need help designing questions that can be answered using statistics. These are questions that include variability and for which data can be gathered. Providing examples and nonexamples can help students, in particular students with disabilities, focus on the elements of an appropriate statistical question. As you read each, decide which are examples of statistical questions, and which are nonexamples:

1. How much change do I have in my pocket?
2. What is the typical amount of loose change a person carries in his or her pocket?
3. What cereal is most healthy?
4. What reasons do people use in selecting gum (e.g., taste, cost, bubble-making quality, long-lasting flavor, good breath)?
5. How long do different kinds of gum keep their flavor?
6. Which store has the best prices?
7. Where will you buy shampoo?

Did you identify the nonexamples of statistical questions? They are questions 1, 3, and 7. Question 3 could be adapted to a statistical question with a more specific focus on what is meant by "healthy." Similarly, question 7 is very broad and would have to be focused in order to collect the data needed to answer it. Facilitating discussions with students about examples and nonexamples, as well as having them develop their own questions, will improve their ability to generate appropriate statistical questions.

Collecting Data

How to collect good data is an important (and sometimes skipped) part of the discussion as students learn statistics. Students may start by just hand-raising or counting events, then move to using ballots with limited and then unlimited response options (Hudson, Shupe, Vasquez, & Miller, 2008). If you do not gather representative data, then you cannot draw valid inferences about the population. For example, if you wonder what movie will be the most popular for a Friday night school event and poll just the sixth-graders or ask several groups of girls at lunch, the data gathered may not represent what the response from the whole school would be.

Often, we think of data collection as a survey, but data can be collected through observation. For example, you can explore the question, "What kinds of cars drive by our school?" You can collect data at different times during the day (e.g., each period, students can record data at the beginning or end of class) to count the types or colors of vehicles, or how many pass in a designated amount of time (e.g., in one minute). Students can also conduct observational data collection events on field trips (Mokros & Wright, 2009) and during evening or weekend activities with their families.

There are two types of data that can be collected—categorical and numeric. Categorical data is (as the name implies) data grouped by labels (categories), such as favorite vacations, colors of cars in the school parking lot, and the most popular suggestion for a mascot for the middle-school team. The mode is often used to describe categorical data, telling what happened most often. (Mode is not mentioned in the *Common Core State Standards*.)

Teaching Tip

Have students look at several sets of data, some categorical and some numeric. Ask them to select one of the measures of central tendency (mode, median, or mean) that they think best describes the center of the data. Students may start to try to find median or mean of categorical data and realize that it is impossible!

Numeric data, by contrast, count things or measures on a continuous scale. Numeric data are ordered numerically, like a number line, and can include fractions and decimals. Data of this type include how many miles to school, the temperature in your town over a one-week period, and the weight of students' backpacks. Students need to explore explicitly the idea that statistical measures such as mean and median involve numeric data (Leavy, Friel, & Mamer, 2009).

Sampling

When asking a question about a small population, like your class, data can be gathered on everyone. But statistics generally does not involve gathering data from the whole population and instead uses a representative sample. CCSS-M identifies learning about using sampling in order to make inferences about a population as a critical area for seventh-graders (CCSSO, 2010).

Sampling must take into consideration variability. For example, a poll on favorite TV shows will produce different answers from a survey of seventh-graders than from a group of teachers or third-graders. It may also vary for girls and for boys or culturally. Answers also may vary based on the day the question is asked or whether a particular show has been recently discussed. To help students determine whether they have identified a representative sample, ask these questions:

- "What is the population for your question?"
- "Who or what is the subject of your question?"

Then ask students to consider how they will gather data that will include representatives across that population. For example, if they are hoping to learn the movie choice for a seventh-grade

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

movie night at the school, they need to poll girls and boys across the seventh-grade teams within the middle school. So, provide opportunities for students to justify whether or not the data is representative and to critique the explanation of others as they describe a representative sample.

Even when it may appear that a sample is representative, it may not be. Unintentional biases can occur, and we cannot always know what subsets might exist within a population. Therefore, *random sampling* is used in statistics. It increases the validity of the results and therefore allows more confidence in being able to make inferences about the population. Activity 15.3 can help students develop an awareness of the importance of sampling.

Activity 15.3

CCSS-M: 6.SP.A.1; 6.SP.A.2; 6.SP.B.3; 7.SP.A.1

How Do We Compare?

Using Scholastic's *Book of Lists: Fun Facts, Weird Trivia, and Amazing Lists on Nearly Everything You Need to Know!* by James Buckley and Robert Stremme, or a similar book or online resource, find a list based on sampling a group of people. Use the How Do We Compare? Activity Page. Read the question to the students and gather data by using the class as the sample. Ask students, "Do you think our class will be a representative sample of the population targeted in this question?" Ask students to offer why the class might or might not be a representative sample. Consider what countries or communities are in the population. Gather data from the class. Compare the two data sets. If appropriate, create a stem-and-leaf plot or other data display to compare the two data sets. After displaying the two sets, ask students what technique they think the authors might have used to sample the population.

Although this activity involved people, the term *population* in statistics is used broadly to mean "group" or "subjects of a study." So, the population could be species of plants, insects, or a type of cars. Activity 15.3 can be replicated with other data. For example, you can look up favorite car colors online and then see if cars that pass by the school form a representative sample of all cars.

Using Existing Data Sources

Data do not have to be collected by a survey or by observation; existing data abound in various places, such as the following sources of print and Web data.

Print Resources

Newspapers, almanacs, sports record books, maps, and various government publications are possible sources of data that may be used to answer student questions.

Nonfiction books can be a source of data for middle-school students. *The Scholastic Book of Lists* was mentioned in Activity 15.3. Books on sports, such as *A Negro League Scrapbook* by Carole Boston Weatherford, have very interesting statistics about historical periods that students can explore and compare.

Web Resources

The Internet provides seemingly limitless data that are often accessed simply by typing the related question into a search. Students may be interested in facts about another country as a result of a social studies unit or a country in the news. Olympic records in various events over the years or data related to environmental issues are other examples of topics around which student questions may be formulated. For these and hundreds of other questions, data can be found on the Internet. Below are several websites with a lot of interesting data.

- **The USDA Economic Research Service** site offers wonderful data sets on the availability and consumption of hundreds of foods. Annual per capita estimates often go back to 1909.
- **Google Public Data Explorer** makes large data sets available to explore, visualize, and interpret.
- **IMDb (Internet Movie Database)** offers information about movies of all genres.
- **Better World Flux** provides data related to the progress of countries and the world over the years, highlighting interesting trends and patterns.
- **The CIA World Factbook** provides demographic information for every nation in the world: population, age distributions, death and birth rates, and information on the economy, government, transportation, and geography.
- **The U.S. Census Bureau** has copious statistical information organized by state, county, or voting district.

Analyzing Data: Graphs

Graphs summarize data that have been collected. How data are organized should be directly related to the question that caused you to collect the data in the first place. For example, suppose students want to know how many songs their friends listen to in one day. The question is, "How many songs do the students in our class listen to in one day?" For data collection, you decide to have each student keep track on a school day (e.g., Tuesday) and come to class on Wednesday with his or her own total. Each student records the number on a sticky note that is placed on the board.

Stop and Reflect

If your sixth-grade class had collected these data, what methods might you have suggested that the students use for organizing and graphing them? Is one of your ideas better than others for answering the question about how many songs?

A dot plot (also called a *line plot*) could be used to illustrate the spread and shape of the data. Or a histogram could be created to capture how many students fall within a range of songs listened to (between 0 and 10, 11 and 20, etc.). Or a box plot could be created to box in the middle 50 percent, focusing attention on the center of the data as well as the range. Each of these displays gives a different snapshot of the data and provides different insights into the question posed.

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Creating Graphs

Students should be involved in deciding how they want to represent their data, but they will need to be introduced to what the options are and when each display can and cannot be used. The value of having students construct their own graphs is not so much that they learn the techniques but that they are personally invested in the data and learn how a graph conveys information. Once a graph is constructed, the most important activity is discussing what it tells the people who see it. Discussions about graphs the students have created help them analyze and interpret other graphs and charts they see in newspapers and on TV.

Teaching Tip

Select graphs from newspapers or websites and ask students, "What can you learn from this graph? What do you not know that you wish you knew?" These questions help students focus on what different graphs can and cannot illustrate.

Standards for Mathematical Practice

6 Attend to precision.

Creating graphs requires care and precision, including determining appropriate scales and labels. But the reason for the precision is so that an audience is able to see at a glance the summary of the data gathered for a particular question. Questioning and assessment should focus on how effectively the graph communicates the findings of the data gathered, which includes, but goes beyond, using appropriate labels.

technology note

Computer programs and graphing calculators can provide a variety of graphic displays. Use the time saved by using technology to focus on the discussions about the information that each display provides! Students can make their own selections from among different graphs and justify their choice based on their own intended purposes. The graphing calculator puts data analysis technology in the hands of every student. The TI-73 calculator is designed for middle-grade students. It will produce eight different kinds of plots or graphs, including pie charts, bar graphs, and picture graphs, and it will compute and graph lines of best fit.

The Internet also offers opportunities to explore different graphs. The NCES Kids' Zone "Create a Graph" provides tools for creating five different graphic displays. The NCTM Illuminations "Data Grapher" and "Advanced Data Grapher" allow you to enter data, select which set(s) to display, and choose the type of representation (e.g., bar graph, pie chart, line graph, etc.).

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Analyzing Graphs

Once a graph has been constructed, engage the class in a discussion of what information the graph tells or conveys—the analysis. Questioning and assessment should focus on how effectively the graph communicates the findings of the data gathered. For example, ask, "What can you tell about our class by looking at this graph of the number of songs listened to in one day?" Graphs convey factual information (e.g., there is a wide variability in the number of songs sixth-graders listen to in one day), and they also provide opportunities to make inferences that are not directly observable in the graph (e.g., most sixth-graders listen to between 20 and 30 songs in a day).

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

The difference between *actual facts* and *inferences that go beyond the data* is an important idea in data analysis. Students can examine graphs found in newspapers or magazines and discuss the facts in the graphs and the message that may have been intended by the person who made the graph. Students' conceptual ability to analyze data, draw conclusions, and make interpretations is often weak (Tarr & Shaughnessy, 2007); discussing and analyzing data is a way to support this higher-level thinking skill.

In the sections that follow, we share options for data displays, with less attention given to the displays that students explore in elementary school (e.g., bar graphs and pie charts) and more to the displays that are the focus of middle school (e.g., displays on a number line).

If students ask questions such as "What is your favorite ice cream flavor?" and "What city would you most like to visit?" the data are *categorical*. In this case, bar graphs and circle graphs are good choices for summarizing the data. Bar graphs and pie charts can also be used for numeric data, but they are the only graphs that can be used for nonnumeric data. If students ask, "How many days a week does a person have ice cream?," the data is *numeric*. Numeric data can be represented on a number line (e.g., temperatures that occur over time, height or weight over age, and percentages of test takers scoring in different intervals). Various data displays can be used for numeric data.

Bar Graphs

Bar graphs are learned and used throughout elementary school. The category of bar graphs includes "real" graphs (real objects used in the graph) and picture graphs. In middle school, bar graphs can be used to collect data quickly; as students come into class and see a question posted on the board, they can pull a sticky note and place it on the bar that is their personal response to the question posed.

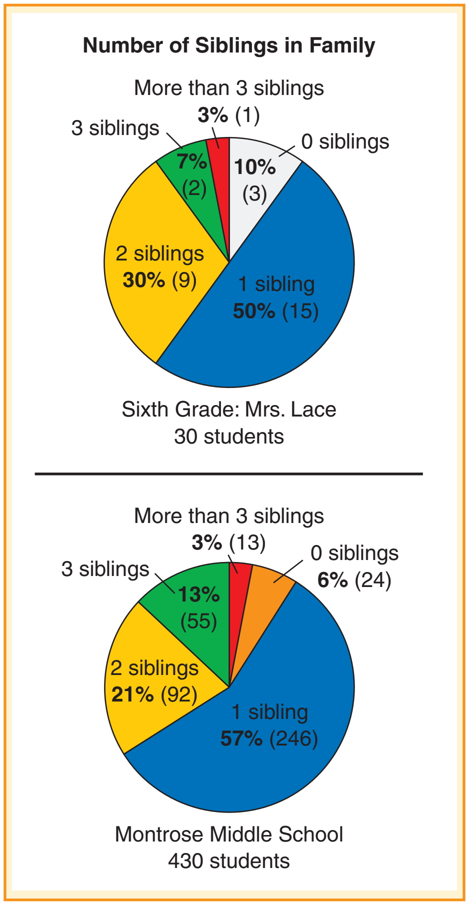
Recall that analyzing data in this way is step 3 of the process of doing statistics. A question is posed (step 1), and data are collected based on the categories that will be graphed (step 2). A class of 25 to 30 students can make a graph in less than 10 minutes. Then the graph can be used as a community builder in a discussion of what is known about the class. Also, the sticky notes can be rearranged in other ways in order to discuss the varying ways data can be summarized in a graph, comparing the pros and cons of different graphs and displays.

Pie Charts/Circle Graphs

Pie charts and *circle graphs* mean the same thing (in fact, sometimes the term pie graph is used); the term circle graph may be more common in curriculum while the term pie chart is used with graphing tools such as Illuminations Data Grapher, TI-73, and electronic spreadsheets. Pie charts are commonly found in newspapers; however, they are less used in statistics because it can be more difficult to make comparisons (angle measures are harder to compare than lengths of bars). Pie charts are useful for comparing two different-sized data sets, for both categorical and numeric data. Figure 15.3, for example, shows a pie chart for classroom data and schoolwide data illustrating the percentages of students with different numbers of siblings. There are several fun and simple ways to make a pie chart, one of which is the focus of Activity 15.4.

Figure 15.3

Pie charts show ratios of part to whole and can be used to compare ratios.



Activity 15.4

CCSS-M: 6.RP.A.1; 6.SP.A.1; 7.SP.A.1

Human Bars to Pies

Determine a question that lends itself to preset categories and is of interest to students, such as a question about their favorite basketball team in the NCAA Tournament's "Elite 8" (or "Final Four"). Post a sign for each team. Ask students to line up in front of the name of the team they like the best. Ask, "About what percent of our class likes [team name]?" Have students estimate the percent for each team (verbally to a partner, or on a note card). Explain that you are going to look at your data again. Ask students to stay in their "row" but to curve their line to form a circle with the rest of the students. Tape the ends of eight (or four) long strings to the floor in the center of the circle, and extend them to the circle at each point where the teams change. Voilà! You now have a life-size circle graph! Ask students if they want to revise their estimates for the percentage of the class that supports each team. If you then place a rational number wheel on the center of the circle, the strings will show the approximate percentages for each part of your graph (Figure 15.4). Note the approximate percentages for each team. Discuss with students the pros and cons of the bar and pie graphs.

Standards for Mathematical Practice

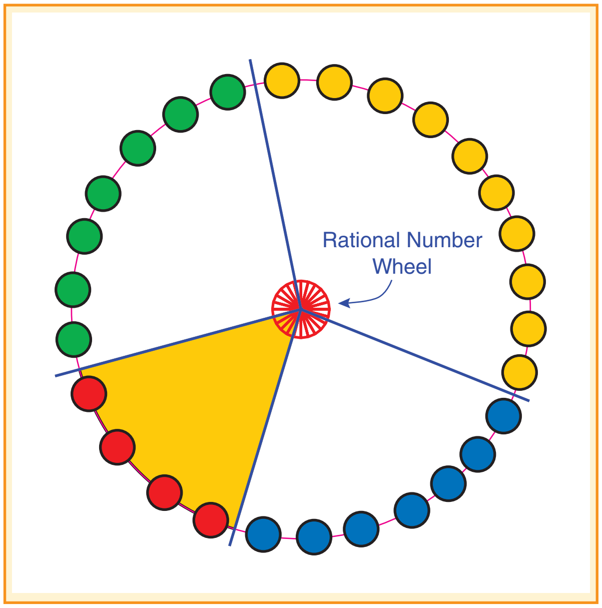
1 Make sense of problems and persevere in solving them.

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Figure 15.4

A human pie chart: Students are arranged in a circle, with string stretched from the center to show the sections.



Activity 15.4 can be done on a smaller scale with different-colored sticky notes or paper clips. Each category is represented by a different color. Then, tape the sticky notes into strips or hook the paperclips, and attach each same-colored strip together end to end. Next, tape (hook) the two ends together to form a circle. Estimate where the center of the circle is, draw lines to the points where the different bars meet, and trace around the full loop.

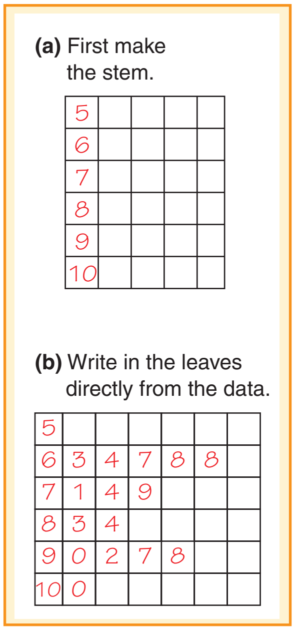
Creating pie charts can be a good connection to percents and proportional reasoning. The numbers of data in each category are added to form the total or whole. Then students find the percent of each part. It is an interesting proportional problem for students to convert between percents and degrees because the one is out of 100 and the other is out of 360. It is helpful to start students with obvious values like 50 percent, 25 percent, and 10 percent before moving to more difficult values. A ratio table with one row for percent and one row for degrees can serve as an important tool to help students reason.

Stem-and-Leaf Plots

Stem-and-leaf plots (sometimes called *stem plots*) are a form of bar graph in which numeric data are graphed and displayed as a list. By way of example, consider the National League baseball teams' total wins for 2015:

Figure 15.5

Constructing a stem-and-leaf plot for 2015 National League MLB wins.



(a) First make the stem.

5				
6				
7				
8				
9				
10				

(b) Write in the leaves directly from the data.

5						
6	3	4	7	8	8	
7	1	4	9			
8	3	4				
9	0	2	7	8		
10	0					

National League - West	Number of Wins	National League - Central	Number of Wins	National League - East	Number of Wins
Los Angeles Dodgers	92	St. Louis Cardinals	100	New York Mets	90
San Francisco Giants	84	Pittsburgh Pirates	98	Washington Nationals	83
Arizona Diamondbacks	79	Chicago Cubs	97	Miami Marlins	71
San Diego Padres	74	Milwaukee Brewers	68	Atlanta Braves	67
Colorado Rockies	68	Cincinnati Reds	64	Philadelphia Phillies	63

If the data are to be grouped by tens, list the tens digits to form the "stem," as shown in Figure 15.5(a). Next, write the ones digits next to the appropriate tens digits, as shown in Figure 15.5(b), ordered from smallest to largest, to form the "leaves." The result shows the shape of data, indicating where the data cluster and where the outliers are. Furthermore, every piece of data can be retrieved from the graph.

To compare two sets of data, create a common stem and extend the leaves in opposite directions (Figure 15.6). The stem in this illustration is grouped by fives instead of tens. Using rows grouped by fives instead of by tens illustrates the spread of the data, perhaps particular grades (e.g., B and B+). Determining how to set up the stem-and-leaf plot depends on the context and on the question being asked. In addition, students can find the range, median, and outliers from the plot.

Teaching Tip

Stem-and-leaf plots are best made on graph paper so that each digit takes up the same amount of space.

Stem-and-leaf plots are also effective for large numbers in data sets. For example, if the data range from 600 to 1,300, the stem can be the numerals from 6 to 13, and the leaves can be made of two-digit numbers separated by commas.

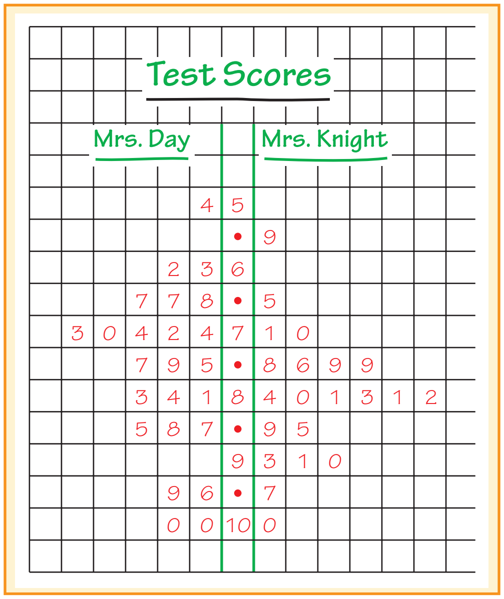
Line Plots and Dot Plots

Line plots and dot plots are among the number line data displays. Both terms are used in the *Common Core State Standards*. Line plots use X's to represent each data point, and dot plots use dots. Line plots are introduced in grade 2 with whole-number units and progress to the display of data in fractions of a unit in grade 5 (CCSSO, 2010, p. 37). Line plots and dot plots are counts of things along a numeric scale on a number line. To make a dot plot, a number line is drawn, and a dot is made above the corresponding value on the line for every corresponding data element. One advantage of a line or dot plot is that every piece of data is shown on the graph. They can therefore be a good first step to creating a box plot or histogram, which are more difficult graphical representations because they do not illustrate each data point (Groth & Bargagliotti, 2012). An example of a dot plot with temperatures is shown in Figure 15.7.

Line and dot plots can and should represent rationale numbers, and measurement is an appropriate context for incorporating fractions. Students might measure and plot, for example, their foot length, their cubit (length of forearm from elbow to finger tips), height of the stack of books they happen to have brought to class, and so on. Data can also be gathered from plants growing, time passing, or weather, as in the next activity.

Figure 15.6

Stem-and-leaf plots can be used to compare two sets of data.

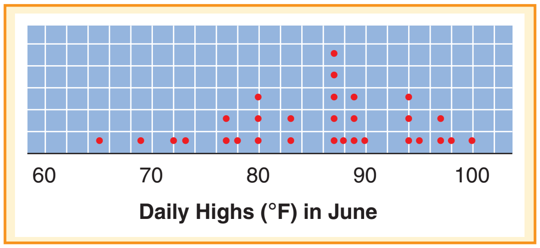


Test Scores

Mrs. Day		Mrs. Knight
4	5	
	•	9
2 3	6	
7 7 8	•	5
3 0 4 2 4	7	1 0
7 9 5	•	8 6 9 9
3 4 1	8	4 0 1 3 1 2
5 8 7	•	9 5
	9	3 1 0
9 6	•	7
0 0	10	0

Figure 15.7

Dot plot summarizing June temperatures.



Activity 15.5

CCSS-M: 5.MD.B.2; 6.SP.B.4; 7.SP.B.3

Storm Plotter

Create a class dot plot (or line plot) and chart the amount of rain (or snow) that falls with each storm (install a rain gauge or you can access the information online). Place the data from each storm on the line/dot plot. This can be done all year and be color-coded by month. As more data is gathered, you can ask, "What do we notice about rain fall in our area? Focus on both variability and center. If you don't want to take months gathering data, instead look up the rain/snow fall in various cities/towns after a storm has passed through and use that to create a storm plot. Students can select locations that represent their home countries or states, or their parents or grandparents. Seventh-graders can explore and compare two different locations.

Line and dot plots also lend themselves to comparisons. For example, if data is gathered from two different groups, separate dot plots can be created, providing a great visual to see the difference in the shape of the data.

Histograms

Although line plots and dot plots are widely used for small data sets, in many real data sets, there is a large amount of data and many different numbers. A dot plot would be too tedious to create and not illustrate the spread of data as effectively. In this case, a histogram is an excellent choice as data are grouped in appropriate intervals. A histogram is a form of bar graph in which the categories are consecutive equal intervals along a numeric scale. The number of data elements falling into a particular interval determines the height or length of that bar. Histograms differ from bar graphs in that bar graphs can be used for categorical data and the bars can be placed in any order without changing the results (Metz, 2010).

Histograms are not difficult in concept but can be challenging to construct: What is the appropriate interval to use for the bar width? What is a good scale to use for the length of the bars? The need for all of the data to be grouped and counted within each interval causes further challenge. Figure 15.8 shows a histogram for the same temperature data used in Figure 15.6. Notice how similar the two displays are in illustrating the spread and clustering of data.

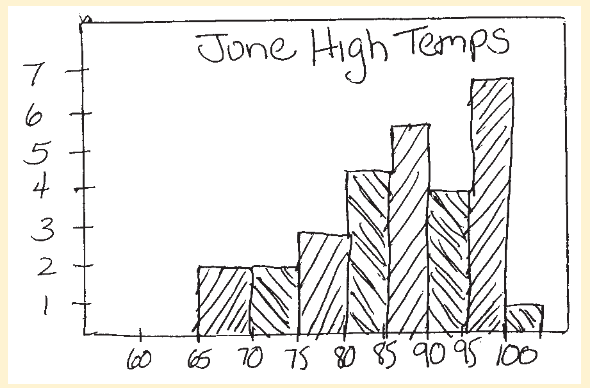


Figure 15.8

Histogram of June high temperatures.

technology note

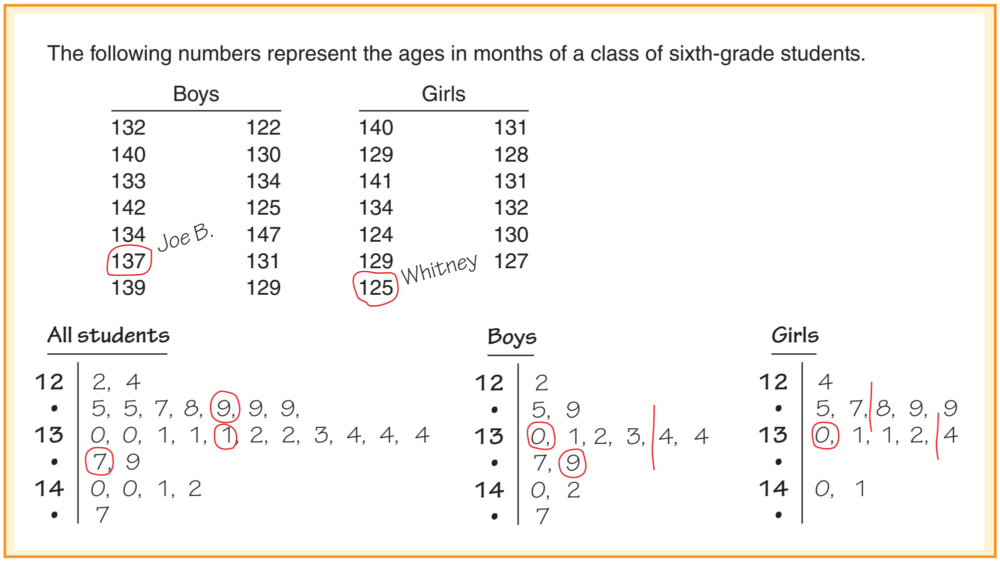
Histograms can be created with graphing calculators, computer software, or online with the NCTM Illuminations Advanced Data Grapher.

Box Plots

Box plots (also called *box-and-whisker plots*) are a method for visually displaying not only the center (median) but also the range and spread of data. Sixth-graders should be able to create and analyze box plots (CCSSO, 2010). In Figure 15.9, the ages in months for 27 sixth-grade

Figure 15.9

Stem-and-leaf plots represent the age of sixth graders. Medians and quartiles are circled or are represented by a vertical bar if they fall between two elements.



The following numbers represent the ages in months of a class of sixth-grade students.

Boys		Girls	
132	122	140	131
140	130	129	128
133	134	141	131
142	125	134	132
134	147	124	130
137 Joe B.	131	129 Whitney	127
139	129	125	

All students

12	2, 4
•	5, 5, 7, 8, 9, 9, 9,
13	0, 0, 1, 1, 1, 2, 2, 3, 4, 4, 4
•	7, 9
14	0, 0, 1, 2
•	7

Boys

12	2
•	5, 9
13	0, 1, 2, 3, | 4, 4
•	7, 9
14	0, 2
•	7

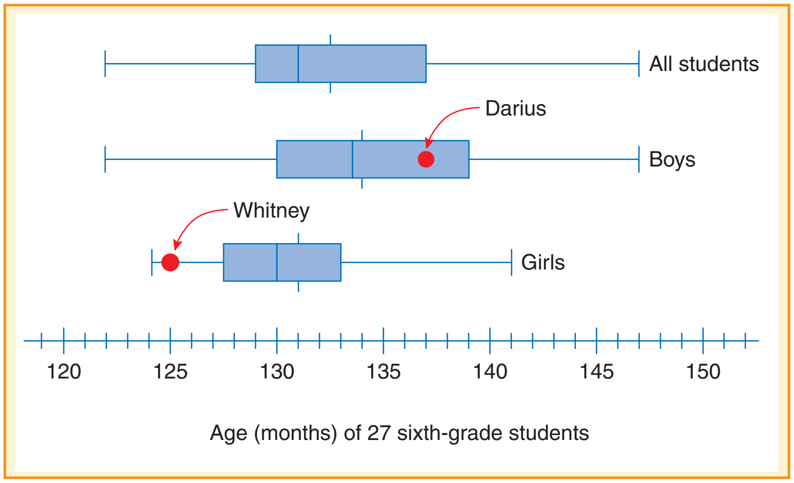
Girls

12	4
•	5, 7, | 8, 9, 9
13	0, 1, 1, 2, | 4
•	
14	0, 1
•	

students are given, along with stem-and-leaf plots for the full class and the boys and girls separately. The stem-and-leaf plot is a good way to prepare for creating a box plot. To find the two quartiles, find the medians of the upper and lower halves of the data. Mark the two extremes, the quartiles, and the median. Then create the box plot on a number line. Box plots for the same data are shown in Figure 15.10.

Figure 15.10

Box plots for sixth-graders (ages in months). In addition to showing how data are distributed, data points of particular interest can be shown.



Each box plot has these three features:

1. A box that contains the "middle half" of the data (the *interquartile range* [IQR]), with one-fourth of the data to the left and one-fourth of the data to the right of the median. The ends of the box are at the *lower quartile* (the median of the lower half of the data) and the *upper quartile* (the median of the upper half of the data).
2. A line is inside the box at the median of the data.
3. Lines (sometimes called the *whiskers*) extending from the left and right ends of the box to the *lower extreme* and *upper extreme* of the data, respectively. Each line, therefore, covers the upper and lower fourths of the data.

Stop and Reflect

Notice that in Figure 15.10, the box for the boys is actually a bit longer than the box for the whole class. How can that be, when there are clearly more students in the full class than there are boys? How would you explain this apparent discrepancy to a class of seventh-graders?

Look at the information these box plots provide at a glance! The box and the lengths of the lines provide a quick indication of how the data are spread out or bunched together. Because the median is shown, this spreading or bunching can be determined for each quarter of the data. The entire class in this example is much more spread out in the upper half than in the lower half. The girls are much more closely grouped in age than either the boys or the class as a whole. The range of the data (the difference between the upper and lower extremes) is represented by the length of the plot, and the extreme values can be read directly. The mean is indicated by the small marks above and below each box. A box plot provides useful visual information to help understand the shape of a data set. Therefore, it is a great choice for looking at data from different disciplines. For example, consider creating a box plot for the age of each president at inauguration (Patterson & Patterson, 2014). This can teach interesting history lessons and the value of using a box plot to analyze data.

To **make a box plot** (**http://www.youtube.com/watch?v=CoVf1jLxgj4**), put the data in order. Next, find the median. This can be done on stem-and-leaf plots. To find the two quartiles, ignore the median itself, and find the medians of the upper and lower halves of the data. Mark the two extremes, the two quartiles, and the median above an appropriate number line. Draw the box and the lines. See also NCTM Illuminations Advanced Data Grapher to create box plots.

Because box plots have so much information and require proportional thinking, students may be challenged to interpret them (Bakker, Biehler, & Konold, 2004). Support making these connections by using contexts that are meaningful and asking questions about the various statistics that are shown on the plot. Understanding of the proportional relationships can be supported through use of percent strips or ratio tables. (See Chapter 11 for more on the use of models to support proportional thinking.)

Remember that a box plot, like any graph, is a tool for learning about the question posed, not an end in itself (McClain, Leckman, Schmitt, & Regis, 2006). Because a box plot offers so much information on the spread and center of the data, much can be learned from careful examination, and particularly from comparing two box plots with related data.

technology note

Graphing calculators and several computer programs draw box plots, making this process even more accessible. The TI-73, TI-84, and TI-Nspire calculators can draw box plots for up to three sets of data on the same axis.

Formative Assessment Note

Students should write about their graphs in a *journal,* explaining what each graph tells and why they selected that type of graph to illustrate the data they have. As you evaluate students' responses, it is important to focus on whether they chose an appropriate representation and have provided a good rationale for its selection that connects back to the context of the data gathered. You might ask a focused question about a graph, such as the previous Stop and Reflect question.

Bivariate Graphs

Standards for Mathematical Practice

4 Model with mathematics.

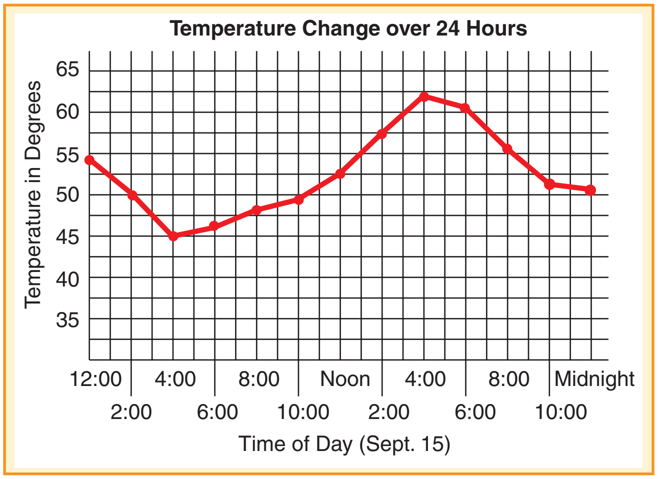
In eighth grade, the focus of statistics is to analyze (CCSSO, 2010). The phrase *bivariate data* may be new to the eighth grade curriculum, but the concept is not. Stated simply, *bivariate* means that two things are varying together (e.g., as the number of people attending the softball game, the number of hot dogs sold at the concession stand). Concepts and activities related to covariation are addressed in Chapter 11 ("Proportional Reasoning") and Chapter 12 ("Exploring Algebraic Thinking, Expressions, and Equations"). In statistics, the focus of bivariate data is on the distribution and related patterns in the covariation. Graphs and tables are effective in displaying these distributions so that patterns or trends are more observable (Kader & Jacobbe, 2013).

Line Graphs

A coordinate axis allows for the plotting of bivariate data. When that data is continuous, a line connects the data points and illustrates the trend in the data. For example, a line graph might be used to show how the length of a flagpole shadow changed from one hour to the next during the day. The horizontal scale would be time, and the vertical scale would be the length of the shadow. Data can be gathered at specific points in time (e.g., every 15 minutes), and these points can be plotted. A straight line can be drawn to connect these points because time is continuous, and data points do exist between the plotted points. See the example in Figure 15.11 for a line graph representing temperature change.

Figure 15.11

Line graph of one day's temperatures.



Scatter Plots

Scatter plots are an emphasis in grade 8 (CCSSO, 2010). Bivariate data can be plotted on a scatter plot, a graph of points on a **Coordinate Grid – Quadrant I** (see Blackline Master 22) with each axis representing one of the two variables. Each pair of numbers from the two sets of data, when plotted, produces a visual image of the data as well as a hint concerning any possible relationships.

Teaching Tip

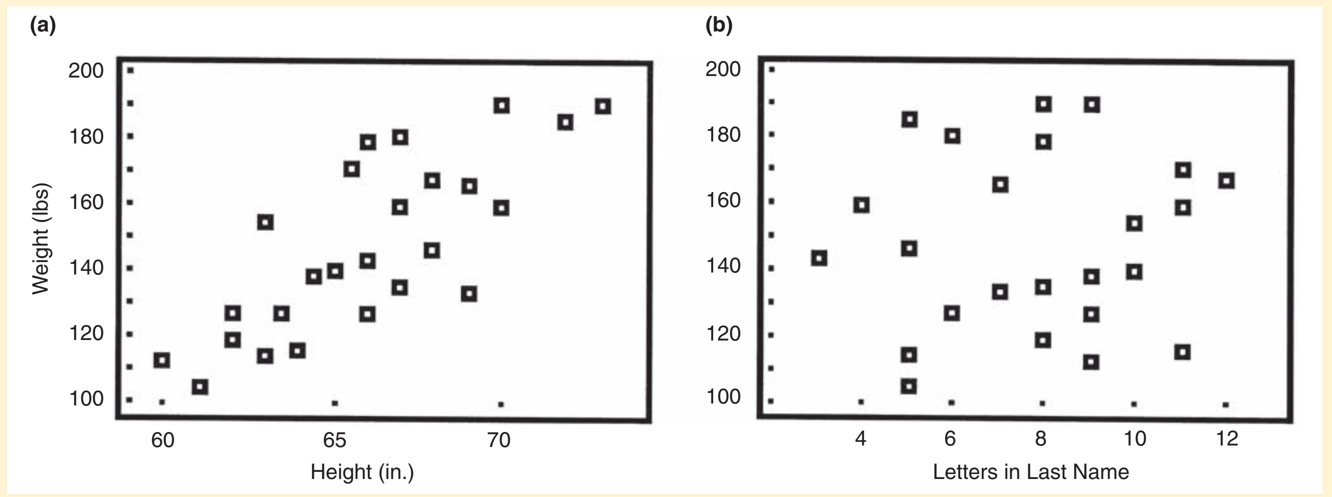
Students tend to connect the dots any time they are plotting points. Instead, they should ask, "Does it make sense to connect the points plotted based on this situation?"

For example, if the following data were gathered from 25 eighth-grade boys: height in inches, weight in pounds, and number of letters in their last name and graphed on scatter plots, you can ask, "Is there a relationship between their height and weight?" and "Is there a relationship between name length and weight?" The two graphs in Figure 15.12 show (a) a scatter plot of height to weight, and (b) a scatter plot of name length to weight.

The scatter plots indicate that there is a relationship in the boys' weights and heights, though there is some variation. But there is no relationship between name length and weight. Encourage students to plot many data sets (using appropriate technology) and look for relationships in the scatter plots, including data sets that suggest linear relationships as well as

Figure 15.12

Scatter plots show potential relationships or lack of relationships.



data sets that indicate no apparent relationship between the variables. Activity 15.6 provides engaging ways to explore bivariate data.

Activity 15.6

CCSS-M: 8.SP.A.1

Is There a Relationship?

Prepare cards with different questions on them about bivariate situations, or use the **Is There a Relationship? Cards** with the examples listed here:

- Distance a toy car rolls and its weight
- Distance a toy car rolls and height of ramp on which it rolls
- Distance a toy car rolls and its position on the ramp when it starts to roll
- Foot length and height
- Shoulder width and height
- Nose length and hand span
- Months of age and height
- Head circumference and wrist circumference
- Minutes watching TV and minutes doing homework
- Number of letters in name and number of times you can say your name in 1 minute

Distribute cards to groups of four students and a **Coordinate Grid** (Blackline Master 22). Ask students to (1) predict whether there is a relationship and (2) determine how they will gather data. Have students gather data and create a scatter plot on paper that they will display to the class. English language learners (ELLs) may be more familiar with metric measures and may benefit from gestures or demonstration to communicate what is being measured. Each group reports on their findings and explains whether they think there is a relationship or not. Other groups listen and determine whether they agree that there is a relationship, and what that relationship is. As an extended experience, have students generate their own questions in which they wonder, "Is there a relationship between *x* and *y*?"

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Best-Fit Lines

Standards for Mathematical Practice

4 Model with mathematics.

If your scatter plot indicates a relationship, it can be simply described in words: "As boys get taller, they get heavier." This may be correct but is not particularly useful. What exactly is the relationship? If I knew the height of a boy, could I predict what his weight might be? Like much of statistical analysis, the value of a statistic is to create a model to predict what has not yet been observed.

The relationship in this case is a ratio between the two measures. If the scatter plot seems to indicate a steadily increasing relationship (as in the height–weight graph) or steadily decreasing relationship, you can find the ratio between the variables by drawing a straight line through the data points that "best" represent the pattern or shape of all of the dots.

Activity 15.7

CCSS-M: 8.SP.A.1; 8.SP.A.2

Spaghetti Lines

Engage students in gathering bivariate data (see list in Activity 15.6). For example, ask students to measure their *foot length* and *their height,* and record the data in a table (posted for the class to see). Give students a copy of a **Coordinate Grid** (Blackline Master

22–Quadrant I). Ask students to plot the coordinates on their own graph, then place a piece of uncooked spaghetti until they feel it is the "Best fit" for the data. Students with special needs may benefit from doing one together to see examples and nonexamples of best fit. Tape it down. Then, ask students to write an equation to represent their line. Invite students to compare and discuss the different equations. For example, ask students to use their equation or line to tell how tall Big Foot might be (e.g., if his foot is 25 inches long).

What determines best fit? From a strictly visual standpoint, the line you select defines the observed relationship and could be used to predict other values not in the data set. The more closely the dots in the scatter plot cluster around the line you select, the greater the confidence you would have in the predictive value of the line. Certainly, you could try to draw a straight line somewhere in the name length–weight graph, but you would not have much confidence in its predictive capability because the dots would be quite dispersed from the line.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

Stop and Reflect

Before reading further, return to the height–weight plot in Figure 15.12(a) and draw a straight line that you think will make a good line of best fit. Why did you draw the line where you did?

Encourage students to use a "mathematical" reason for why a line might be best. A good line is one around which most dots cluster—that is, one from which the distances of all of the points are minimal. This general notion of least distance to the line for all points is the basis for an algorithm, which is introduced in high school, that will always produce a unique line for a given set of points.

technology note

Using a graphing calculator, students can enter their data into the table feature, plot it on the graph, and then find the line of best fit. If students have already drawn a line by hand and written the equation of that line, then the calculator provides a good opportunity to compare equations to see whether their estimated line is close to the actual line of best fit.

Analyzing Data: Measures of Center and Variability

Although graphs provide visual images of data, measures of center and variability of the data are also important ways to summarize, analyze, and describe the data. Measures of center include *mean*, *median*, and *mode*, and measures of variability include *range* and *mean absolute deviation*. This is a critical area in grade 6 of the *Common Core State Standards*, and a review of the CCSS-M reveals that the emphasis in teaching mean, median, and mode is *not* on how to find each (or which one is which) but on selecting the appropriate measure based on the context and the population. Students can get an idea of the importance of these statistics by exploring the ideas informally.

Measures of Center

The term *average* is heard quite frequently in everyday usage. Sometimes it refers to an exact arithmetic average, as in "the average daily rainfall." Sometimes it is used quite loosely, as in "she is about average height." In either situation, an average is a single number or measure that describes a set of numbers. Students' understanding of average may be any of the following: average as mode (what is there most of?), average as something reasonable, average as the standard algorithm for finding the mean, and average as mathematical equilibrium (Garcia & Garret, 2006).

Mode

The mode is the most frequently occurring value in the data set. The mode is the least frequently used as a measure of center because data sets may not have a mode or may have more than one mode, or the mode may not be descriptive of the data set.

Median

The median is the middle value in an ordered set of data. Half of all values lie at or above the median, and half at or below. The median is easier to understand and to compute, and it is not affected, as the mean is, by one or two extremely large or extremely small values outside the range of the rest of the data. The most common misconception in using the median emerges when students neglect to order the numbers in the data set from least to greatest. The median and the mean first appear as standards in the sixth grade in the CCSS-M (CCSSO, 2010).

Mean

Ask an adult what the mean is, and they are likely to tell you something like this: "The mean is when you add up all the numbers in the set and divide the sum by the number of numbers in the set." This is not what the mean *is*; this is how you calculate the mean. This is a reminder of the traditionally procedurally oriented curriculum and the need to shift to a stronger understanding of statistics. Another limited conception about the mean is that it is considered *the* way to find a measure of center regardless of the context (not considering median as a viable option) (McGatha, Cobb, & McClain, 1998). In fact, in the *Common Core State Standards*, sixth-graders are expected to determine when the mean is appropriate and when another measure of center (e.g., the median) is more appropriate: "Students recognize that a data distribution may not have a definite center and that different ways to measure center yield different values" (CCSSO, 2010, p. 39). The next section focuses on developing the concept of the mean.

Understanding the Mean: Two Interpretations

There are actually two different ways to think about the mean. First, it is a number that represents what all of the data items would be if they were *leveled off*. In this sense, the mean represents all of the data items. Statisticians prefer to think of the mean as a *central balance point*. This concept of the mean is more in keeping with the notion of a measure of the "center" of the data or a measure of central tendency. Both concepts are discussed in the following sections.

Leveling Interpretation

Suppose that the average number of family members for the students in your class is 5. One way to interpret this is to think about distributing the entire collection of moms, dads, sisters, and brothers to each of the students so that each has a "family" of the same size. Or, consider the average score of 93 for four tests; this is like spreading the total of all of your points evenly across the four tests. It is as if each student had a family of the same size and all the test scores were the same. An added benefit of this explanation of the mean is that it connects to the algorithm for computing the mean.

Activity 15.8

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

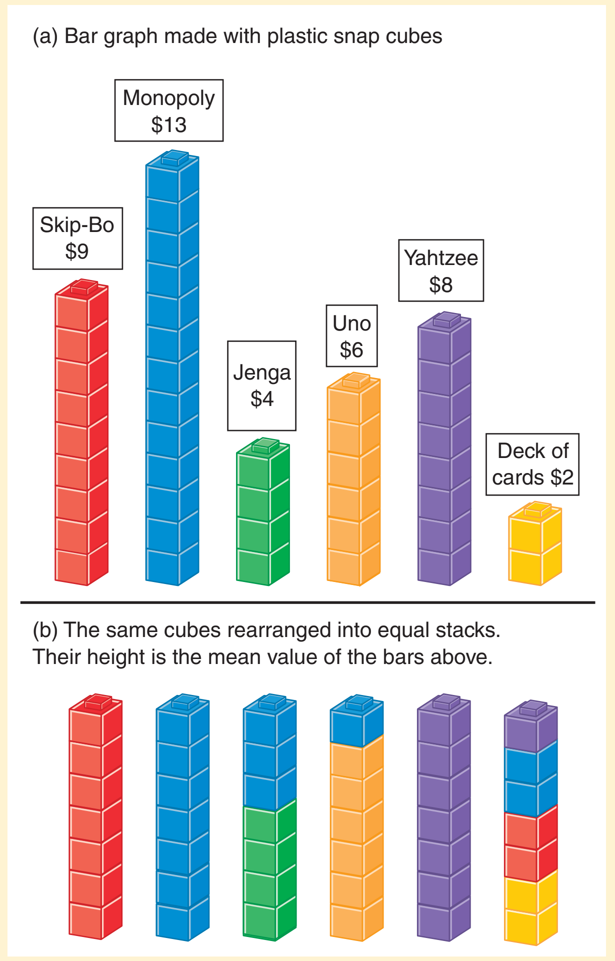
Mean Cost of Games

Post a copy of **Game Costs** Activity Page. Have students make a bar graph of some data using connecting cubes (one cube per dollar). Choose a situation with five or six values. For example, Figure 15.13(a) shows cube stacks for the price of each game. The task for students is to use the stacks of cubes (bars) to determine what the price would be if all of the games were the same price. Encourage students to use various techniques to rearrange the cubes to "level" the prices, or make the price the same for each item (see Figure 15.13(b)). Be sure that ELLs understand the meaning of "leveling" the bars.

Explain to students that the size of the leveled bars is the mean of the data—the amount that each item would cost if all items cost the same amount but the total of the prices remained fixed. Follow "Leveling the Bars" with the next activity to help students develop an algorithm for finding the mean.

Figure 15.13

Understanding the mean as a leveling of the data.



Activity 15.9

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

The Mean Foot

Pose the following question: "What is the mean length of our feet in inches?" This context needs to be clear to ELLs because *foot* is not being used as a measurement unit, but as an object to measure. Also, consider measuring in centimeters rather than inches. Have each student cut a strip of cash register tape that matches the length of his or her foot. Students record their names and the length of their feet in inches on the strips. Suggest that before finding a mean for the class, you first get means for smaller groups. Put students into groups of four, six, or eight (use even numbers). In each group, have the students tape their foot strips end to end. The task for each group is to come up with a method of finding the mean without using any of the lengths written on the strips. They can only use the combined strip. Each group will share their method with the class. From this work, they will devise a method for determining the mean for the whole class. For students with disabilities, help them fold the strip to see how to divide a cash register strip into equal lengths.

To evenly distribute the inches for each student's foot among the members of the group, they can fold the strip into equal parts so that there are as many sections as students in the group. Then they can measure the length of any one part.

How can you find the mean for the whole class? Suppose that there are 23 students in the class. Using the strips already taped together, make one very long strip for the whole class. It is not reasonable to fold this long strip into 23 equal sections. But if you wanted to know how long the resulting strip would be, how could that be done? The total length of the strip is the sum of the lengths of the 23 individual foot strips. To find the length of one section if the strip were actually folded in 23 parts, simply divide by 23. In fact, students can mark off "mean feet" along the strip. There should be close to 23 "feet" of equal length. This dramatically illustrates the algorithm for finding the mean.

Balance Point Interpretation

Statisticians think about the mean as a point on a number line where the data on either side of the point are balanced. To help think about the mean in this way, it is useful to think about the data placed on a dot plot. Notice that what is important includes how many pieces of data are on either side of the mean *and* their distances from the mean.

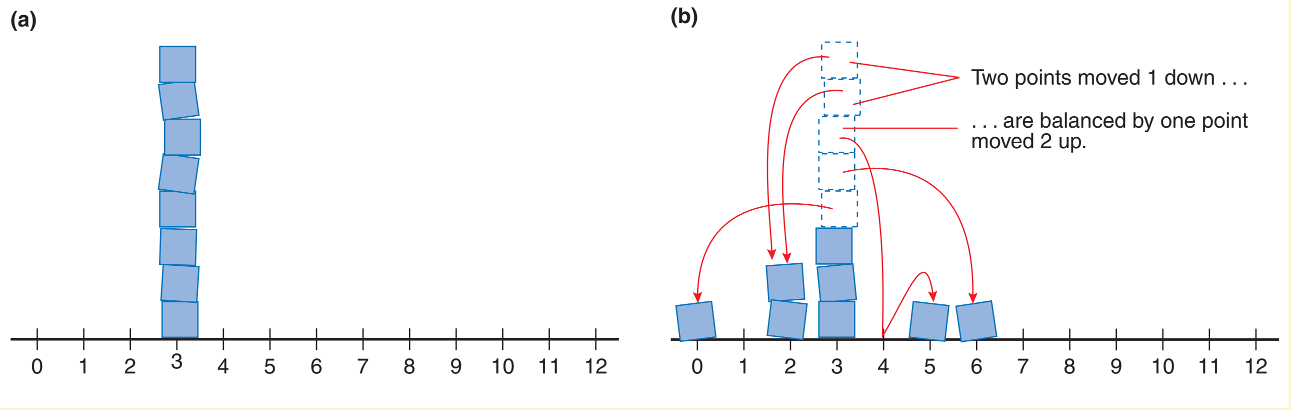
To illustrate, draw a number line on the board, and arrange eight sticky notes above the number 3, as shown in Figure 15.14(a). Each sticky note represents one family. The notes are positioned on the line to indicate how many pets each family owns. Stacked like this indicates that all families have the same number of pets (3). The mean is 3 pets. But different families are likely to have different numbers of pets. So, we could think of eight families with a range of numbers of pets. Some may have 0 pets, and some may have 10 pets or even more. How could you change the number of pets for these eight families so that the mean remains 3? Students will suggest moving the sticky notes in opposite directions, probably in pairs. This will result in a symmetrical arrangement (e.g., one sticky note moves down to 1 and one moves up to 5). But what if one of the families has 8 pets (a move of five spaces from 3)? This might be balanced by moving two families to the left—one family three spaces to the 0 and one family two spaces to the 1. Figure 15.13(b) shows one way the families could be rearranged to maintain a mean of 3. To help them reason about the balancing of the mean, ask students, "Can you find at least two other distributions of the families, each having a mean of 3?"

Standards for Mathematical Practice

2 Reason abstractly and quantitatively. ▶

Figure 15.14

(a) If all data points are the same, the mean is that value. (b) By moving data points away from the mean in a balanced manner, different distributions can be found that have the same mean.



Use the next activities to find the mean or balance point given the data.

Activity 15.10

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Balancing Cubes

Give students a ruler, a block such as a pattern block piece, and cubes such as Unifix cubes. Have students balance the ruler on the pattern block. Notice that the 6-inch mark of the ruler is at the center. Explain that students are going to be creating data sets with a mean of 6 (O'Dell, 2012). Ask students to place four cubes on the ruler so that there is a balance point (mean) of 6. Students might, for example, place a cube on 4 (2 away from the mean of 6) and then one on the 8 to balance it). You can increase the challenge by asking students the following: use only one data point on each side, use exactly 5 cubes, add one cube that keeps the balance, move two cubes to maintain the balance, place cubes with a wide distribution or with a narrow distribution.

Activity 15.11

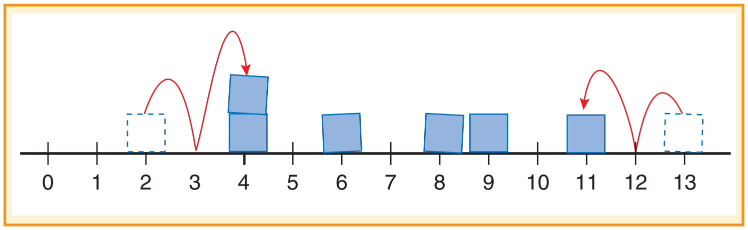
CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Balance Point Sticky Notes

Use the **Balance Point** Activity Page, or just have students draw a number line from 0 to 13 with about two inches between the numbers. Use six small sticky notes to represent the prices of six games as shown in Figure 15.13. Have students place a light pencil mark on the line where they think the mean might be. Ask students to determine the mean by moving the sticky notes in toward the "center." That is, the students are to find out what price (point on the number line) balances out the six prices. For each move of a sticky one space to the left, a different sticky must be moved one space to the right. Eventually, all sticky notes should be stacked above the same number, the balance point or mean.

Figure 15.15

Move data points toward the center or balance point without changing the balance around that point. When you have all points at the same value, that value is the mean.



Stop and Reflect

Stop and try this exercise yourself. What do you notice about how you can move the values?

After any pair of moves that keeps the distribution balanced, you actually have a new distribution of prices with the same mean. The same was true when you moved the sticky notes out from the mean when they were all stacked on the same point.

Changes in the Mean

The balance approach to finding the mean clearly illustrates that different data distributions can have the same mean. Especially for small sets of data, the mean is significantly affected by extreme values. For example, suppose another game with a price of $20 is added to the six we have been using in the examples.

Formative Assessment Note

Consider using a *diagnostic interview* to assess whether students are able to determine the best measure of center to use in a given situation, such as the average height of students in the class. You can begin with general questions such as these:

What is an average? What does the mean represent?

What does the median represent?

What is the difference between the mean and the median?

What is each useful for?

Then move to more analytical questions:

Which should we use for this set of data?

Might we use a different measure of center in another class?

When you've found the average height of the students in our class, is it possible that no one is that height? Why?

Choosing a Measure of Center

As mentioned earlier, the context in statistics is important. The context of a situation will determine whether mean or median is the measure you want to use. For example, in reporting home prices (see first page in this chapter), the median is quite different from the mean, with the mean higher. Which better portrays the cost of housing? Very expensive homes can drive the mean up, so typically the median is a more common measure for describing average housing costs. Activities 15.12 and 15.14 provide strategies for engaging students in selecting a measure of center.

Activity 15.12

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Which Measure of Center?

Prepare possible questions to investigate, like the ones listed here.

- **How many pencils does a sixth-grader have?**
- **What is the cost of used cars in our area?**
- **What is the height of a typical cereal box?**
- **What is the average monthly cost of a mobile phone?**

The students' task is to decide which measure of center makes the most sense and be able to justify their decision. The first question can be explored by gathering classroom data (Johnson, 2011), and then selecting and justifying which measure of center makes sense. For another situation, you can assign a topic to a group. Each group does the following: (a) selects which measure of center they think makes the most sense for their topic, (b) prepares a data set that illustrates their point, and (c) prepares a justification of why they picked their measure. Students with special needs may benefit from being given several sets of data for their topic as a way to consider which measure might be best. Also discuss the spread and overall shape of the data gathered (e.g., Does the height of cereal crowd the center? Is there a lot of variation in the phone bills?).

Exploring how new data affect each measure of center provides valuable insights to students in comparing the pros and cons of each. Let's revisit the Mean Cost of Games (Activity 15.8).

Activity 15.13

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

Average Cost of More Games

Distribute copies of the Average Game Costs Activity Page and post the Game Costs Activity Page. Ask students to predict how the mean, median, and mode will change if a new game costing $20 is part of the game set. Repeat asking, What if the $2 game is removed? Then, ask students to figure out the price of a game that was added that increased the mean to $9. For a full lesson, see Expanded Lesson: Playing with Measures of Central Tendency.

Having to make decisions using statistics is an authentic reason for choosing a measure of center, the focus of the next activity.

Activity 15.14

CCSS-M: 6.SP.A.2; 6.SP.A.3; 6.SP.B.5

You Be the Judge

Use the You Be the Judge Activity Page. The gymnastics coach can send only one person to the all-star state competition. She wants to select the student with the best average and most consistency for the season. The table below gives overall scores for the eight most recent competitions. Whom should she pick?

Meet	Jenna	Miah	Leah
1	9	9	5
2	3	9	6
3	10	7	7
4	9	8	6
5	7	7	9
6	5	9	8
7	10	9	10
8	9	8	10

Ask students questions such as the following: "Which measure of center seems to be the fairest way to judge the competition? What variability do you notice for each person? Which person would you pick and why?"

In addition to selecting the measure that makes sense, students need to understand how characteristics of a data set (e.g., distribution of data, outliers) affect the mean, median, and mode. In seventh grade and beyond, students must be able to compare the characteristics of one data set to another (CCSSO, 2010). For example, students can gather data on the time it takes fizz to die down for two different brands of soda (repeated trials for each brand) (Kader and Jacobbi, 2013).

technology note

Data can be found online and used for comparison with class data. For example, CNN recently reported that the average youth owns seven pairs of jeans. Mooney and Bair (2011) used this data to compare to data gathered by their class, which varied from the reported average, and discussed what might cause the deviation. Also, students can efficiently investigate how the mean and median are affected by each piece of data using the Mean and Median applet on NCTM Illuminations.

Variability

Although measures of center are a long-standing topic, measures of variability also need explicit attention in the curriculum (Franklin et al., 2005; Kader & Jacobbe, 2013; Scheaffer, 2006). Increased attention to variability is needed, and this may not be adequately addressed in textbooks that have tended to focus on measures of center. In CCSS-M, variability is introduced in grade 6 and is a major focus of grade 7 (CCSSO, 2010). Sixth-graders should "recognize that a measure of variability (interquartile range or mean absolute deviation) can also be useful for summarizing data because two very different sets of data can have the same mean and median yet be distinguished by their variability" (CCSSO, 2010, p. 39–40). Shaughnessy (2006) summarized the findings on what students should know about variability in the following list, starting with basic notions and progressing to more sophisticated ideas:

1. Focusing only on outliers or extremes (but not on the full distribution of the data)
2. Considering change over time (which can lead into discussions of other types of variation)
3. Examining variability as the full range of data (range is everything that occurs, but it doesn't reveal the frequency of different events within the range)
4. Considering variability as the likely range or expected value
5. Looking at how far data points are from the center (e.g., the mean)
6. Looking at how far off a set of data is from some fixed value

In order to be prepared to teach students variability beyond outliers and extremes, it is important to know about the way that variability occurs in statistics.

The *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report* (Franklin et al., 2005) discusses three levels of statistical thinking that, though developmental in nature, can be roughly mapped to elementary, middle, and high school curriculum. This report elaborates on what is in the CCSS-M, providing guidance on how to teach statistics, make meaningful connections, and chart appropriate learning trajectories (Groth & Bargagliotti, 2012). At the first level, the focus is on variability within a group—for example, the varying lengths of students' names, varying family sizes, and so on. When students create a bar graph of class data and compare the data collected, they are discussing the variability within a group.

At the second level, variability within a group continues, but groups of data are also considered. Students might compare the variability of fifth-graders' favorite music choices with eighth-graders' music choices, an example of variability between groups. In addition, middle-school students study how the change in one variable relates to change in another variable—yes, algebra! Students also explore sampling variability (Franklin et al., 2005). When students flip a coin 10 times as a sample, they may get 5 heads and 5 tails, but they also may get many other results (even 0 heads and 10 tails).

At the third level, students examine natural and induced variability. For example, plants grow at different rates (natural variability). But, experiments might compare two groups of plants that are in two different gardens, examining the impact of various factors, such as fertilization, amount of sunlight, and amount of water (induced variation). This is at the heart of doing statistics (Franklin & Garfield, 2006).

Variability can be analyzed by looking at data in a table; for example, looking at the frequency of occurrences of categorical data (Kader & Jacobbe, 2013). Figure 15.16 shows the frequency and relative frequency of students' favorite Saturday activity. Students first submitted their favorite activity on a sticky note and stuck them to the white board. These were sorted into six categories.

Figure 15.16

Frequency and relative frequency describe the variability in the data.

Activity	Frequency	Relative Frequency
Play a sport	7	$\frac{7}{28} = .25 = 25\%$
Go to the movies	3	$\frac{3}{28} = .107 = 11\%$
Read	3	$\frac{3}{28} = .107 = 11\%$
Play outside	6	$\frac{6}{28} = .214 = 21\%$
Hang out with friends	4	$\frac{4}{28} = .143 = 14\%$
Play with electronics	5	$\frac{5}{28} = .178 = 18\%$
TOTAL	28	100%

One way to help students understand variability is to ask questions about variability in the discussion of data. Friel, O'Conner, and Mamer (2006), using the context of heart rates, suggest the following questions as examples of how to get students to focus on variability:

- If the average heart rate for 9- to 11-year-olds is 88 beats per minute, does this mean every student this age has a heart rate of 88 beats per minute? (Note that the range is actually quite large—from 60 to 110 beats per minute.)
- If we found the heart rates for all students in the class (of 30), what might the distribution of data look like?
- If another class (of 30) were measured, would their distribution look like the one for our class? What if they had just come in from the gym?
- Would the distribution of data from 200 students look like the distribution of data from the two classes?

Comparing different data sets or playing a game repeatedly also provides the opportunity for students to analyze the spread of data and think about variability in data (Franklin & Mewborn, 2008; Kader & Mamer, 2008).

Range

Range is a *measure* of variability. Range of a data set is the difference between the highest and the lowest data points, or it can be expressed simply by stating the minimum and the maximum values. The *interquartile range* of the data is connected to the box plot previously described. It is the difference between the lower and upper quartiles (Q3 − Q1), or the range of the middle 50 percent of the data. A small interquartile range means that there is a lot of clustering around the median.

Let's look at an example.

The data set below is the number of hours seventh-graders spent playing sports or playing outside over the weekend (the data have already been placed in order).

0 0 0 1 3 4 4 4 5 5 5 5 6 6 7 8 8 9 10 10

Find the interquartile range. What does the result tell you about the variation in the data set?

In this case, the median is 5 (because the 10th and 11th values of the 20 values are both 5). This is also referred to as quartile 2 (Q2). Quartile 1 (Q1) is the median of the lower half of the data. This median is the average of the fifth and sixth values in the data set, which is 3.5. Quartile 3 (Q3) is the median of the upper half of the data set; in this case, it falls between 7 and 8, so it is 7.5. The interquartile range is 7.5–3.5, or 4. For hours spent exercising, the interquartile range is fairly small, showing that there is a lot of clustering around the center of the data.

Mean Absolute Deviation

Whereas the range relates to the median, the Mean Absolute Deviation (MAD) relates to the mean and tells how spread out the data is (Kader and Jacobbi, 2013). In other words, a large MAD, means that there is a lot of deviation (difference) between data points and the mean, so the data are spread out. MAD is introduced in grade 6 in the CCSS-M, with the intent being that it is explored in an informal manner to develop a deeper understanding of variability. Let's use the data set above to explore mean absolute deviation.

Use the data set below to find the mean absolute deviation:

0 0 0 1 3 4 4 4 5 5 5 5 6 6 7 8 8 9 10 10

What does the result tell you about the variation in the data set?

Figure 15.17

A dot plot illustrates the data on hours spent exercising over the weekend.

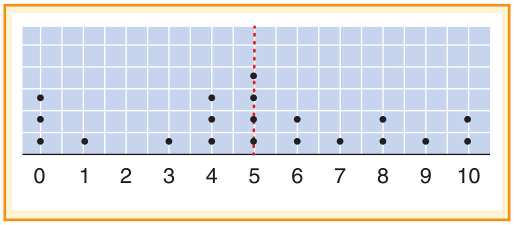


Figure 15.17 places the data in a dot plot. Dot plots can be used to illustrate absolute deviations from the mean (Hudson, 2012/2013). Students draw a vertical line at the mean and draw or observe how far away each data point is (absolute deviation). The mean of the data set is 5 hours.

The first step in finding the mean absolute deviation is to find the *deviation* (difference) of each data point from the mean. Figure 15.18(a) illustrates these differences in a dot plot. The *absolute deviation* is the distance from the mean, which means the positive difference (see Figure 15.18(b)). Finally, the *mean of the absolute deviation* is the mean of all these differences (see Figure 15.18(c))—in this case, 2.4. Did you notice that you started with the end of the phrase *mean absolute deviation*, by first finding the deviation, then the absolute deviation, and finally the mean absolute deviation? Pointing this out can help students, in particular students with disabilities, focus on the meaning of what they are doing, and why.

In context, this value indicates that the average distance from the mean hours of exercise is about two and a half hours.

Teaching Tip

A good way to have students focus on variability is to have two data sets, one with a very small absolute mean deviation and one with a large one. Focus attention on how this measure helps interpret the data set.

Figure 15.18

Dot plots illustrate the difference from the mean of each data point for hours spent exercising over the weekend.

(a) Deviation (from the mean)

					0					
–5				–1	0					
–5				–1	0	1		3		5
–5	–4		–2	–1	0	1	2	3	4	5
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

(b) Absolute deviation (from the mean)

					0					
5				1	0					
5				1	0	1		3		5
5	4		2	1	0	1	2	3	4	5
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

(c) Mean absolute deviation (from the mean)

$$\frac{5+5+5+4+2+1+1+1+1+0+0+0+0+1+1+2+3+3+4+5+5}{20} = \frac{48}{20} = 2.4$$

Interpreting Results

Interpretation is the fourth step in the process of doing statistics. As seen in the sample test items shown earlier, sometimes questions focus on mathematical ideas rather than statistical ideas. Although it is helpful to ask mathematical questions, it is essential to ask questions that are statistical in nature. That means the questions focus on the context of the situation and seeing what can be learned or inferred from the data. This final activity, based on Wilburne and Kulbacki (2014), is an engaging way to involve students in designing their own data analysis and then interpreting their results.

Activity 15.15

CCSS-M: 6.SP.A.1; 6.SP.A.2; 6.SP.B.5

How Many Words on the Page?

Select a book that is very popular with students. Project a picture of a page where most of the page is covered up, showing only about eight lines of text. Ask students to use their statistical reasoning skills to explore this situation and determine how many words are on this page. Explain that, in the end, they will have to argue for their result, explaining what methods they used to determine their answer. Debrief by talking about the variability in data, and about the pros and cons of different measures of center.

Interpreting data questions should not just focus on the specific question under investigation but also focus on key ideas of statistics, such as variability, center of the data, and shape of the data. During interpretation, students might want to create a different data display to get a different look at the data, or gather data from a different population to see whether their results are representative.

Different researchers have recommended questions that focus on statistical thinking (Franklin et al., 2005; Friel et al., 2006; Russell, 2006; Shaughnessy, 2006). Here are some ideas from their lists to get you started on having meaningful discussions interpreting data:

- What do the numbers (symbols) tell us about our class (or other population)?
- If we gathered the same kind of data from another class (population), how would that data look? If we asked a larger group, how would that data look?
- How do the numbers in this graph (population) *compare* to this graph (population)?
- Where are the data "clustering"? How much of the data are in the cluster? How much are *not* in the cluster? About what percent is or is not in the cluster?
- What kinds of variability might need to be considered in interpreting these data?
- Would the results be different if . . . [change of sample/population or setting]? (Example: Would gathered data on word length in a third-grade book be different from those in a seventh-grade book? Would a science book give different results from a reading book?)
- How strong is the association between two variables (scatter plot)? How do you know? What does that mean if you know x? If you know y?
- What does the graph *not* tell us? What might we infer?
- What new questions arise from these data?
- What is the maker of the graph trying to tell us?

These prompts apply across many data displays. It certainly should be a major focus of instruction. Consider it the *After* phase of your lesson, though some of these questions will be integrated in the *During* phase as well.

Our world is inundated with data, from descriptive statistics to different graphs. It is essential that we prepare students to be literate about what can be interpreted from data and what cannot be interpreted from data, what is important to pay attention to and what can be discarded as misleading or poorly designed statistics. This is important for success in school, as well as for being a mathematically literate citizen.

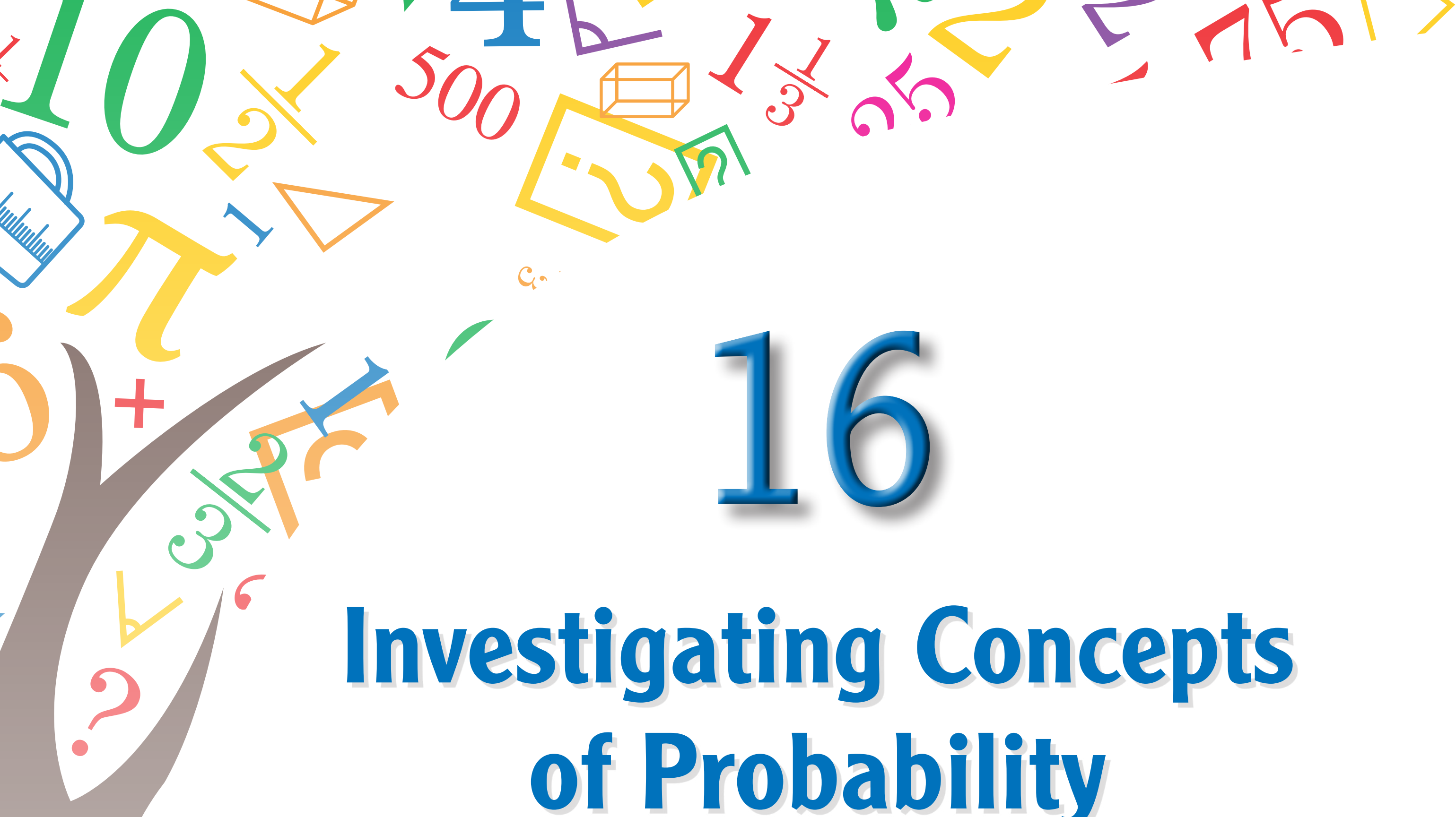
Literature Connections

200% of Nothing: An Eye-Opening Tour Through the Twists and Turns of Math Abuse and Innumeracy *Dewdney, 1993*

This middle-school-friendly book has explanations of the many ways that "statistics are turned" to mislead people. Because the examples are *real*, provided by readers of *Scientific American*, this book is an excellent tool for showing how important it is to be statistically literate in today's society. Reading the examples can launch a mathematics project into looking for errors in advertisements and at how overlapping groups (as in a Venn Diagram) can be reported separately to mislead readers. (See Bay-Williams and Martinie, 2009, for more ways to use this book.)

If the World Were a Village: A Book about the World's People (Second Edition) *Smith, 2011*

This book explores global wealth, culture, language, and other influences. Each beautiful two-page spread shares the statistics for the topic (e.g., language). This book can give rise to other questions about the world, which can be researched and interpreted into the village metaphor. An article that links this idea to a project exploring concepts of statistics using 100 students is a great follow-up (Riskowski, Olbricht, & Wilson, 2010).



16

Investigating Concepts of Probability

BIG IDEAS

1 Chance has no memory. The chance occurrence of six heads in a row has no effect on whether another head will occur on the next toss of the coin. That chance remains 50–50.

2 The probability that a future event will occur can be characterized along a continuum from impossible (0) to certain (1). A probability of $\frac{1}{2}$ indicates an even chance of the event occurring.

3 The relative frequency of outcomes through experiments can be used as an estimate of the probability of an event. The larger the number of trials, the better the estimate will be. The results for a small number of trials may be quite different from those obtained in the long run.

4 For some events, the exact probability can be determined by an analysis of the event itself. A probability determined in this manner is called a *theoretical probability*.

5 *Simulation* is a technique used for answering real-world questions or making decisions in complex situations in which an element of chance is involved. To see what is likely to happen in the real event, a model must be designed that has the same probabilities as the real situation.

References to probability are all around us: The weather forecaster predicts a 60 percent chance of snow, medical researchers predict that people with certain diets have a high chance of heart disease, investors calculate the risks of specific investments, and so on. Simulations of complex situations are frequently based on probabilities and then used in making decisions about such situations as airplane safety under different weather circumstances, highway traffic patterns after new housing has been built, and disaster plans.

Realistic concepts of chance require considerable development before students are ready to construct formal ideas about the probability of an event. Optimally, this development occurs as students consider and discuss with their peers the outcomes of a wide variety of probabilistic situations. In the *Common Core State Standards* (CCSSO, 2010), probability begins in middle school and is further developed in high school:

> *Grade 7:* Understand the meaning of probability; find probability by implementing experiments; design a probability model; and find probability of compound events.

Introducing Probability

Probability does not appear in the *Common Core State Standards* expectations until seventh grade. Notions of chance and fairness, however, develop informally at younger ages, as students play games and consider events that are likely or not likely. Although such intuition can be a positive thing, in probability it can also be a preconception that works against understanding the randomness of events (Abu-Bakare, 2008). Because middle school is the first time probability is addressed in the standards, we begin this chapter with some introductory-level experiences.

Likely or Not Likely?

Probability is a ratio (see Chapter 11) that compares desired outcomes to total possible outcomes. Probability is about how likely an event is. Therefore, a good place to begin is with a focus on possible and not possible (Activity 16.1) and later *impossible*, *possible*, and *certain* (Activity 16.2). In preparation for these activities, discuss the meaning of *impossible* and *certain*. These experiences can be woven into discussions across the curriculum, as in Activity 16.1, which is an idea for connecting to children's literature, or in science and social studies, as in Activity 16.2.

Activity 16.1

CCSS-M: 7.SP.C.5

Events in Lyrics: Possible or Not Possible?

Create a table, labeling one column "Impossible" and the other "Possible." Use literature or poems that include possible and impossible events. Rhymes, such as "Hey, Diddle, Diddle" can be fun for middle school students as they debate whether something in the rhyme is possible (a cow jumping over the moon). Songs and raps are also engaging. Record each statement in the appropriate column.

Activity 16.2

CCSS-M: 7.SP.C.5

Is It Likely?

Ask students to judge various events as *impossible, possible*, or *certain*. Consider these examples:

- **It will rain tomorrow.**
- **Drop a rock in water, and it will sink.**

- The sun will rise tomorrow morning.
- A hurricane/tornado will hit our town.
- In an election, candidate A will be elected.
- If you ask people who the first U.S. president was, they will know.
- You will have two birthdays this year.
- You will be in bed by 11:00 p.m.

For each event, ask students to justify their choice of how likely they think it is. Notice that the last two ideas are about the students. This is an opportunity to bring in students' identities and cultures. Ask students to work with their families to write down family events that are certain, impossible, or possible. Encourage native language use, as appropriate, for ELLs. For students with disabilities, use a strip of cash register tape, and label the ends with the words *impossible* and *certain* to assist them in organizing their thinking. Write the events listed above on cards so that students can place them along the strip.

The key idea to developing chance or probability on a continuum is to help students see that some of these possible events are more likely or less likely than others. For instance, if Erik is a very fast runner and is in a race, the chance of being in first is not certain but is very likely.

The use of random devices (tools) that can be analyzed (e.g., spinners, number cubes, coins to toss, colored cubes drawn from a bag) can help students make predictions about how likely a particular occurrence is. Begin with the use of random devices with which students can count the outcomes. Colored dots can be stuck on the sides of a wooden cube to create different color probabilities. Color tiles (e.g., eight red and two blue) can be placed in opaque bags. Students draw a tile from the bag and then return it after each draw.

Standards for Mathematical Practice

5 Use appropriate tools strategically.

technology note

Science Netlinks offers an interactive tool, Marble Mania, for exploring probability of different colored marbles in a bag. You can determine how many and what color marbles to place in the bag. An advantage of such digital resources is that you can run a large number of different trials in a short amount of time. In addition, the National Library of Virtual Manipulatives (NLVM) has a large collection of probability tools in their Data Analysis and Probability Manipulatives collection, organized across grade bands. There are also a number of apps for generating random numbers (e.g., randomness iOS and numerous dice rolling apps for iOS/android)

The process of exploring how likely an event is maps to the *before*, *during*, *after* lesson plan model. In the *before* phase, students make predictions of what they think will be likely; in the *during* phase, students experiment to explore how likely the event is; and in the *after* phase, students compile and analyze the experimental results to determine more accurately how likely the event is.

The following dice activities have unequal outcomes. However, students may not initially connect that having more of something means it is more likely. A common initial misconception is that there is a one-in-three chance of each of the values (1, 2, and 3) because each one is possible.

Activity 16.3

CCSS-M: 7.SP.C.5; 7.SP.C.6

1-2-3 How Likely?

Make number cubes with sides labeled as follows: 1, 1, 2, 3, 3, 3. Ask students to predict what number they might get when they roll the cube. What is likely? What is impossible? Or, more specifically, which row will fill the fastest or will the rows fill at equal rates? Have students roll the cube and record the results in a bar graph (See **1-2-3 How Likely?** Activity Page). Students mark an X in the column for 1, 2, or 3 each time the cube shows that value and stop when one row is full. After they stop, students reflect on how likely each number is.

Activity16.4

CCSS-M: 7.SP.C.5; 7.SP.C.6; 7.SP.C.8a

1-2-3 How Likely Are Sums?

This game requires two cubes labeled as above. It is a more difficult task because it considers the probability of two events (two dice rolls). Students take turns rolling the two cubes and recording the sums of the two numbers. Before the game begins, ask students to predict which row will fill the fastest or if the rows will fill at equal rates. Ask students to keep track of their data on **1-2-3 How Likely Are Sums?** Activity Page. Students roll the cubes until one of the rows is full then discuss with their partner what happened and how likely they now think each sum is.

After exploring either 1-2-3 How Likely activity, ask, "Which numbers 'won' the most and the least often?" and "If you play again, which number would you pick to win and why?" In Activity 16.4, although an outcome of 1 is impossible, all of the other outcomes, 2 through 6, are possible. A sum of 4 is the most likely. Sums of 2 or 3 are not likely.

Area representations, such as spinners, are more challenging because students cannot count the possible outcomes as readily (Abu-Bakare, 2008). Activity 16.5 uses spinners and counting spins to build this connection.

Activity 16.5

CCSS-M: 7.SP.C.6; 7.SP.C.7a, b

Race to Ten

Use the **Race to Ten** Activity Page. Refer to the spinner and ask: "If we count spins that land on red and ones that land on blue, which one will reach ten first?" Two players take turns spinning the spinner, each time placing an X in the matching column. The Activity Page includes a spinner that is three-fourths red and one-fourth blue (give students a paperclip to use as a pointer), but this activity can be played with different spinners (see Figure 16.1). Before playing, each student predicts which color will win, blue or gold (or red). Play continues until one color reaches ten. After the activity, discuss which color won and why. Ask students to explain how likely it is for each color to win.

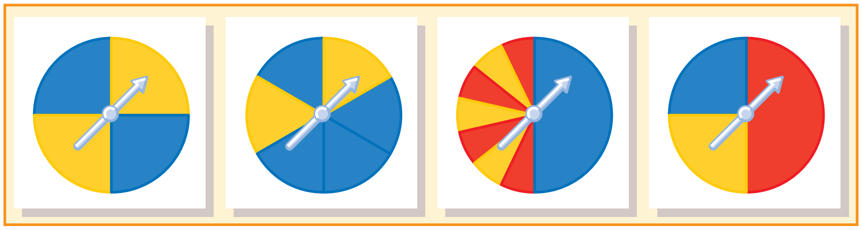
Teaching Tip

A two-color spinner or probability wheel can be constructed easily by cutting out two circles of different colors, cutting a radius in each, and sliding one into the other. Then use a paperclip as the spinner (for three options, use three different colored circles).

Repeat this activity with a variety of spinners, ones that have two colors with the same area—and colors covering different areas, as shown Figure 16.1. This activity provides an opportunity to explore how likely an event is with an area model, but because 10, the total goal, is a small number, students may have surprising results. This issue is addressed in the section "The Law of Large Numbers."

Figure 16.1

Possible spinners for “Race to Ten.”



Students do not always see that the first spinner, third spinner, or a spinner partitioned into just two sections (50% blue, 50% red), have the same chance of getting blue (Cohen, 2006; Nicolson, 2005). Therefore, it is important to use spinners that are partitioned in different ways. Spinner faces can easily be made to adjust the chances of different outcomes. For virtual spinners, NCTM Illuminations “Adjustable Spinner” can be designed to have any number of sections of any size and can be virtually spun any number of times.

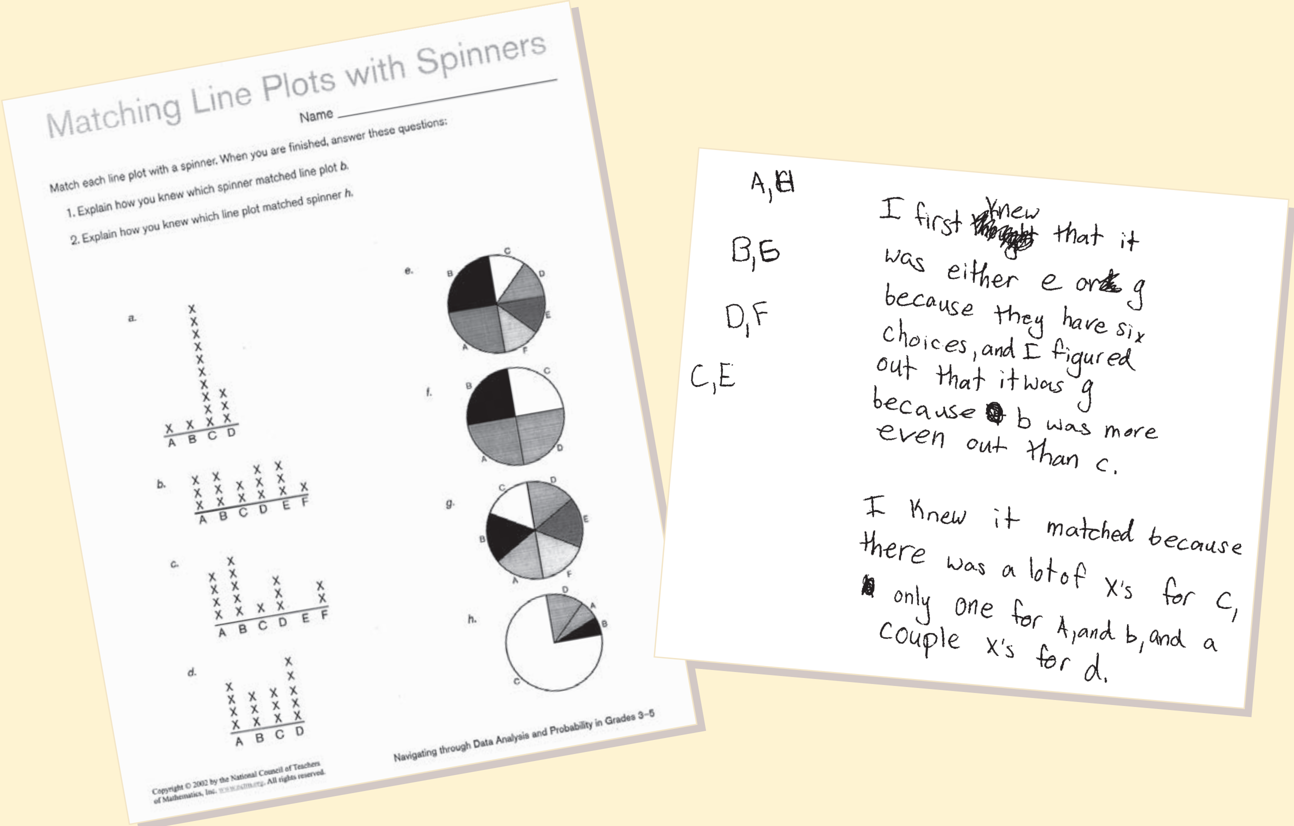
An effective way to connect the idea that the larger area or region on the spinner is more likely to have a spin land there is to have students use frequency charts to record data. In Figure 16.2, a student explains how she knows which frequency table goes with which circle graph.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Figure 16.2

Activity and student explanations connecting frequency charts to spinners.



Matching Line Plots with Spinners

Name ________________

Match each line plot with a spinner. When you are finished, answer these questions:

1. Explain how you knew which spinner matched line plot *b*.
2. Explain how you knew which line plot matched spinner *h*.

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Navigating through Data Analysis and Probability in Grades 3–5

A, H

B, G

D, F

C, E

I first ~~thought~~ knew that it was either e or g because they have six choices, and I figured out that it was g because b was more even out than c.

I knew it matched because there was a lot of X's for C, only one for A, and b, and a couple X's for d.

Source: Matching Lineplots with Spinners page from Chapin, S., Kozial, A., MacPherson, J., & Rezba, C. (2002). *Navigating Through Data Analysis and Probability in Grades 3–5*. Reston, VA: NCTM, p. 116. Copyright © 2002 by the National Council of Teachers of Mathematics. All rights reserved.

Formative Assessment Note

Diagnostic interviews can uncover student misconceptions or preconceptions about the probability of an event. Ask students about the probability using color tiles or dice (countable objects). For example, ask, "If there are 3 red and 1 blue tile in this bag and I draw one out, what do you think I will get?" and, "If I draw four times and put the tile back each time, what do you think I will get?" Ask about the chance of rolling selected outcomes on a die. Some students may think 5 is a more likely outcome on a die than a 2 because 5 is bigger than 2. Or students may think a 1 is not as likely as rolling a 5 on a die, perhaps because they are familiar with a game in which 1 is desirable. The 1 is not likely compared to the combined possibility of the other five choices, but it is as likely as any other number (Nicolson, 2005; Watson & Moritz, 2003). Finally, ask about how likely outcomes are in an area representation such as a spinner. These questions will help you know whether you need to focus on counting or area representations, and what kinds of questions or experiences to prepare in order to help students understand that probability is based on knowing all the possible outcomes and how likely each one is.

The Probability Continuum

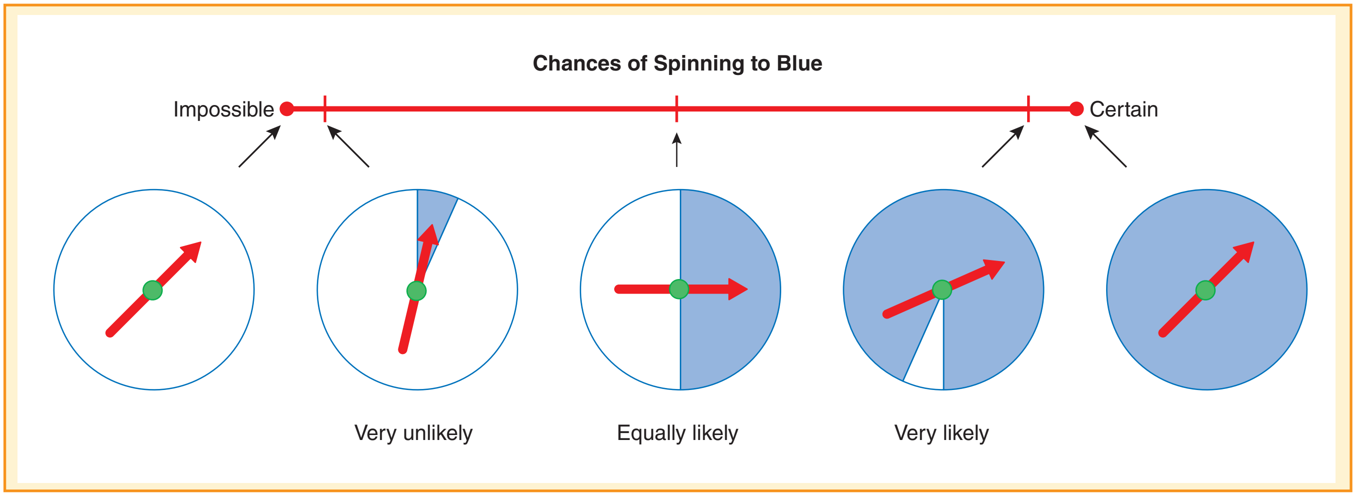
The number line is an important representation across mathematical concepts, and it is emphasized across the content strands in the *Common Core State Standards*. Probability is no exception. Presenting probability on a number line from 0 (impossible) to 1 (certain) provides a visual representation of how likely an event can be. The number line can be connected to spinners, as illustrated in Figure 16.3. Post the probability continuum in the classroom, where it can be used as a reference for other opportunities to talk about how likely something is (see the events in Activity 16.2 for a start to the many things you could ask). Some things change in their probability; for example, the chance of a snow day could be posted and moved from day to day.

Teaching Tip

The probability continuum serves to estimate the relative size of fractions and therefore strengthens rational number sense. Ask students comparison questions related to probability, such as "Which is more likely?" Connect to the comparison of fraction strategies.

Figure 16.3

The probability continuum. Use these spinner faces to help students see how chance can be at different places on a continuum between impossible and certain.



In order to deepen students' understanding of the probability continuum, select a particular mark along the continuum for example, $\frac{1}{4}$ and have them create a spinner with a color that is about that likely to occur. This can also be done with counters, as in Activity 16.6.

Activity 16.6

CCSS-M: 7.SP.C.6

Design and Test Bags

Provide each pair or group of students with a copy of the Design a Bag Activity Page. Ask each group to choose a designated color (e.g., red) for their tiles so that the probability of selecting that color is the probability they have been assigned. Once they have colored the tiles on the Activity Page to match their fraction (e.g., $\frac{1}{3}$ are red, for example), the students trade papers. With the new Activity Page, the students do the following:

1. **Mark the probability line at the point they think matches the number of tiles colored red.**
2. **Use actual colored tiles and brown paper bags based on what is colored on the "Design a Bag" they received. They draw tiles from this bag (with replacement) 50 times. Remind students to shake their bag each time to ensure random sampling.**
3. **Determine the fraction of times they have drawn a red tile.**
4. **Return the papers to the group that colored the Design a Bag and find out what the original probability was. (They will have to decide if their fraction is close to the assigned fraction. For example, is $\frac{17}{50}$ close to $\frac{1}{3}$?)**

At the end of the activity, have students explain (on the back of the handout or in their journals) how they decided where to place their mark on the probability line. ELLs may benefit from sentence frames, such as, "In our bag there were ___ red and ___ blue tiles. We first thought _____. After we did our experiment, we thought ___________. We picked this probability because __________.

Standards for Mathematical Practice

3 Construct viable arguments and critique the reasoning of others.

This activity engages students in conjecturing about how likely an event is, experimenting, and comparing their predictions with experimental outcomes as they continue to explore and refine their conjectures about theoretical probability. This builds a strong foundation for the more advanced probability techniques they will develop in seventh grade and beyond.

Activity 16.7

CCSS-M: 7.SP.C.6; 7.SP.C.7a, b

Mystery Bags

This activity builds from Activity 16.6. Ask each original group to select a probability (e.g., $\frac{1}{4}$, $\frac{1}{6}$, $\frac{3}{4}$, $\frac{3}{8}$), design an actual opaque bag using 24 colored cubes or tiles, such that $\frac{1}{6}$ of the tiles are red if they selected the probability of $\frac{1}{6}$ (and $\frac{5}{6}$ are not red). Students need to give their bag a name so they can get it back! On a secret card, they record the probability they selected and tuck it away. Groups trade bags. The new group cannot look in the bag. The new group conducts 10 draws (with replacement) and on paper writes a prediction for how many tiles they think are red and what they think the probability is of drawing red. Trade bags again (giving opaque bag to a third group). This time, each group conducts 30 draws (with replacement) and again records the name of their bags and their prediction for how many tiles they think are red and what they think the probability is of drawing red. Return bags to original owners. During share time, each group holds up their bag, hears the prediction from the group that drew 10 times and from the group that drew 30 times, and then reveals the answer. As a whole class, discuss the connection between probability and how many draws are needed to make a good prediction of what is in the bag.

You may notice that probability is closely connected to fractions. Although probability is not explicitly in the curriculum prior to grade 7, fractions are heavily emphasized in grades 3–7 (CCSS-M, 2010). Activities such as the Design and Test Bags and Mystery Bags are excellent for thinking about relative size of fractions, comparing fractions, equivalence of fractions, and multiplication of a whole number by a fraction.

Theoretical Probability and Experiments

The *probability* of an event is a measure of the event occurring (Franklin et al., 2005). Students to this point have only been asked to place events on a continuum from impossible to certain or to compare the probability of one event with another. So how do you measure chance of an event? In many situations, there are actually two ways to determine this measure.

Probability has two distinct types. The first type involves any specific event whose probability of occurrence is known (e.g., that a fair die has a $\frac{1}{6}$ chance of producing each number). When the probability of an event is known, probability can be established theoretically by examining all the possibilities. For example:

What is the probability of rolling a three with a fair die?

When the probability of an event is known, probability can be established theoretically by examining all the possibilities.

In the second case, probability can be established only through empirical data or evidence from past experiments or data collection (Colgan, 2006; Nicolson, 2005). Examples include:

What is the probability that Jon V. will make his free throws (based on his previous record)?

What is the chance of rain (based on how often it has rained under similar conditions)?

Although this type of probability is less common in the school curriculum, it is the most applicable to fields that use probability and therefore important to include in your teaching (Franklin et al., 2005). In both cases, experiments or simulations can be designed to explore the phenomena being examined. (Sometimes, in school curriculum, this is referred to as *experimental probability*, but because this terminology is not employed by statisticians it is not used here.)

Some experiments have outcomes that are equally likely, whereas other experiments do not. With coin flips, there are two possible outcomes that are equally likely, so each has a probability of $\frac{1}{2}$. Hence, the theoretical probability of obtaining a head is $\frac{1}{2}$. When all possible outcomes of a simple experiment are equally likely, the probability of an event can be expressed as follows:

$$\frac{\text{Number of Outcomes in the Event}}{\text{Number of Possible Outcomes}}$$

Consider the shift in meaning of the question, “Is this coin fair?” This is a statistics problem that can only be answered by doing an experiment and establishing the frequency of heads and tails over the long run (Franklin et al., 2005). The answer requires empirical data and the probability will be as follows:

$$\frac{\text{Number of Observed Occurrences of the Event}}{\text{Total Number of Trials}}$$

Because it is impossible to conduct an infinite number of trials, we can only consider the relative frequency for a very large number of trials as an approximation of the theoretical probability. This emphasizes the notion that probability is more about predictions over the long term than about predictions of individual events.

Theoretical Probability

A problem-based way to introduce theoretical probability is to engage students in an activity with an unfair game. In the following activity, the results of the game will likely be contrary to students' intuitive ideas. This in turn will provide a real reason to analyze the game in a logical manner and find out why things happened as they did—theoretical probability.

Activity 16.8

CCSS-M: 7.SP.C.6; 7.SP.C.7a

Fair or Unfair?

Three students toss two like coins (e.g., two pennies or two nickels), and they record points according to the following rules: Player A gets 1 point if the coin toss results in "two heads"; player B gets 1 point if the toss results in "two tails"; and player C gets 1 point if the toss results are "mixed" (one head, one tail). The game is over after 20 tosses. The player who has the most points wins. Have students play the game at least two or three times. After each game, the players are to stop and discuss if they think the game is fair and make predictions about who will win the next game.

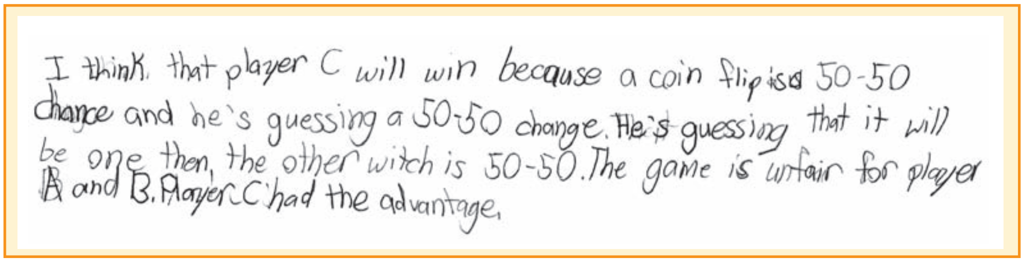
When the full class has played the game several times, conduct a discussion on the fairness of the game. Challenge students to make an argument based on the data as to whether the game is fair, and why. For ELLs, discuss the meaning of *fair* before beginning the game, and review the term when asking students to create an argument.

A common analysis of the game in Activity 16.8 might go something like this: At first, students think that because there are three outcomes—two tails, one head and one tail, and two heads—that each has an equal chance, so the game should be fair. However, after playing, students find that player C (who gets a point for a mixed result) appears to have an unfair advantage (especially if they have been played several games or the class has pooled its data). This observation seems to contradict the notion that all the outcomes are equally likely. As a follow-up to Activity 16.8, challenge students to design their own fair game. See **Expanded Lesson: Design a Fair Game** for details.

When students from fifth through eleventh grades performed a similar two-coin task and were asked for the probability of getting a head and a tail with two coins, similar misconceptions were found (Rubel, 2006, 2007). About 25 percent of the students said the probability was $\frac{1}{3}$ because one of three things could happen: two heads, one of each, or two tails. Although about half answered correctly, many of these students used faulty reasoning, explaining that they picked that answer because there is a 50–50 chance in any experiment. See, for example, Figure 16.4.

Figure 16.4

Correct conclusion but incomplete reasoning on "Fair or Unfair?"



I think. that player C will win because a coin flip is a 50-50 chance and he's guessing a 50-50 change. He's guessing that it will be one then, the other witch is 50-50. The game is unfair for player A and B. Player C had the advantage.

Figure 16.5

Four possible outcomes of flipping two coins.

First Coin	Second Coin
Head	Head
Head	Tail
Tail	Head
Tail	Tail

In order to help students connect how likely an event is to the possible outcomes, encourage students to analyze the situation and generate all the possible outcomes—for example, use a table such the one in Figure 16.5. Getting a head and a tail in two of the four possible outcomes. Figure 16.6 provides an example of a student's correct explanation for "Fair or Unfair." This theoretical probability is based on a logical analysis of the experiment, not on experimental results.

"Rock-paper-scissors" is a great context for exploring fair games and possible outcomes. It can be played in the normal way or adapted so that "same" scores 1 point for one player and "different" scores 1 point for the other player. Challenge students to determine whether this is a fair game (Ellis, Yeh, & Stump, 2007–2008).

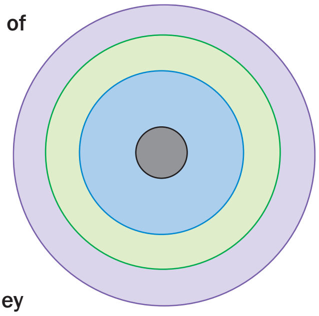
Area has important connections to probability. In the CCSS-M, students explore both area of circles and probability in grade 7. The following activity is an excellent way to integrate these two important ideas.

Activity 16.9

CCSS-M: 7.G.B.4; 7.SP.C.6; 7.SP.C.7b

Chance of Hitting the Target?

Project a target such as the one illustrated here with concentric circles having radii of 2 inches, 6 inches, 8 inches, and 10 inches, each area shaded in a different color. Ask students to determine the fraction and percent of each colored region in the circle. Note: For students who struggle or students with disabilities, have each circle cut out separately so that they can see that each region is the area of the larger circle minus the area of the circle one size smaller. Ask students to discuss the probability for landing on the center (assuming all throws land on the circle and are thrown randomly!). Ask students to discuss why data may or may not match the percent of the area that is covered (e.g., people with good aim will be able to hit the smaller areas more often). Then, have students propose what point values they would assign to each region.

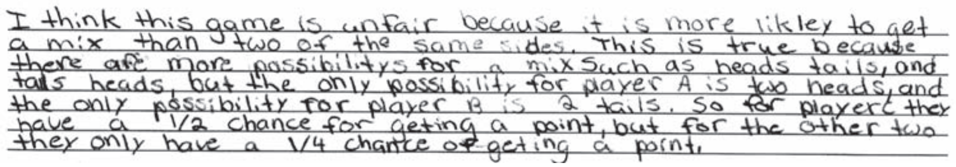


Students may assign values in various ways. For example, they may think the skinny outer circle is harder to land on and give it more points than other sections, even though the area of that region may be more. Allow them time to share their reasoning and to critique others' ways of assigning points.

Dartboards can be made many different ways, not just in the traditional way (Williams & Bruels, 2011). For example, using a **10-by-10 grid**, any combination of shapes can be drawn in such a way that the area can be calculated. Students can determine the area of various regions and use the areas to determine the probability of different outcomes, an excellent connection between measurement and probability.

Figure 16.6

A student's reasoning for "Fair or Unfair?" that connects outcomes to probability.



I think this game is unfair because it is more likley to get a mix than two of the same sides. This is true because there are more possibilitys for a mix such as heads tails, and tails heads, but the only possibility for player A is two heads, and the only possibility for player B is 2 tails. So for player C they have a 1/2 chance for geting a point, but for the other two they only have a 1/4 chance of geting a point.

Experiments

As noted previously, some probabilities cannot be determined by the analysis of possible outcomes of an event; instead, they can be determined only through gathering empirical data. The data may be preexisting or may need to be established through an experiment, with a sufficiently large number of trials conducted to become confident that the resulting relative frequency is an approximation of the theoretical probability. For example, the probability of a hurricane is based on historical data. The United States Landfalling Hurricane Web Project has a link called "Interactive Landfall Probability Display" (Hurricane Predictor) that provides such probabilities for hurricanes (by state and county).

The following activities are examples of situations in which an experiment is the only way to establish how likely an outcome is.

Activity 16.10

CCSS-M: 7.SP.C.7b

Drop It!

In this activity, students drop an object to explore how likely various outcomes are. The number of possible outcomes varies with the different objects. Any object can be used. Here are a few ideas to try:

1. ***Cup toss.*** **Provide a small plastic cup to each pair of students. Ask them to list the possible ways that the cup could land if they tossed it in the air and let it fall on the floor. Which of the possibilities (upside down, right side up, or on the side) do they think is most likely and which least likely? Why? Have students toss the cup 20 times, each time recording how it lands. Students should agree on a uniform method of tossing the cups to ensure unbiased data (e.g., dropping the cups from the same height). Record each pair's data in a class chart. Discuss the differences, and generate reasons for them. Have students predict what will happen if they pool their data. Pool the data and compute three ratios: one for each type of landing (upside down, right side up, or on the side) to the total number of tosses. The relative frequency of the combined data should approximate the actual probability.**
2. ***Toy animal drop.*** **Bring in small plastic toys that can land in different ways. Repeat the first activity. (See Nelson and Williams, 2009, for an exploration with toy pigs.)**
3. ***Falling kisses.*** **Using Hershey Kisses, conduct an experiment to see how often they land directly on the base (Gallego, Saldamando, Tapia-Beltran, Williams, & Hoopingarner, 2009). Alternative foods include Hershey's Rolo Caramels, or for more healthy options, consider fish crackers (direction in which they face).**

Standards for Mathematical Practice

8 Look for and express regularity in repeated reasoning.

In these experiments, there is no practical way to determine the results before you start. However, once you have results for 200 tosses (empirical data), you will undoubtedly feel more confident in predicting the results of the next 100 tosses. After gathering data for 1000 trials, you will feel even more confident. In other words, the more tosses that are made, the more confident you become. For example, in dropping the cup, after 100 or so trials, you may have determined a probability of $\frac{4}{5}$, or 80 percent, for the cup to land on its side.

The Law of Large Numbers

The phenomenon in which the relative frequency of an event becomes a closer approximation of the actual probability or the theoretical probability as the size of the data set (sample) increases is referred to as the law of large numbers. The larger the size of the data set, the more representative the sample is of the population. In thinking about statistics, a survey of 1,000 people provides more reliable and convincing data about the larger population than does a survey of 5 people. The larger the number of trials (people surveyed), the more confident you can be that the data reflect the larger population. The same is true when you are attempting to determine the probability of an event through data collection.

Although critical to understanding probability, this concept is difficult for students to grasp. Students commonly think that a probability should play out in the short term, a misconception sometimes referred to as "the law of small numbers" (Flores, 2006; Tarr, Lee, & Rider, 2006).

Teaching Tip

Comparing small data sets with large data sets is one way to help students think more deeply about the fact that the size of the trial matters.

Comparing small data sets to large data sets is one way to help students think more deeply about the fact that the size of the trial matters. The next two activities are designed with this purpose in mind.

Activity 16.11

CCSS-M: 7.SP.C.6

Get All 6

Ask students to list the numbers 1 through 6 at the bottom of a frequency table. Students roll a die and mark an X over each number until they have rolled each number at least once. Repeat five or six times. Discuss how the frequency charts compare in each case. Students will see that in some cases there were many fours, for example, or that it took 25 rolls before all numbers were rolled, whereas in other cases they got all the numbers in only 10 rolls. Now, pool all the data and discuss the relative frequencies for the numbers that emerge. Focus discussion on the fact that in the short run, data varies a lot—it is over the long run that the data "evens out." This activity can also be done on a graphing calculator (see Flores, 2006 for details) or using software. Students with disabilities can benefit from graphing using tools so that they stay focused on the mathematics and not the details of creating a frequency chart.

Truly random events often occur in unexpected groups; a fair coin may turn up heads five times in a row. A 100-year flood may hit a town twice in 10 years. Using random devices such as spinning spinners, rolling dice, or drawing cubes from a bag (physically or virtually through online apps) gives students an intuitive feel for the imperfect distribution of randomness. The next activity is designed to help students with this difficult idea.

Activity 16.12

CCSS-M: 7.SP.C.6

What Are the Chances?

Use a copy of What Are the Chances? Activity Page. Provide pairs of students with a spinner face that is half red and half blue. Discuss the chances of spinning blue. Mark the halfway point on the continuum of impossible to certain and draw a vertical line down through all of the lines below this point. Then have the students in each pair spin their

spinner 10 times, tallying the number of red and blue spins. Mark the number of blue spins on the second line. For example, if there are 3 blue and 7 red spins, place a mark at about 7 on the 0-to-10 number line. If the result of the 10 spins is not exactly 5 and 5, discuss possible reasons why this may be so.

Repeat 10 more times. Add the count from the first 10 spins, and again mark the total in the right-hand box of the third line. Repeat at least two more times, continuing to add the results of new spins to the previous results. Using a graphing calculator or applet, even 1,000 trials are possible in a short amount of time. Ask students to reflect on what they notice in each number line.

The successive number lines used in “What Are the Chances?” each have the same length, and each represent the total number of trials. When the results are plotted on any one number line, the position shows the fraction of the total spins as a visual portion of the whole line. With more trials, the marks will get closer and closer to the $\frac{1}{2}$ mark. Notice that 240 blue spins out of 500 is 48 percent, or very close to one-half. This is true even though there are 20 more red spins (260) than blue.

The **What Are the Chances?** Activity Page and the process of accumulating data in stages can be used for other experiments. For example, try using this approach with the cup toss experiment in Activity 16.10. Rather than drawing a vertical line before collecting data, decide on the best guess at the actual probability after the number of trials has become large, and then draw the vertical line in the appropriate position. Observe and record on the number lines 10 additional trials, 20 additional trials, and 50 additional trials. Compare these smaller data sets with the larger data set. Write the probabilities as percents and as fractions to show the connection between these representations.

Formative Assessment Note

Use the following *performance assessment* to assess students’ ideas about long-run results versus short-run results. Have students write about their ideas.

> **Margaret spins the spinner 10 times. Blue turns up on 3 spins. Red turns up on 7 spins. Margaret says that there is a 3-in-10 chance of spinning to blue. Carla then spins the same spinner 100 times. Carla records 53 spins to blue and 47 spins to red. Carla says that the chance of spinning to blue on this spinner is about even. Who do you think is correct—Margaret or Carla? Explain. Draw a spinner that you think they may have been using.**

Look for evidence that students understand that the result of 10 spins is not very good evidence of the probability and that the result of 100 spins tells us more about how likely each color is, and therefore what percent of the spinner might be blue or red. Also, to assess whether students understand the big idea that chance has no memory, have students write in their journal about the following:

> **Duane has a lucky coin that he has tossed many, many times. He is sure that it is a fair coin with an even chance of heads or tails. Duane tosses his coin six times and heads come up six times in a row. Duane is sure that the next toss will be tails because he has never been able to toss heads seven times in a row. What do you think the chances are of Duane tossing heads on the next toss? Explain your answer.**

In this case, you are looking for the idea that each toss of the coin is independent of prior tosses.

Why Use Experiments?

Actually conducting experiments and examining outcomes in teaching probability are important in helping students address common misconceptions and build a deeper understanding of why certain things are more likely than others. Specifically, experiments:

- Model real-world problems that are actually solved by conducting experiments (doing simulations). See, for example, "Undersea Treasure" at The Futures Channel Website, which provides a probability map to locate sunken ships that contains gold treasure.
- Provide a connection to counting strategies (lists, tree diagrams) to increase confidence that the probability is accurate.
- Provide an experiential background for examining the theoretical model. When you begin to sense that the probability of two heads is $\frac{1}{4}$ instead of $\frac{1}{3}$ through experiments, the analysis in Figure 16.5 seems more reasonable.
- Help students see how the ratio of a particular outcome to the total number of trials begins to converge to a fixed number. (For an infinite number of trials, the relative frequency and theoretical probability would be the same.)
- Help students learn more than students who are not doing experiments (Gurbuz, Erdem, Catlioglu, & Birgin, 2010).

Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Try to use an experimental approach whenever possible, posing interesting problems to investigate. If a theoretical analysis is possible (e.g., as in the two-coin experiment in "Fair or Unfair?"), it should also be examined and the results compared with the expected outcome.

Use of Technology in Experiments

Random outcomes can be generated by computer applications, notebook apps, and calculators. Calculators can produce random numbers that can then be interpreted relative to the possible outcomes in the experiment. For example, if the final digit is odd, you can assign it to represent one outcome, and if it is even to represent a second outcome. If there are four outcomes, you can look at the remainder when the last two digits are divided by 4 (i.e., the remainder will be 0, 1, 2, or 3) and assign a remainder to each outcome. In addition, some calculators, like the TI-73, TI-83, and TI-84, can run the free Probability Simulation App, an interactive tool that simulates tossing coins, rolling number cubes, using spinners, and generating random numbers.

Standards for Mathematical Practice

5 Use appropriate tools strategically.

Computer applets can be used to flip coins, spin spinners, or draw numbers from a hat virtually. The NCTM Illuminations website has a series of lessons, "Probability Explorations," in which students explore probability through virtual experiments and can also graph the results. As long as students accept the results generated by the technology as truly random or equivalent to those of hands-on devices, these virtual devices have the advantage of being quick, more motivating to some students, and accessible when natural devices (e.g., spinners with various partitions) are not. Web-based tools such as the National Library of Virtual Manipulatives "Spinners" have the advantage of generating many more trials in much less time. Because of the speed at which an experiment can be done, these digital devices afford the opportunity to explore probability across a variety of tools (virtual dice, coins, cards, etc.), including the use of graphic displays of the trials. Also, in a virtual world, dice can be "loaded" and used to challenge students' thinking: "Are these fair dice? How can you find out?" (Beck & Huse, 2007; Phillips-Bey, 2004).

Sample Spaces and the Probability of Compound Events

Understanding the concepts of *sample space* and *event* is central to understanding probability. The *sample space* for an experiment or chance situation is the set of all possible outcomes for that experiment. For example, if a bag contains two red, three yellow, and five blue tiles, the sample space consists of all ten tiles. An *event* is a subset of the sample space. The event of drawing a yellow tile has three elements or outcomes in the sample space, and the event of drawing a blue tile has five elements in the sample space. For rolling a single common die, the sample space consists of the numbers 1 through 6. A two-event experiment requires two (or more) actions to determine an outcome. Examples include rolling two dice, drawing two tiles from a bag, and the combination of the occurrence of rain and forgetting your umbrella.

When two-event experiments are explored, there is another factor to consider: Does the occurrence of the event in one stage have an effect on the occurrence of the event in the other? In the following sections, we will consider two-event experiments of both types—those with independent events and those with dependent events. You can **watch** (**https://www.youtube.com/watch?v=2cIIwwh0Kz0**) a lesson in which sixth graders engage in simulations of dependent and independent events using bags of objects.

Independent Events

In Activity 16.8, "Fair or Unfair?," students explored the results of tossing two coins. The toss of one coin had no effect on the toss of the other. Tossing a coin twice is an example of *independent events*; the occurrence or nonoccurrence of one event has no effect on the other.

Let's explore rolling two dice and adding the results. Suppose that your students gather data on the sums that they get for two dice. The results might be recorded in a dot plot, as in Figure 16.7(a). These events (sums) do not appear to be equally likely, and in fact the sum of 7 appears to be the most likely outcome. To explain this, students might look for the combinations that make 7: 1 and 6, 2 and 5, and 3 and 4. But there are also three combinations for 6 and for 8. It seems as though 6 and 8 should be just as likely as 7, and yet they are not.

Now suppose that the experiment is repeated. This time, for the sake of clarity, suggest that the students roll two dice of different colors and that they keep the tallies in a chart like the one in Figure 16.7(b).

The results of a large number of dice rolls indicate what one would expect—namely, that all 36 cells of this chart are equally likely (however, the sums are not equally likely–why?). Compare the sums of 6, 7, and 8, the most common sums. Notice that for 7, red 3, green 4 is different from red 4, green 3, and there are 3 such pairs for a total of 6 outcomes that result in a sum of 7 out of a total of the sample space (36), for a probability of $\frac{6}{36}$, or $\frac{1}{6}$. The sums for 6 and for 8 each include a double, and therefore each sum has five outcomes (not 6), for a probability of $\frac{5}{36}$. The color-coded dice can help students see how possibilities are counted, addressing the fact that while 3 + 4 is the same as 4 + 3, these are each separate outcomes in rolling dice.

To create the sample space for two independent events, use a chart or diagram that keeps the two events separate and illustrates all possible combinations. The matrix in Figure 16.7(b) is effective when there are only two events.

Teaching Tip

Using dice (or counters) of different colors can help students focus on each event separately and therefore more clearly see the different possible outcomes.

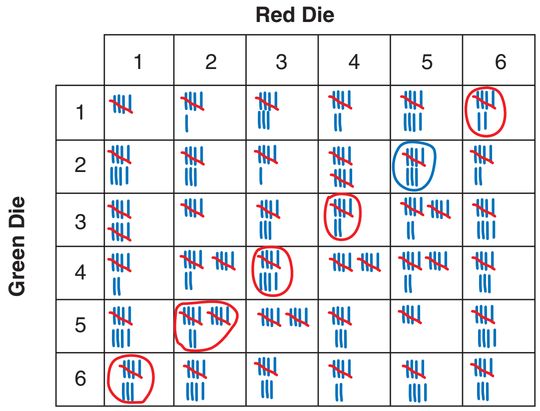
Figure 16.7

Exploring the frequencies (dot plot) and possible outcomes (matrix) for the sum of two dice.

(a) A dot plot is used to gather data on frequency of the sum of two dice (one group's data).

Number of times sum appears	Sum										
	2	3	4	5	6	7	8	9	10	11	12
10											
9											
8											
7											
6											
5					●						
4					●		●				
3			●	●	●	●	●				
2			●	●	●	●	●	●			
1	●	●	●	●	●	●	●	●	●	●	

(b) A matrix is used for documenting the possible outcomes for the sum of two dice (class data).



A tree diagram (Figure 16.8) is another method of determining sample spaces that can be used with any number of events. For example, consider the context of creating an ice cream cone. You can choose a waffle cone or a regular cone, ice cream that is dipped or not dipped, and then choice of three single flavors. This can be simulated with coins and a spinner, as illustrated in Figure 16.8.

Stop and Reflect

Use a chart and/or a tree diagram to analyze the sum of two number cubes, each with sides 1, 1, 2, 3, 3, and 3. (These were the cubes used in Activities 16.3 and 16.4.) What is the probability of each sum, 1 through 6? How might these tools support student understanding of sample space and the probability of independent events?

A common process to help students connect sample space with probability is to ask them first to make a prediction of the probability of the event, second to conduct an experiment with a large number of trials, and third to compare the prediction with what happened. Then ask students to create the sample space and see how it compares with the prediction and the results of the experiment. Games provide an excellent context for such explorations, as described in Activity 16.13.

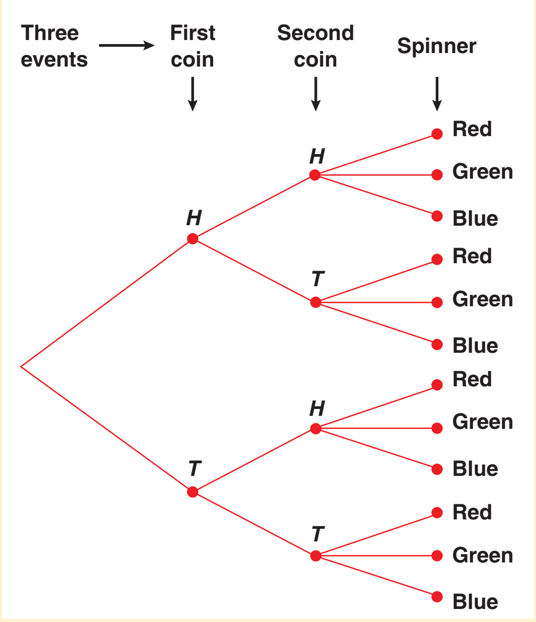
The following are additional examples of probabilities of independent events. Any one of these could be explored as part of a full lesson.

- Rolling an even sum with two dice
- Spinning blue twice on a spinner
- Having a tack *or* a cup land up when each is tossed once
- Getting *at least* two heads from tossing four coins
- Rolling two dice and getting a difference that is *no more than* 3

Words and phrases such as *and*, *or*, *at least*, and *no more than* may cause students some confusion and therefore require explicit attention. Of special note is the word *or* because its meaning in everyday usage is generally not the same as its strict logical meaning in mathematics. In mathematics, *or* includes the case of *both*. For example, in the tack-cup toss experiment, the event of "tack or cup landing up" includes tack (only) up, cup (only) up, and *both* tack *and* cup up.

Figure 16.8

A tree diagram showing all possible outcomes for two coins and a spinner that is $\frac{1}{3}$ red, $\frac{1}{3}$ green, and $\frac{1}{3}$blue.



Area Representation

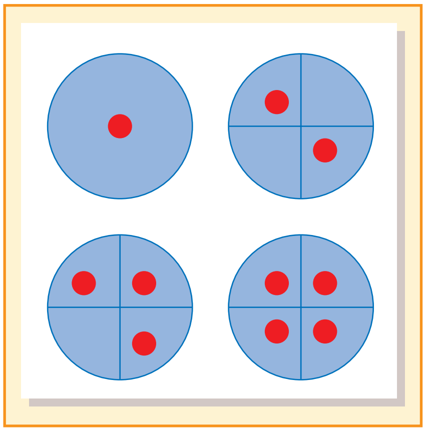
One way to determine the theoretical probability of a multistage event is to list all possible outcomes and count the number of outcomes that make up the event. This is effective but has some

Activity 16.13

CCSS-M: 7.SP.C.7a; 7.SP.C.8a

Lu-Lu

This Hawaiian partner game involves taking turns tossing four stones and calculating the resulting score. (You can create stones like the ones pictured here by getting glass stones from a craft store and marking dots on one side.) Player 1 tosses the four stones (marked on only one side). If all 4 are face up, they score 10 and take a second turn, adding 10 to their second sum. If all 4 are not face up, the player re-tosses the face-down stones and adds any face up stones to their first toss. High score wins.



After they have played, ask what they notice about the sums they are getting:

What sums are possible?

What sums are common?

What are all the outcomes (possible combinations of stones)?

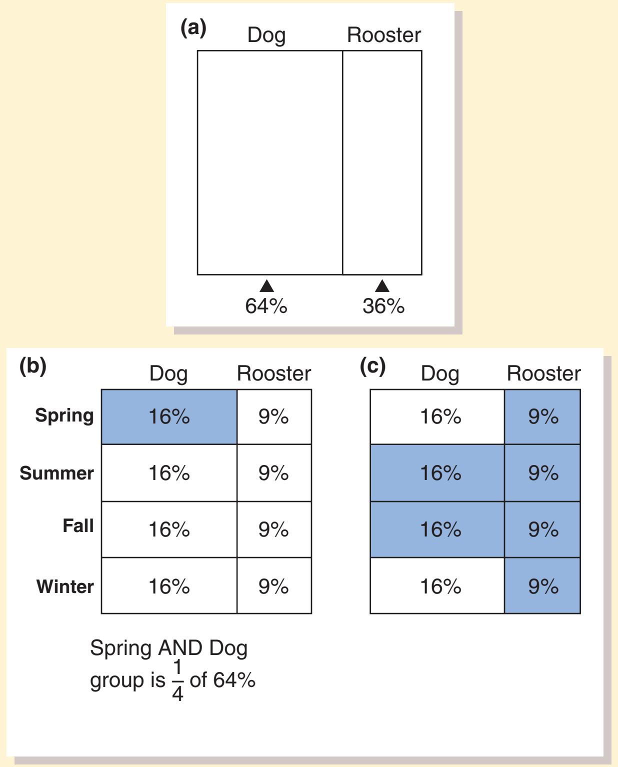
What is the probability of each score?

Students from different cultures may have games from their native countries or communities. These can be used to explore probability. (See McCoy, Buckner, Munley, 2007, for more on this game.)

limitations. First, a list implies that all outcomes are equally likely. Second, lists can get tedious when there are many possibilities. Third, students can lose track of which possibilities they have included in the list and may leave off some of the possibilities. For all of these reasons, an area representation is a good option for determining probability, the focus of Activity 16.14.

Figure 16.9

An area representation for determining probabilities of simple and compound events.



Activity 16.14

CCSS-M: 7.SP.C.8a, b

Are You a Spring Dog?

Before doing this activity, determine which Chinese birth year animals are likely to be represented in the classroom (e.g., the dog and the rooster). Spend some time discussing the Chinese birth year animals with students. (This would be particularly timely at the Chinese New Year!) Begin by finding out what percentage of the class is represented by each animal. Ask, "If I name one of the Chinese birth year animals, what is the probability it will be *your* birth year animal?" Illustrate this percentage by partitioning a rectangle, as in Figure 16.9(a). (This particular illustration finds that 64 percent of the students in the class were born in the year of the dog, and 36 percent were born in the year of the rooster.) Ask, "If I name one of the seasons, what is the probability it will be *your* season?" Ask students to illustrate their response by partitioning and shading a rectangle (Figure 16.9(b)). Then ask, "What is the probability of being both a spring and a dog?"

Teaching Tip

The area representation is accessible to a range of learners because it is less abstract than equations or tree diagrams.

In Figure 16.9(b), you can visually see that students in the year-of-the-dog and spring groups make up $\frac{1}{4}$ of 64 percent, or 16 percent of the population. This should look very familiar because the same process is used for multiplying fractions.

The area representation is also effective in solving "or" situations. Consider the question, "What is the probability you were born in fall or summer, or that you are a rooster?" The shading for this example is illustrated in Figure 16.9(c). Half of the students are born in summer or fall, and 36 percent are born in the year of the rooster. Students can add the percentages in the boxes, or they can think about the two situations separately: 50 percent are born in summer or fall and 36 percent are born in the year of the rooster. The sum of these two separate results would be 86 percent, but some students are "both" and have therefore been double-counted (see overlap in diagram). In this example, the overlap (students falling in "both") is 18 percent. Therefore, the population that is born in summer or fall or born in the year of the rooster is 50 + 36 − 18 = 68 percent of the population. This can be generalized to the following model for the probability of two independent events:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Designing a spinner is a challenging and engaging way for students to think about the probability of independent events (Ely & Cohen, 2010). The following activity is challenging; an area representation can help students reason about how to design their spinner.

Standards for Mathematical Practice

2 Reason abstractly and quantitatively.

Activity 16.15

CCSS-M: 7.SP.C.6; 7.SP.C.7a; 7.SP.C.8a, b

Design a Winning Spinner

Explain that each student is going to create a winning spinner, which means that when it is spun twice, the sum will be on a number strip with values 2, 3, 4, 5, 6, 7, and 8. Students create their own spinner, partitioning the circle however they like and writing a number in each sector. Students with disabilities may benefit from having a spinner already prepared for the first round of the activity, then adapt it themselves for the second and third rounds. Once students have their spinner, they pair with someone else and play the game with their own spinner. Student A spins twice and adds the two values. If the sum is 5, student A covers 5 on his or her number strip. Student B takes a turn. The first partner to cover all numbers on his or her strip wins. Play three rounds. Next, ask students to redesign their spinner, find a new partner, and play three more rounds. If possible, repeat a third time. Afterward, discuss how they designed a winning spinner.

In going backward (from the desired outcome to the spinner), students can build a deeper understanding of how to determine the probability of independent events.

Dependent Events

A dependent event is a second event whose result depends on the result of a first event. For example, suppose that there are two identical boxes. One box contains one genuine dollar bill and two counterfeit bills, and the other box contains one genuine and one counterfeit bill (you do not know which box is which). You may choose one box and from that box select one bill without looking. What are your chances of getting a genuine dollar bill? Here, there are two events: selecting a box and selecting a bill. The probability of getting a dollar in the second event depends on which box is chosen in the first event. The events are *dependent*. The next activity is an engaging way to explore probability of dependent events.

Activity 16.16

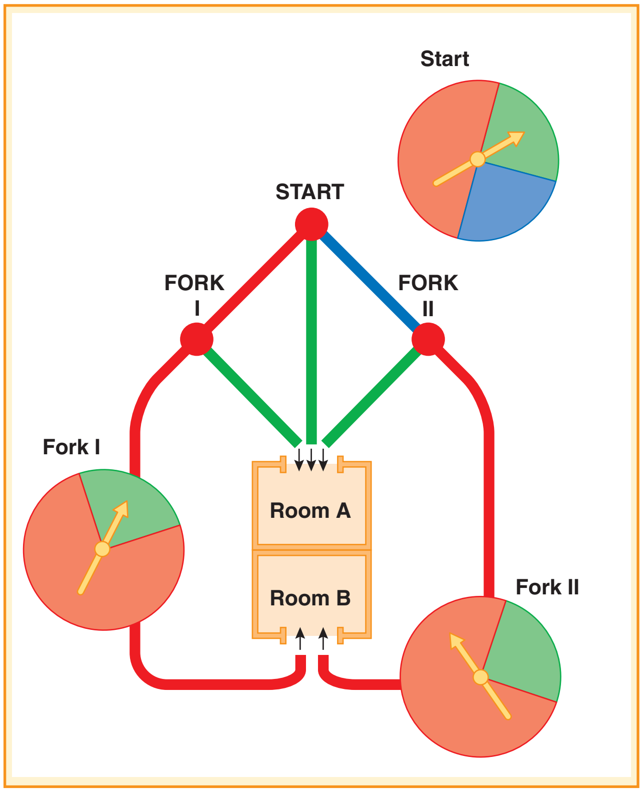
CCSS-M: 7.SP.C.8a, b

Keys to a New Car

Pose the following problem: In a game show, you can win a car—if you make it through the maze to the room where you have placed the car key. You can place the key in either Room A or Room B (see maze in Figure 16.10). At the start and at each fork in the path, you must spin the indicated spinner and follow the path it points to. In which room should you place the key to have the best chance of winning the car?

Figure 16.10

Should you place your key in Room A or Room B to have the best chance at winning?



You can also use the area representation to determine the probability for dependent events. Figure 16.11 illustrates the "Keys to a New Car" task.

Stop and Reflect

How would the area representation for the car problem be different if the spinners at Fork I and Fork II were $\frac{1}{3}$ A and $\frac{2}{3}$ B spinners? What questions like this one can you ask students in order to help them think about how one event depends on the next?

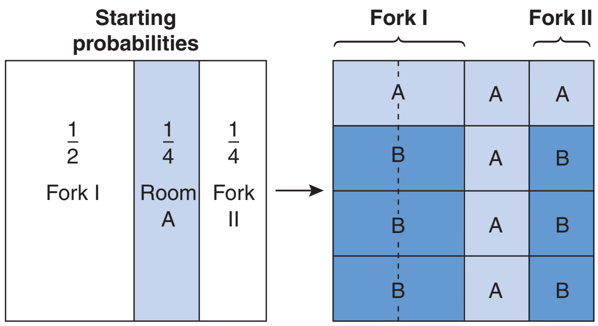
Figure 16.12 shows a tree diagram for the "Keys to a New Car" problem, with the probability of each path in the maze written on the "branch" of the tree. The tree diagram is more abstract than the area approach, but it applies to a wider range of situations. Each branch of the tree diagram in Figure 16.12 matches with a section of the square in Figure 16.11. Use the area approach to explain why the probability for each complete branch of the tree is determined by multiplying the probabilities along the branch.

Teaching Tip

Having students describe the connection between the area representation and the tree diagram can help build meaning for the tree diagram approach, which can be used in any multiple-events probability task.

Figure 16.11

Using an area representation to solve the maze problem.



At Fork I, $\frac{3}{4}$ of the time you will go to Room B.

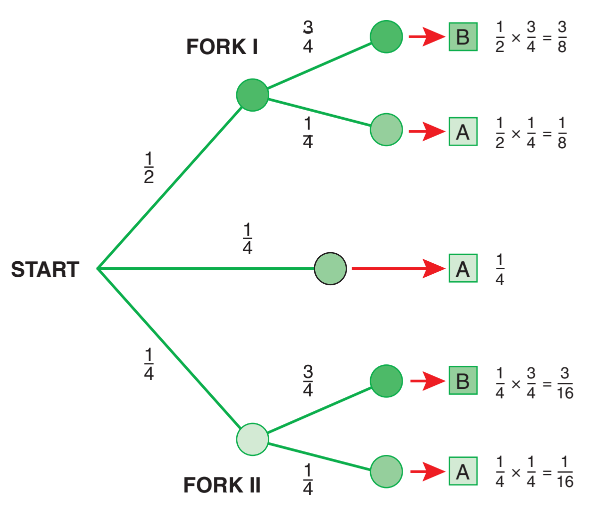
(Note: Not $\frac{3}{4}$ of the square but $\frac{3}{4}$ of the times you go to Fork I.)

At Fork II, $\frac{3}{4}$ of these times (or $\frac{3}{16}$ of total time) you will go to Room B.

Therefore, you will end up in Room A $\frac{7}{16}$ of the time and Room B $\frac{9}{16}$ of the time.

Figure 16.12

A tree diagram is another way to model the outcomes of two or more dependent events.



Simulations

Simulation is a technique used for answering real-world questions or making decisions in complex situations where an element of chance is involved. Many times, simulations are conducted because it is too dangerous, complex, or expensive to manipulate the real situation. To see what is likely to happen in the real event, a model must be designed that has the same probabilities as the real situation. For example, in designing a rocket, a large number of related systems all have some chance of failure that might cause serious problems with the rocket. Knowing the probability of serious failures will help determine whether redesign or backup systems are required. It is not reasonable to make repeated tests of the actual rocket. Instead, a model that simulates all of the chance situations is designed and run repeatedly with the help of a computer. The computer model can simulate thousands of flights, and an estimate of the chance of failure can be made.

Activity 16.17

CCSS-M: 7.SP.C.8b, c

Probability of Getting Water

Show students an illustration of a water pump system like the one illustrated in Figure 16.13. Explain that the five pumps that connect A and B are aging, and it is estimated that at any given time, the probability of pump failure is $\frac{1}{2}$. If a pump fails, water cannot pass that station. For example, if pumps 1, 2, and 5 fail, water flows only through pumps 4 and 3. Ask students to discuss how likely they think it is that water will make it through the pump. Have students mark how likely on a probability continuum. Follow the steps for teaching a simulation (described below). After students have completed their simulation revisit the important probability questions:

- **What is the probability that water will flow at any time?**
- **On average, about how many stations need repair at any time?**

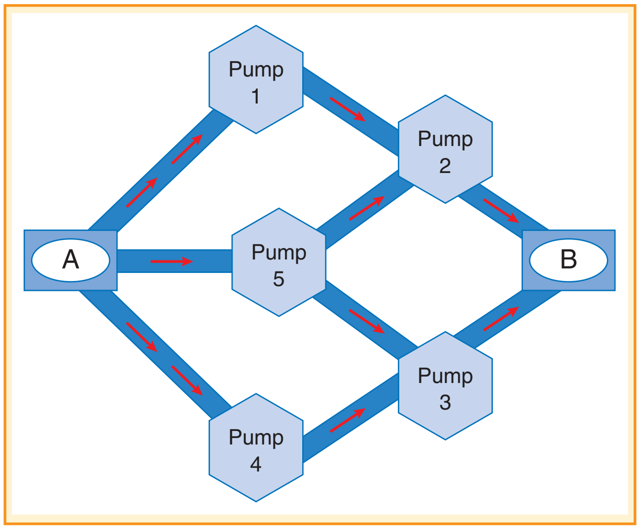
Shodor's Project Interactivate offers a realistic simulation of actual forest fires. The simulation, titled "A Better Fire," uses a virtual "die" to see whether a tree should be planted for each square. Then the fire is set and allowed to burn.

For any simulation, the following steps can serve as a useful guide. Here, the steps are explained for use with Activity 16.17.

1. *Identify key components and assumptions of the problem.* The key component in the water problem is the condition of a pump. Each pump is either working or not working. In this problem, the assumption is that the probability that a pump is working is $\frac{1}{2}$.
2. *Select a random device for the key components.* Any random device can be selected that has outcomes with the same probability as those of the key component—in this case, the pumps. Here a simple choice might be tossing a coin, with heads representing a working pump.
3. *Define a trial.* A *trial* consists of simulating a series of key components until the situation has been completely modeled one time. In this problem, a trial could consist of tossing a coin five times, each toss representing a different pump (heads for pump is working and tails for pump is not working).
4. *Conduct a large number of trials and record the information.* For this problem, it would be useful to record the number of heads and tails in groups of five because each set of five is one trial and represents all of the pumps.
5. *Use the data to draw conclusions.* There are four possible paths for the water, each flowing through two of the five pumps. As they are numbered in the drawing, if any one of the pairs 1–2, 5–2, 5–3, and 4–3 is open, it makes no difference whether the other pumps are working. By counting the number of trials in which both coins in at least one of the four pairs come up heads, we can estimate the probability of water flowing. To answer the second question, the number of tails (pumps not working) per trial can be averaged.

Figure 16.13

Each of these five pumps has a 50 percent chance of failure. What is the probability that some path from A to B is working?



Here are a few more examples of problems for which a simulation can be used to gather empirical data.

In a true-or-false test, what is the probability of getting 7 out of 10 questions correct by guessing alone? (*Key component*: Answering a question. *Assumption*: Chance of getting the correct answer is $\frac{1}{2}$.)

Simulation option: Flip a coin 10 times for one trial.

In a group of five people, what is the chance that two were born in the same month? (*Key component*: Month of birth. *Assumption*: All 12 months are equally likely.)

Simulation option: Use 12-sided dice or 12 cards. Draw/roll one, replace, and draw/roll again.

Casey's batting average is .350. What is the chance he will go hitless in a complete, nine-inning game? (*Key component*: Getting a hit. *Assumptions*: The probability of a hit for each at-bat is .35. Casey will get to bat four times in the average game.)

Simulation option: Use a spinner that is 35 percent shaded. Spin four times for one trial.

Krunch-a-Munch cereal packs one of five games in each box. About how many boxes should you expect to buy before you get a complete set? (*Key component*: Getting one game. *Assumption*: Each game has a $\frac{1}{5}$ chance.)

Simulation option: Use a spinner with five equal sections. Spin until all sections occur at least once. Record how many spins it took (this is one trial). Repeat. The average length of a trial answers the question. *Extension*: What is the chance of getting a set in eight or fewer boxes?

Students often have trouble selecting an appropriate random device for simulations. Spinners are an obvious choice because areas can be adjusted to match probabilities. Coins or two-color counters are useful for probabilities of $\frac{1}{2}$. A standard die can be used for probabilities that are multiples of $\frac{1}{6}$. There are also dice available online and on smart phones with 4, 8, 12, and 20 sides. Many calculators include a key that will produce random numbers that can be used to simulate experiments (e.g., 1 means true, 2 means false). Usually, the random numbers generated are between 0 and 1, such as 0.8904433368. How could a list of decimals like this replace flipping a coin or spinning a spinner? Suppose each was multiplied by 2. The results would be between 0 and 2, as shown here:

$$0.8904433368 \times 2 = 1.7808866736$$
$$0.0232028877 \times 2 = 0.0464057754$$
$$0.1669322714 \times 2 = 0.3338645428$$

If you focus on the ones column, you have a series of zeros and ones that could represent heads and tails, boys and girls, true and false, or any other pair of equally likely outcomes. For three outcomes, the same as a $\frac{1}{4}/\frac{1}{4}/\frac{1}{2}$ spinner, you might decide to look at the first two digits of the number and assign values from 0 to 24 and from 25 to 49 to the two one-quarter portions and values from 50 to 99 to the one-half portion. The NCTM Illuminations Adjustable Spinners can be set up for such simulations. Alternatively, each randomly generated number could be multiplied by 4 and the decimal part ignored, resulting in random numbers 0, 1, 2, and 3. These could then be assigned to the desired outcomes.

In this activity, consider how you would design a simulation.

Activity 16.18

CCSS-M: 7.SP.C.8b, c

Chance of Triplet Girls

Ask students, "What is the chance that a woman having triplets will end up with all girls?" Record estimates. Ask students to create a simulation to model this problem using the five steps previously described. Encourage students to use various tools to simulate (flipping three coins, using a random number generator, spinning a two-color spinner three times, etc.). After examining the results, ask questions to relate the predictions to the results. This may lead to creating a tree diagram of the options to make sense of the results.

Common Misconceptions about Probability

Tasks like "Chance of Triplet Girls" can lead to interesting follow-up questions: "Are three girls less likely or more likely than two girls and a boy? If a family already has two girls, what do you think they will have for their third child?" (Tillema, 2010). These questions connect to two of the common misconceptions students have related to probability: commutativity confusion and gambler's fallacy. These misconceptions, and two shared earlier, are discussed briefly here.

1. *Commutativity confusion.* Students may think that two girls and one boy is one possible outcome. But notice that if you list the eight possible combinations for three children, you have three girls only once (GGG) but two girls and one boy three times (GGB, GBG, BGG). Two girls and one boy is three times as likely. We refer to this as *commutativity confusion* because students, knowing that 3 + 4 is the same as 4 + 3, think that that one boy and two girls is one event, not three. Students need to think about all the ways an event can occur when they try to determine how likely an event is.
2. *Gambler's fallacy.* The gambler's fallacy is the notion that what has already happened (two girls) influences the event. Students will argue a boy is more likely because there are already two girls. Similarly, students think that if a tossed coin has had a series of four heads, it is more likely on the fifth toss to be tails (Ryan & Williams, 2007). But a coin has no memory, and the probability of heads or tails is still 50–50, just as the probability of a girl is still 50-50.
3. *Law of small numbers.* This misconception is like the gambler's fallacy in that it relates to small samples, but with this misconception, students expect small samples to be like the greater population (Flores, 2006; Tarr et al., 2006). This was discussed in the section "The Law of Large Numbers" (see Activities 16.11 and 16.12). So, in the case of the three girls, it is not so unusual—it is just a very small data set, so it is not likely to resemble the larger population.
4. *Possibility counting.* In this case, students see what the possibilities are, and assume each is equally likely. If a dice has a 1, 2, 3, then each has a one-third chance of occurring. Similarly, if a spinner has a red and blue section, then each has a 50–50 chance of occurring, even if there is more red on the spinner than blue. Activity 16.3 (1-2-3 How Likely?), Activity 16.4 (1-2-3 How Likely Are Sums?) and Activity 16.5 (Race to 10) are designed to help students focus on frequency rather than possible outcomes in determining probability of an event.

Whether doing simulations, experiments, or theoretical probability, it is important for students to use many models (lists, area, tree diagrams) and discuss developing conceptions and misconceptions explicitly. In addition to being more interesting, teaching probability in this way allows students to understand important concepts that have many real-world implications. This final activity is designed to apply different representations and to apply simulations.

Activity 16.19

CCSS-M: 7.SP.C.8a, b, c

Money in Two Piggy Banks

Use the **Money in Two Piggy Banks** Activity Page or post the following problem for students:

> Use an area representation and a tree diagram to determine the probability for the following situation:
>
> In a game at the County Fair, the game leader puts one $5 bill and three $1 bills in Piggy Bank 1. In Piggy Bank 2, he puts one $5 bill and one $1 bill. To play the game, you get to take one bill from Piggy Bank 1 (without looking) and put it in Piggy Bank 2. After mixing Piggy Bank 2, you get to take one bill from that bank. The game costs $2 to play. Should you spend your money?

Ask students to illustrate the theoretical probability using (1) an area representation and (2) a tree diagram. This can be done by having some groups work on one, and other groups work on the other and then comparing and sharing their diagrams. Or, everyone can create both. After these illustrations are complete, ask students how they might design a simulation to test this game. Invite students to explain their simulation and present it. Ask other students to determine if the simulation accurately models the game.

Literature Connections

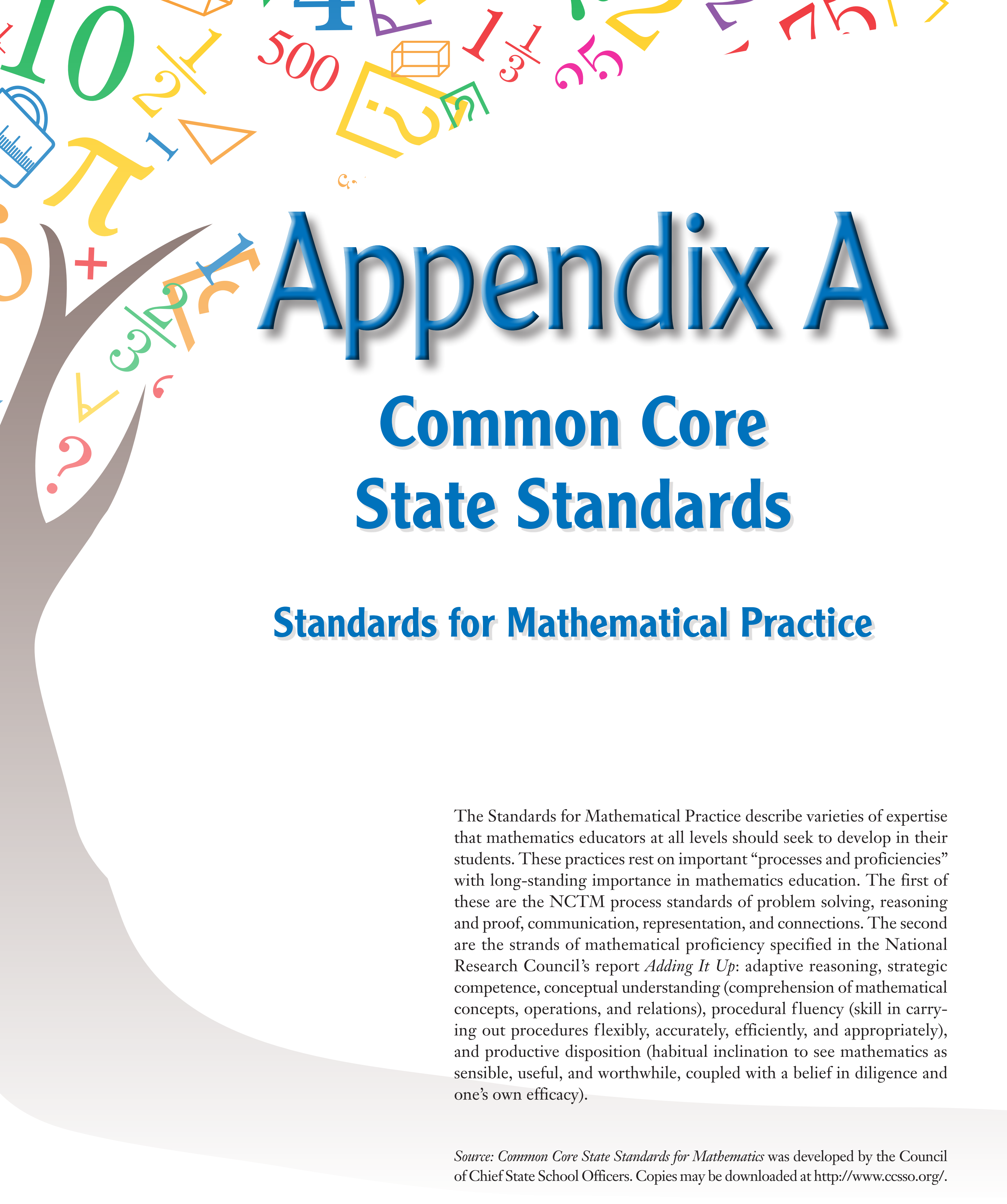
The books described here offer both fanciful and real-life data for investigating probability. Also, these books can be paired with activities in this chapter.

Go Figure! A Totally Cool Book about Numbers *Ball, 2005*

This wonderful book could be placed in every chapter of this book. About 40 different topics are covered, one of which is called "Take a Chance." This two-page spread is full of interesting contexts for probability, including a match-dropping experiment and genetics.

Do You Wanna Bet? Your Chance to Find Out about Probability *Cushman, 2007*

The two characters in this book, Danny and Brian, become involved in everyday situations both in and out of school. Each situation involves an element of probability. For example, two invitations to birthday parties are for the same day. What is the chance that two friends would have the same birthday? In another situation, Danny flips heads several times and readers are asked about Brian's chances on the next flip. Students can create simulations to examine some of the ideas.



Appendix A

Common Core State Standards

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with long-standing importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations, and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

Source: Common Core State Standards for Mathematics was developed by the Council of Chief State School Officers. Copies may be downloaded at http://www.ccsso.org/.

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose.

Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

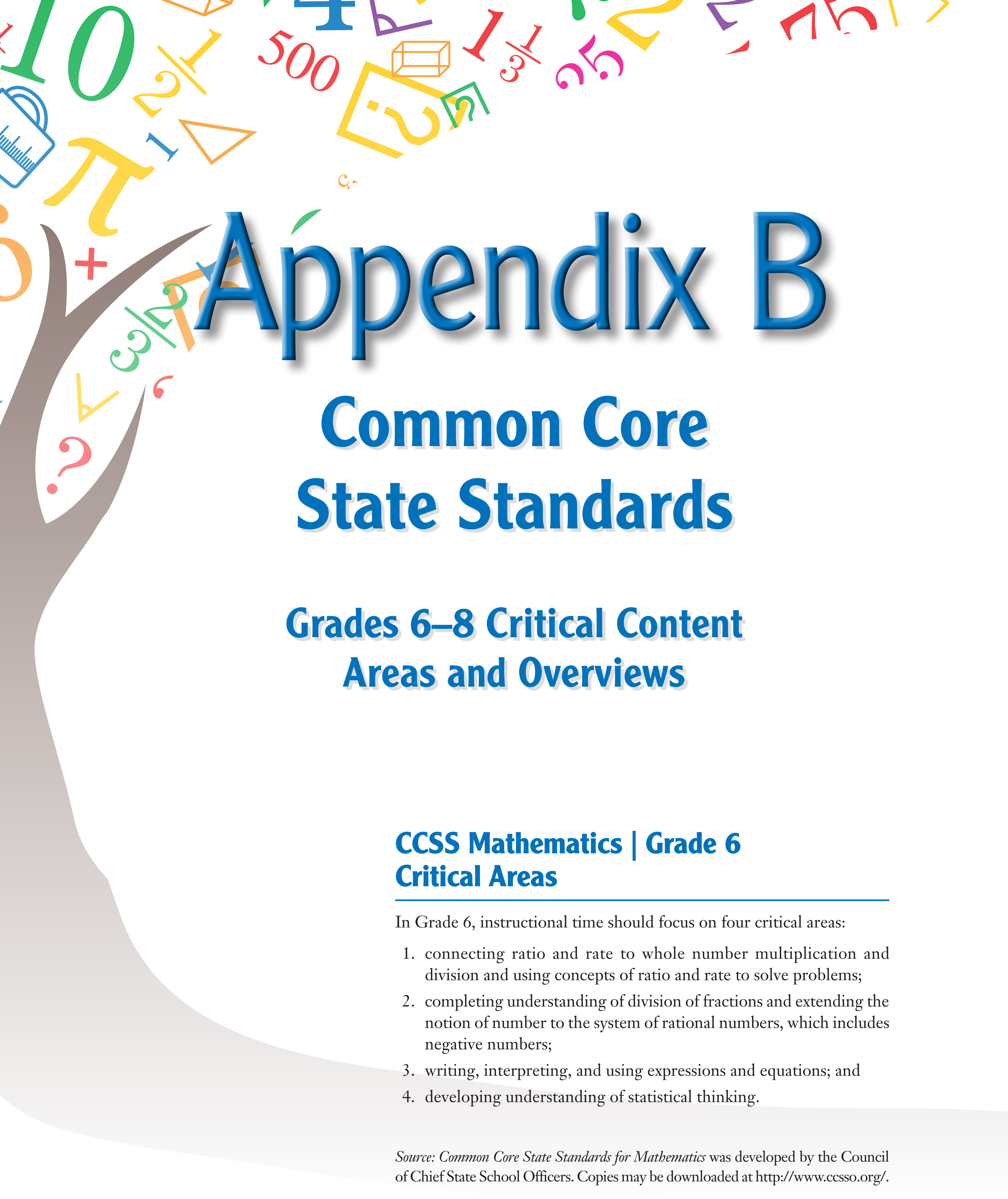
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.



Appendix B

Common Core State Standards

Grades 6–8 Critical Content Areas and Overviews

CCSS Mathematics | Grade 6 Critical Areas

In Grade 6, instructional time should focus on four critical areas:

1. connecting ratio and rate to whole number multiplication and division and using concepts of ratio and rate to solve problems;
2. completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers;
3. writing, interpreting, and using expressions and equations; and
4. developing understanding of statistical thinking.

Source: Common Core State Standards for Mathematics was developed by the Council of Chief State School Officers. Copies may be downloaded at http://www.ccsso.org/.

1. ***Students use reasoning about multiplication and division to solve ratio and rate problems about quantities.*** By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students connect their understanding of multiplication and division with ratios and rates. Thus students expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions. Students solve a wide variety of problems involving ratios and rates.
2. ***Students use the meaning of fractions, the meanings of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense.*** Students use these operations to solve problems. Students extend their previous understandings of number and the ordering of numbers to the full system of rational numbers, which includes negative rational numbers, and in particular negative integers. They reason about the order and absolute value of rational numbers and about the location of points in all four quadrants of the coordinate plane.
3. ***Students understand the use of variables in mathematical expressions.*** They write expressions and equations that correspond to given situations, evaluate expressions, and use expressions and formulas to solve problems. Students understand that expressions in different forms can be equivalent, and they use the properties of operations to rewrite expressions in equivalent forms. Students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships between quantities.
4. ***Building on and reinforcing their understanding of number, students begin to develop their ability to think statistically.*** Students recognize that a data distribution may not have a definite center and that different ways to measure center yield different values. The median measures center in the sense that it is roughly the middle value. The mean measures center in the sense that it is the value that each data point would take on if the total of the data values were redistributed equally, and also in the sense that it is a balance point. Students recognize that a measure of variability (interquartile range or mean absolute deviation) can also be useful for summarizing data because two very different sets of data can have the same mean and median yet be distinguished by their variability. Students learn to describe and summarize numerical data sets, identifying clusters, peaks, gaps, and symmetry, considering the context in which the data were collected.

Students in Grade 6 also build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and volume. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposing them into pieces whose area they can determine. They reason about right rectangular prisms with fractional side lengths to extend formulas for the volume of a right rectangular prism to fractional side lengths. They prepare for work on scale drawings and constructions in Grade 7 by drawing polygons in the coordinate plane.

Grade 6 Overview

Ratios and Proportional Relationships

- Understand ratio concepts and use ratio reasoning to solve problems.

The Number System

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Compute fluently with multidigit numbers and find common factors and multiples.
- Apply and extend previous understandings of numbers to the system of rational numbers.

Expressions and Equations

- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason about and solve one-variable equations and inequalities.
- Represent and analyze quantitative relationships between dependent and independent variables.

Geometry

- Solve real-world and mathematical problems involving area, surface area, and volume.

Statistics and Probability

- Develop understanding of statistical variability.
- Summarize and describe distributions.

CCSS Mathematics | Grade 7 Critical Areas

In Grade 7, instructional time should focus on four critical areas:

1. developing understanding of and applying proportional relationships;
2. developing understanding of operations with rational numbers and working with expressions and linear equations;
3. solving problems involving scale drawings and informal geometric constructions, and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and
4. drawing inferences about populations based on samples.

1. ***Students extend their understanding of ratios and develop understanding of proportionality to solve single- and multi-step problems.*** Students use their understanding of ratios and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease.

Students solve problems about scale drawings by relating corresponding lengths between the objects or by using the fact that relationships of lengths within an object are preserved in similar objects. Students graph proportional relationships and understand the unit rate informally as a measure of the steepness of the related line, called the slope. They distinguish proportional relationships from other relationships.

2. ***Students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers.*** Students extend addition, subtraction, multiplication, and division to all rational numbers, maintaining the properties of operations and the relationships between addition and subtraction, and multiplication and division. By applying these properties, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero), students explain and interpret the rules for adding, subtracting, multiplying, and dividing with negative numbers. They use the arithmetic of rational numbers as they formulate expressions and equations in one variable and use these equations to solve problems.
3. ***Students continue their work with area from Grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects.*** In preparation for work on congruence and similarity in Grade 8 they reason about relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections. They solve real-world and mathematical problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes and right prisms.
4. ***Students build on their previous work with single data distributions to compare two data distributions and address questions about differences between populations.*** They begin informal work with random sampling to generate data sets and learn about the importance of representative samples for drawing inferences.

Grade 7 Overview

Ratios and Proportional Relationships

- Analyze proportional relationships and use them to solve real-world and mathematical problems.

The Number System

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Expressions and Equations

- Use properties of operations to generate equivalent expressions.
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Geometry

- Draw, construct and describe geometrical figures and describe the relationships between them.
- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

Statistics and Probability

- Use random sampling to draw inferences about a population.
- Draw informal comparative inferences about two populations.
- Investigate chance processes and develop, use, and evaluate probability models.

CCSS Mathematics | Grade 8 Critical Areas

In Grade 8, instructional time should focus on three critical areas:

1. formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations and systems of linear equations;
2. grasping the concept of a function and using functions to describe quantitative relationships;
3. analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem.

1. ***Students use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems.*** Students recognize equations for proportions ($y/x = m$ or $y = mx$) as special linear equations ($y = mx + b$), understanding that the constant of proportionality (m) is the slope, and the graphs are lines through the origin. They understand that the slope (m) of a line is a constant rate of change, so that if the input or x-coordinate changes by an amount A, the output or y-coordinate changes by the amount $m \cdot A$. Students also use a linear equation to describe the association between two quantities in bivariate data (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question and to interpret components of the relationship (such as slope and y-intercept) in terms of the situation.

Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

2. ***Students grasp the concept of a function as a rule that assigns to each input exactly one output.*** They understand that functions describe situations where one quantity determines another. They can translate among representations and partial representations of functions (noting that tabular and graphical representations may be partial representations), and they describe how aspects of the function are reflected in the different representations.

3. ***Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems.*** Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons. Students complete their work on volume by solving problems involving cones, cylinders, and spheres.

Grade 8 Overview

The Number System

- Know that there are numbers that are not rational, and approximate them by rational numbers.

Expressions and Equations

- Work with radicals and integer exponents.
- Understand the connections between proportional relationships, lines, and linear equations.
- Analyze and solve linear equations and pairs of simultaneous linear equations.

Functions

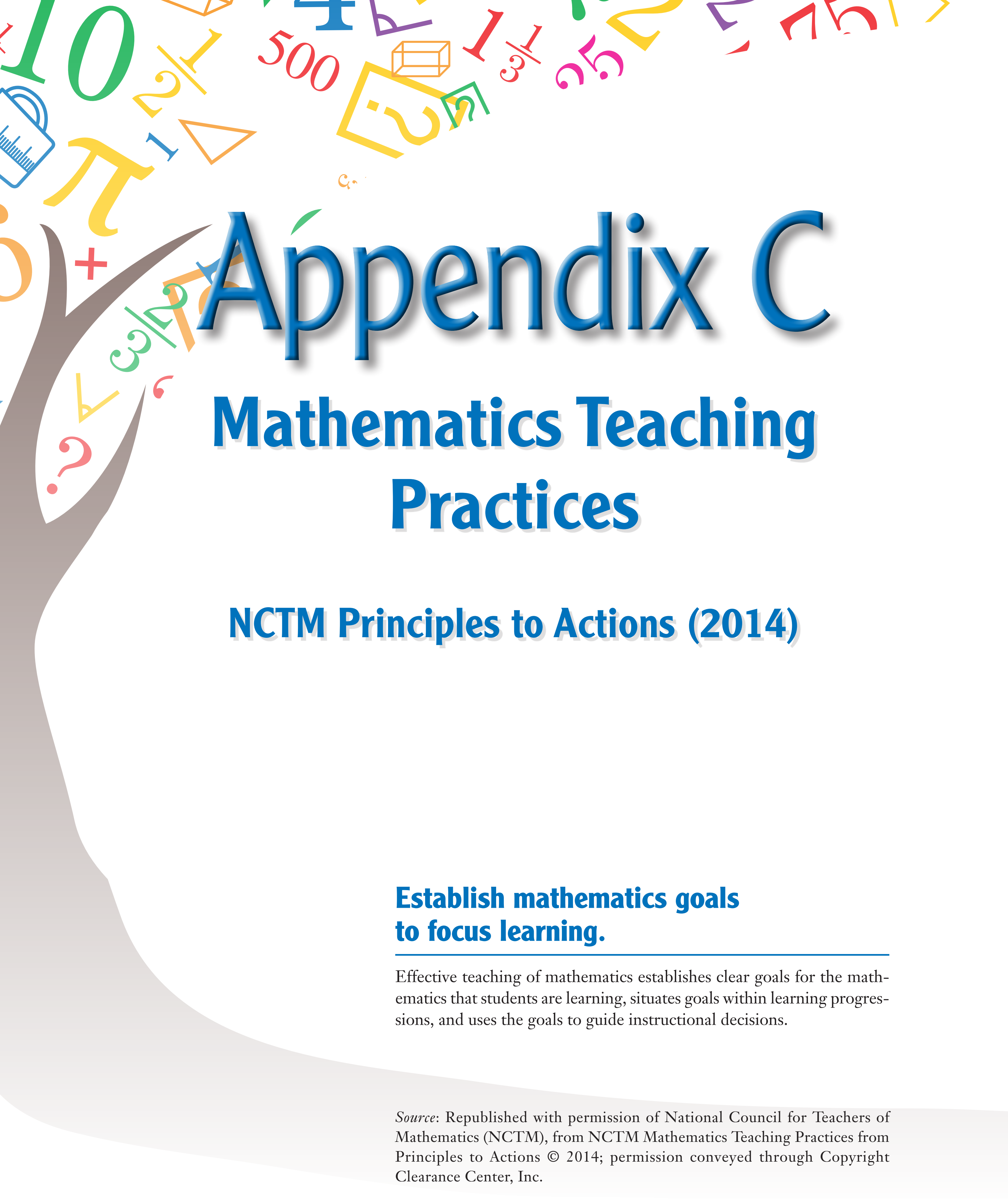
- Define, evaluate, and compare functions.
- Use functions to model relationships between quantities.

Geometry

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Understand and apply the Pythagorean Theorem.
- Solve real-world and mathematical problems involving volume of cylinders, cones and spheres.

Statistics and Probability

- Investigate patterns of association in bivariate data.



Appendix C

Mathematics Teaching Practices

NCTM Principles to Actions (2014)

Establish mathematics goals to focus learning.

Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

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Implement tasks that promote reasoning and problem solving.

Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Use and connect mathematical representations.

Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

Facilitate meaningful mathematical discourse.

Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Pose purposeful questions.

Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

Build procedural fluency from conceptual understanding.

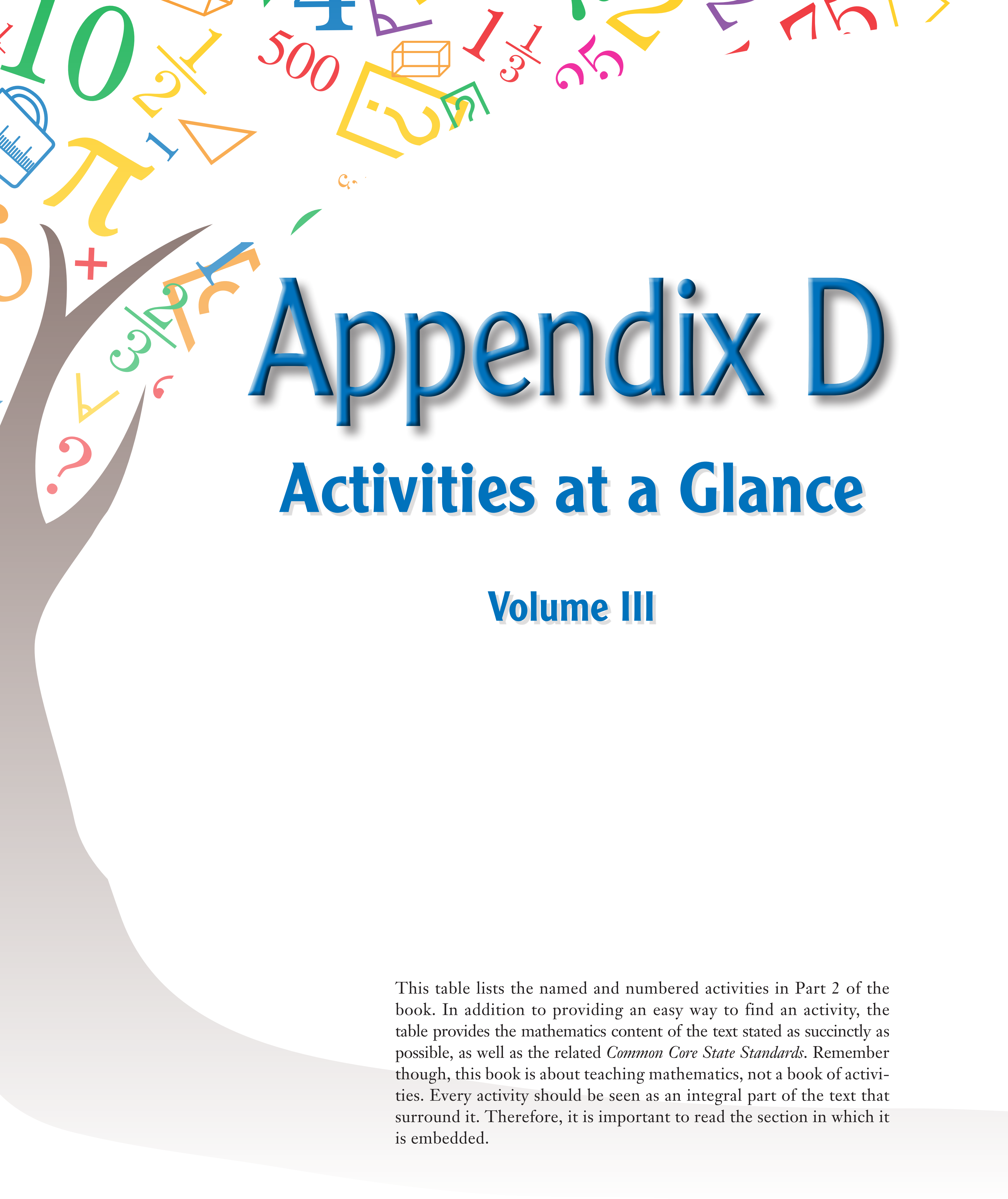
Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Support productive struggle in learning mathematics.

Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Elicit and use evidence of student thinking.

Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.



Appendix D

Activities at a Glance

Volume III

This table lists the named and numbered activities in Part 2 of the book. In addition to providing an easy way to find an activity, the table provides the mathematics content of the text stated as succinctly as possible, as well as the related *Common Core State Standards*. Remember though, this book is about teaching mathematics, not a book of activities. Every activity should be seen as an integral part of the text that surround it. Therefore, it is important to read the section in which it is embedded.

Chapter 8 Fraction concepts and computation.

Activity		Mathematical Content	CCSS-M	Page
8.1	Who Is Winning?*	Develop fractional concepts using a linear model	3.NF.A.2a,b; 3.NF.A.3a,b,d; 6.NS.C.6c	120
8.2	Class Fractions	Develop concepts of fractions using a set model	3.NF.A.1; 3.NF.A.3b	121
8.3	Fourths or Not Fourths?*	Develop understanding of fractional parts	1.G.A.3; 2.G.A.3; 3.NF.A.1	122
8.4	What Fraction Is Colored?	Determine shaded region on number strips where not all partitions are shown	3.NF.A.1; 3.NF.A.2a, b	123
8.5	How Far Did Nicole Go?*	Determine shaded region on number strips where not all partitions are shown	3.NF.A.1; 3.NF.A.2a, b; 6.NS.C.6c	123
8.6	A Whole Lot of Fun!*	Partition and iterate to determine lengths of paper strips	3.NF.A.1; 3.NF.A.2a, b	125
8.7	Calculator Fraction Counting	Explore counting by a fraction unit	3.NF.A.1; 5.NF.B.7b; 6.NS.A.1	127
8.8	About How Much?	Estimate and explore the relative size of fractions	3.NF.A.1; 3.NF.A.2a,b	129
8.9	Making Stacks*	Introduction to equivalent fractions using pattern blocks	3.NF.A.1; 3.NF.3a, b, c	130
8.10	Dot-Paper Equivalencies*	Develop an understanding of the concept of equivalent fractions	3.NF.A.1; 3.NF.3a, b, c	130
8.11	Lego Land: Building Options	Explore fraction equivalencies in model that can be area or set	4.NF.B.3a,b; 5.NF.A.1	131
8.12	Apples and Bananas*	Explore equivalence using a set model	3.NF.A.1; 3.NF.3a, b, c	132
8.13	Garden Plots*	Develop an equivalent fraction algorithm	3.NF.3b; 4.NF.A.1	133
8.14	Which Fraction Is Greater?*	Use reasoning strategies to compare fractions	4.NF.A.2	134
8.15	Gardening Together*	Explore addition and subtraction of fractions in an area model	4.NF.B.3a, d; 5.NF.A.1; 5.NF.A.2	138
8.16	Jumps on the Ruler*	Explore addition and subtraction of fractions in a linear model	4.NF.B.3a, d; 5.NF.A.1; 5.NF.A.2	139
8.17	Over or Under 1*	Develop estimation of sums and differences of fractions	4.NF.B.3a; 5.NF.A.2	140
8.18	Cups of Milk	Estimate addition and subtraction problems using a linear context (measuring cup)	4.NF.B.3a; 5.NF.A.2	141
8.19	Can You Make It True?*	Develop the concept of estimating with benchmark fractions	4.NF.B.3a; 5.NF.A.2	142
8.20	Common Multiple Flash Cards*	Practice finding common multiples	4.OA.B.4; 5.NF.A.1	144
8.21	Hexagon Wholes	Explore multiplication of a fraction by a whole number (e.g., $3 \times \frac{1}{4}$)	4.NF.B.4a, b	146
8.22	How Big Is the Banner?	Explore multiplication of a whole number by a fraction (e.g., $\frac{1}{2} \times 6$)	5.NF.B.4a, b; 5.NF.B.4a, b	147

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Chapter 8 Fraction concepts and computation. (*continued*)

Activity		Mathematical Content	CCSS-M	Page
8.23	Quilting Pieces	Develop multiplication of fractions with an area model	5.NF.B.4b; 5.NF.B.5b	150
8.24	Playground Problem	Explore multiplication of fractions and the commutative property	5.NF.B.4b; 5.NF.B.6	151
8.25	Can You See It?	Explore the impact of multiplying by a number less than one and a number more than one	5.NF.B.3; 5.NF.B.5a, b	152
8.26	Fractions Divided by Whole Number Stories*	Solve problems involving a fraction divided by a whole number	5.NF.B.7a, c	155
8.27	Sandwich Servings	Solve problems involving whole numbers divided by unit fractions	5.NF.B.7b, c	156
8.28	How Much in One Whole Set?*	Solve problems involving whole numbers divided by unit fractions with a sharing or partitioning interpretation of division	5.NF.B.7a, c	157
8.29	How Much for One?	Explore fractions divided by unit fractions, and explore problems in a sharing or partitioning interpretation of division	5.NF.B.7a, c; 6.NS.A.1	158
8.30	The Size Is Right: Division*	Estimate division problems involving fractions	5.NF.B.7a, b, c; 6.NS.A.1	159

Chapter 9 Decimal concepts and computation.

Activity		Mathematical Content	CCSS-M	Page
9.1	Calculator Decimal Counting	Develop an understanding of the patterns in decimal notation	4.NF.C.6; 5.NBT.A.1	165
9.2	Shifting Units*	Practice shifting the unit in place value	4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a	169
9.3	Build It, Name It*	Practice connecting the decimal notation to a physical model	4.NF.C.6; 5.NBT.A.1	171
9.4	Familiar Fractions to Decimals*	Develop a conceptual connection between fractions and decimal notations	4.NF.C.7	171
9.5	Estimate and Verify*	Develop a conceptual connection between fractions and decimal notations	4.NF.C.7	172
9.6	Decimals and Fractions—Double Line Up	Develop a conceptual connection between fractions and decimal notations	4.NF.C.6	173
9.7	Close to a Familiar Fraction*	Connect decimals and fractions that are approximately equivalent	4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a	175
9.8	Best Match*	Practice estimation of decimal numbers with simple fractions	4.NF.C.6	175
9.9	Line 'Em Up*	Develop an understanding of the way that decimal numbers are ordered	4.NF.C.6; 4.NF.C.7; 5.NBT.A.1; 5.NBT.A.3a, b	177
9.10	Close Decimals	Develop an understanding of the relative size of decimal numbers	5.NBT.A.3b; 6.NS.C.6	178
9.11	Zoom*	Explore the density of decimals	5.NBT.C3b; 6.NS.C.6	178

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Activity		Mathematical Content	CCSS-M	Page
9.12	Represent and Review*	Build the connection between computation, models and context	5.NBT.B.7	181
9.13	Where Does the Decimal Go? Multiplication	Use estimation to place the decimal point in multiplication	5.NBT.B.7; 6.NS.B.3	183
9.14	Where Does the Decimal Go? Division	Use estimation to place the decimal point in division	5.NBT.B.7; 6.NS.B.3	186
9.15	Percent Memory Match*	Explore the relationship between percents and circle graphs	6.RP.A.3c	187

Chapter 10 The number system.

Activity		Mathematical Content	CCSS-M	Page
10.1	Stacks of Coins	Explore the order of operations conceptually	3.OA.D.8; 6.EE.A.2c	195
10.2	Guess My Number	Practice writing an expression using order of operations	6.EE.A.2a, b, c; 6.EE.B.6	197
10.3	Order of Operations: True or False Equations*	Explore order of operations	3.OA.D.8; 6.EE.A.3; 6.EE.A.4	197
10.4	Entering Expressions	Determine how to enter expressions into calculator to preserve order of operations	6.EE.A.3; 6.EE.A.4	200
10.5	Find the Error*	Identify errors that involve order of operations	8.EE.A.1	201
10.6	How Far away from the Sun?*	Explore scientific notation and relative size of numbers	8.EE.A.4	203
10.7	At a Snail's Pace	Explore scientific notation of very small numbers in context	8.EE.A.4	204
10.8	Exploring Powers of Ten	Develop an understanding of scientific notation and other expressions for powers of 10	8.EE.A.3	204
10.9	What Is Her Net Worth?	Develop the concept of positive and negative numbers	6.NS.C.5	206
10.10	(American) Football Statistics	Develop the concept of positive and negative numbers	6.NS.C.5; 6.NS.C.6a	208
10.11	Greater, Less, Equal or Don't Know?*	Build an understanding for symbols (minus and parentheses) used in expressions	6.NS.C.5; 6.NS.C.6a	209
10.12	Find the Zero*	Develop strategy for adding with negative numbers	7.NS.1a, b, c	212
10.13	Creating Stories and Conjectures for Operations with Negative Numbers	Estimate the result of operations involving negative numbers	7.NS.A.2a, b	215
10.14	How Many in Between?*	Develop understanding that there is always another fraction between two values (density of numbers)	6.NS.C.5; 7.NS.A.2d	218
10.15	Repeater or Terminator?	Develop the connection between repeating versus terminating decimal numbers	7.NS.A.2d	219
10.16	Edges of Squares and Cubes	Develop the concepts of square roots and cube roots	8.NS.A.1; 8.EE.A.2	221
10.17	Wheel of Theodorus	Explore irrational numbers through classic art project	8.NS.A.2; 8.EE.A.2	222

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Chapter 11 Ratios and proportional relationships.

Activity		Mathematical Content	CCSS-M	Page
11.1	Stocking The Pond*	Solve real world problems involving ratios	6.RP.A.1; 6.RP.A.2; 6.RP.A.3	227
11.2	Birthday Cupcakes	Develop the concept of a composed unit and multiplicative comparison	6.RP.A.1; 6.RP.A.2; 6.RP.A.3	228
11.3	Which Has More?	Develop the distinction between additive and multiplicative comparisons	6.RP.A.1	230
11.4	Weight Loss	Explore the difference between additive and multiplicative comparison	6.RP.A.1	230
11.5	Pencil to Pencil	Develop the distinction between additive and multiplicative comparison	6.RP.A.3a; 7.RP.A.2b	233
11.6	Different Objects, Same Ratios*	Explore ratio and proportion concepts in measurement	6.RP.A.1; 6.RP.A.2	234
11.7	Look-Alike Rectangles*	Develop the concepts of ratio and proportion in the context of similar rectangles	6.RP.A.1; 6.RP.A.2a; 6.RP.A.3a	235
11.8	Scale Drawings*	Develop the concepts of ratio and proportion in the context of similar 2-D figures	6.RP.A.1; 7.RP.A.3a	237
11.9	Rectangle Ratios–Graph It!*	Explore proportions through graphs	7.RP.A.2a, b, c, d	239
11.10	Dripping Faucets	Apply algebraic thinking to develop proportional reasoning	6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.2b, c	240
11.11	Comparing Lemonade Recipes	Explore proportional situations and compare ratios	6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2a	244
11.12	Creating Paint Swatches	Explore proportional situations and compare ratios	6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2a	245
11.13	Which Camp Gets More Pizza?	Compare ratios in a problem-solving context	6.RP.A.1; 6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2a	245
11.14	Solving Proportional Problems Using Ratio Tables*	Develop proportional reasoning through scaling	6.RP.A.3a, b; 7.RP.A.1; 7.RP.A.2b, c; 7.RP.A.3	247
11.15	You and the Zoo	Use double number lines to solve situations involving proportions	6.RP.A.3a, d; 7.RP.A. 2b; 7.RP.A.3	250
11.16	Making Sense of Percent Stories	Use double number lines to solve situations involving percents	6.RP.3c; 7.RP.A.2c; 7.RP.A.3	251

Chapter 12 Algebraic thinking: expressions, equations, and functions.

Activity		Mathematical Content	CCSS-M	Page
12.1	Birds in the Backyard*	Translate a pattern into number and variable expressions	6.EE.A.2a; 6.EE.C.9; 8.F.B.4	256
12.2	Different Ways to Zero	Write equivalent expressions	6.EE.A.3; 7.NS.A.1; 7.EE.A.1	259
12.3	Convince Me Conjectures*	Make and test generalizations about whole number operations (and properties)	5.OA.A.1; 6.EE.A.4; 7.NS.A.1d	261

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Activity		Mathematical Content	CCSS-M	Page
12.4	Broken Calculator: Can You Fix It?	Explore properties of odd and even numbers	5.OA.A.1; 6.EE.B.6	261
12.5	Hurricane Names	Explore the structure of repeating patterns analytically	5.OA.B.3; 6.EE.B.2a	263
12.6	Predict How Many*	Develop functional relationships in growing patterns	6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.2; 8.F.A.4	265
12.7	Two of Everything*	Explore a variety of functions using an input–output context	5.OA.B.3; 6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.4	267
12.8	Perimeter Patterns*	Generalize and graph geometric growing patterns (functions)	6.EE.C.9; 7.EE.B.4a; 8.F.A.1; 8.F.A.2	269
12.9	Bottles and Volume Graphs	Explore graphical representations of functional relationships	5.G.A.2; 8.F.B.5	271
12.10	Sketch a Graph*	Explore graphical representations of functional relationships	6.EE.D.9; 8.F.B.5	272
12.11	Border Tiles*	Explore a pattern to develop a generalization	6.EE.A.1; 6.EE.A.2a, b, c; 6.EE.A.3; 6.EE.A.4	277
12.12	What Do You Know about the Shapes?	Develop an understanding of the equal sign	6.EE.A.2a; 6.EE.B.5; 6.EE.B.6; 7.EE.B.4a, b	280
12.13	Tilt or Balance?*	Develop understanding of the equal sign and the less-than and greater-than symbols	2.NBT.A.4; 4.NBT.A.1; 5.NBT.A.3a, b; 6.EE.A.4	281
12.14	True or False?	Explore equivalence, develop relational understanding, and reason about rational numbers	5.NF.A.1; 6.EE.A.4; 7.NS.A.1d	282
12.15	What's Missing?	Develop relational understanding of equations and inequalities and reason about rational numbers	5.OA.A.2, 6.EE.A.3, 6.EE.B.5	283
12.16	Writing True/False Statements	Write equations and inequalities	5.OA.A.2, 6.NS.B.C, 6.EE.A.3, 6.EE.B.5	285
12.17	Telephone: Transforming Words to Equations/Inequalities	Translate from words to symbols and back through a classic game	6.EE.A.2a; 6.EE.B.6; 7.EE.B.4	286
12.18	Ball Weights	Explore variables in context	6.EE.A.4, 7.EE.A.2, 8.EE.C.8b	287
12.19	Solving the Mystery*	Explore properties and equivalent expressions	5.OA.A.2; 6.EE.A.2a; 7.EE.A.2	289
12.20	How Many Gallons Left?	Develop functional relationships in real-world contexts	7.EE.A.2; 7.EE.B.4a; 8.F.B.4	293
12.21	Designing the Largest Box	Develop functional relationships from formulas	5.MD.C.5b; 6.EE.A.2c; 6.G.A.2	296

Chapter 13 Developing geometry concepts.

Activity		Mathematical Content	CCSS-M	Page
13.1	Property Lists for Quadrilaterals*	Describe properties of classes of quadrilaterals	3.G.A.1; 4.G.A.2; 4.G.A.3; 7.G.A.2	301

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Chapter 13 Developing geometry concepts. (*continued*)

Activity		Mathematical Content	CCSS-M	Page
13.2	Minimal Defining Lists*	Develop logic and reasoning to minimally define shapes	4.G.A.2; 5.G.B.3; 7.G.A.2	302
13.3	Whats My Shape?*	Describe and classify shapes	3.G.A.1; 4.G.A.2; 7.G.A.2	305
13.4	Triangle Sort*	Classify triangles by properties	5.G.B.3; 5.G.B.4; 7.G.A.2	310
13.5	Angle Sums in a Triangle*	Draw conjectures for the sum of the interior angles of a triangle	5.G.B.3; 7.G.A.2; 8.G.A.5	312
13.6	Triangle Midsegments*	Make conjectures about measures of a triangle	7.G.A.2; 8.G.A.5	313
13.7	Mystery Definition*	Develop defining properties of special classes of shapes	4.G.A.2; 5.G.B.3; 5.G.B.4; 7.G.A.2	313
13.8	Diagonals of Quadrilaterals*	Determine properties of diagonals of two-dimensional shapes	5.G.B.3; 5.G.B.4; 6.RP.A.1; 7.G.A.2	315
13.9	True or False?	Explore informal deductive statements concerning properties of shapes	5.G.B.4; 7.G.A.2	316
13.10	Can You Build It?*	Draw shapes based on descriptions and properties	6.G.A.3; 7.G.A.2	318
13.11	The Pythagorean Relationship*	Explore the pythagorean relationship geometrically	8.G.B.6; 8.G.B.7	321
13.12	Finding Pythagorean Triples: 3-4-5 in Disguise	Investigate triangles that are pythagorean triples	8.G.B.6; 8.G.B.7	322
13.13	Motion Man*	Perform transformations (translations, rotations, reflections)	8.G.A.1a, b, c	324
13.14	Motion Man–Double Move*	Perform two-step transformations	8.G.A.1a, b, c; 8.G.A.2	327
13.15	Are They Congruent?*	Explore congruence and transformations on coordinate axis	8.G.A.1; 8.G.A.2	327
13.16	Dot to Dot	Locate points on a coordinate axis	5.G.A.2; 6.G.A.3	328
13.17	Four in a Row	Locate points on a coordinate axis (quadrant i)	5.G.A.2;	329
13.18	Simon Says	Locate points on a coordinate axis	5.G.A.2; 6.G.A.3	329
13.19	Coordinate Slides*	Perform translations on the coordinate axis	6.G.A.3; 8.G.A.1; 8.G.A.3	329
13.20	Coordinate Reflections*	Perform reflections on the coordinate axis	8.G.A.1; 8.G.A.3	330
13.21	Pentomino Rotations and Reflections*	Perform rotations and reflections on the coordinate axis	8.G.A.1	331
13.22	Polygon Dilations*	Explore impact of multiplying coordinates by a constant (it is *not* a rigid motion)	6.G.A.3; 7.G.A.1; 8.G.A.1; 8.G.A.3	331
13.23	Is It a Square Deal?*	Measure distance on a coordinate plane	6.G.A.4	334
13.24	Developing the Distance Formula*	Develop the formula for the distance between two points	8.G.B.8	334
13.25	Net Challenges	Explore nets (surface area) of various three-dimensional shapes	6.G.A.4	336
13.26	Building Views*	Explore two-dimensional and three-dimensional representations of objects	6.G.A.4; 7.G.A.1	336
13.27	Three-Dimensional Drawings*	Develop spatial visualization skills	6.G.A.4; 7.G.A.1	337
13.28	Slicing 3-D Shapes	Explore the 2-D shapes formed by slicing 3-D shapes	7.G.A.3	338

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Chapter 14 Exploring measurement concepts.

Activity		Mathematical Content	CCSS-M	Page
14.1	Guess the Unit*	Choose appropriate units of measure	5.MD.C.3; 7.G.B.6	343
14.2	Estimation Exploration	Estimate different measurable attributes	6.G.A.1; 6.G.A.2; 7.G.B.4; 8.G.C.9	346
14.3	Estimation Scavenger Hunt*	Find real objects with given measures	6.G.A.1; 6.G.A.2; 7.G.B.4; 8.G.C.9	346
14.4	E-M-E Sequences*	Use one measure to estimate another measure	6.G.A.1; 6.G.A.2; 7.G.B.4; 8.G.C.9	347
14.5	A Unit Angle*	Develop concept of angle measure	4.MD.C.5; 7.G.B.5	348
14.6	Wax Paper Protractor Projects	Create a protractor and measure angles with it	7.G.A.2	349
14.7	Angle Sums of Polygons*	Make conjectures about angle sums of polygons	7.G.A.2, 7.G.B.5, 8.G.A.5	349
14.8	Angle Relationships	Explore supplementary, complementary, vertical and adjacent angles	7.G.B.5	351
14.9	In Search of Missing Angles*	Apply knowledge of supplemental, complementary, vertical and adjacent angles to solve missing angle tasks	7.G.B.5	351
14.10	Cover and Compare	Develop an understanding of units to measure area	4.MD.A.3; 5.NF.B.4b; 6.G.A.1	354
14.11	Fixed Perimeters or Areas*	Explore the relationship between area and perimeter of rectangles	5.NF.B.4b	354
14.12	Area of a Parallelogram	Develop the area formula for parallelograms	6.G.A.1	357
14.13	Area of a Triangle*	Develop the area formula for triangles	6.G.A.1	358
14.14	Making "To-Go" Boxes	Create nets for various-sized rectangular prisms	6.G.A.4; 7.G.B.6	360
14.15	Ratio of Diameter to Circumference	Measure various-sized circles to discover ratio (pi)	7.G.B.4	361
14.16	Fixed Volume: Comparing Prisms*	Explore the relationship between volume and surface area of prisms	5.MD.C.5A; 5.MD.C.5b; 6.G.A.2; 7.G.B.6	363
14.17	Which Silo Holds More?	Compare volume of different-shaped cylinders	6.G.A.4; 7.G.B.7	363
14.18	Box Comparison—Cubic Units	Develop formula for volume of a rectangular prism	5.MD.C.4; 5.MD.C.5; 6.G.A.2; 7.G.B.6	364

Chapter 15 Working with data and doing statistics.

Activity		Mathematical Content	CCSS-M	Page
15.1	Who's in Our Village?	Gather and analyze local data to compare to the wider population	6.SP.A.1; 7.SP.B.3	373
15.2	What Can We Learn about Our Community?*	Generate questions about the community, gather data, and analyze it	6.SP.A.1; 6.SP.B.4; 6.SP.B.5	374
15.3	How Do We Compare?*	Develop an awareness of the importance of sampling	6.SP.A.1; 6.SP.A.2; 7.SP.A.1; 7.SP.B.3	376
15.4	Human Bars to Pies*	Create pie charts for categorical data	6.RP.A.1; 6.SP.A.1; 7.SP.A.1	379

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Chapter 15 Working with data and doing statistics. (*continued*)

Activity		Mathematical Content	CCSS-M	Page
15.5	Storm Plotter	Explore line plots using real life data	5.MD.B.2; 6.SP.B.4; 7.SP.B.3	381
15.6	Is There a Relationship?*	Explore and analyze bivariate data	8.SP.A.1	386
15.7	Spaghetti Lines*	Conceptually estimate the line of best fit	8.SP.A.1; 8.SP.A.2	386
15.8	Mean Cost of Games*	Develop the concept of the mean as using a leveling interpretation	6.SP.A.2; 6.SP.A.3; 6.SP.B.5	389
15.9	The Mean Foot	Apply the leveling interpretation of the mean and connect to algorithm	6.SP.A.2; 6.SP.A.3; 6.SP.B.5	389
15.10	Balancing Cubes	Explore a balancing point interpretation of the mean	6.SP.A.2; 6.SP.A.3; 6.SP.B.5	391
15.11	Balance Point Sticky Notes*	Explore a balancing point interpretation of the mean	6.SP.A.2; 6.SP.A.3; 6.SP.B.5	391
15.12	Which Measure of Center?	Decide which measure of center, median or mean, makes more sense given various contexts	6.SP.A.2; 6.SP.A.3; 6.SP.B.5	392
15.13	Average Cost of More Games*	Explore how new data impact each measure of center	6.SP.A.2; 6.SP.A.3; 6.SP.B.5	393
15.14	You Be the Judge*	Determine which measure of center should be used	6.SP.A.2; 6.SP.A.3; 6.SP.B.5	393
15.15	How Many Words on the Page?	Design strategy for gathering and analyzing data to solve a real life problem	6.SP.A.1; 6.SP.A.2; 6.SP.B.5	397

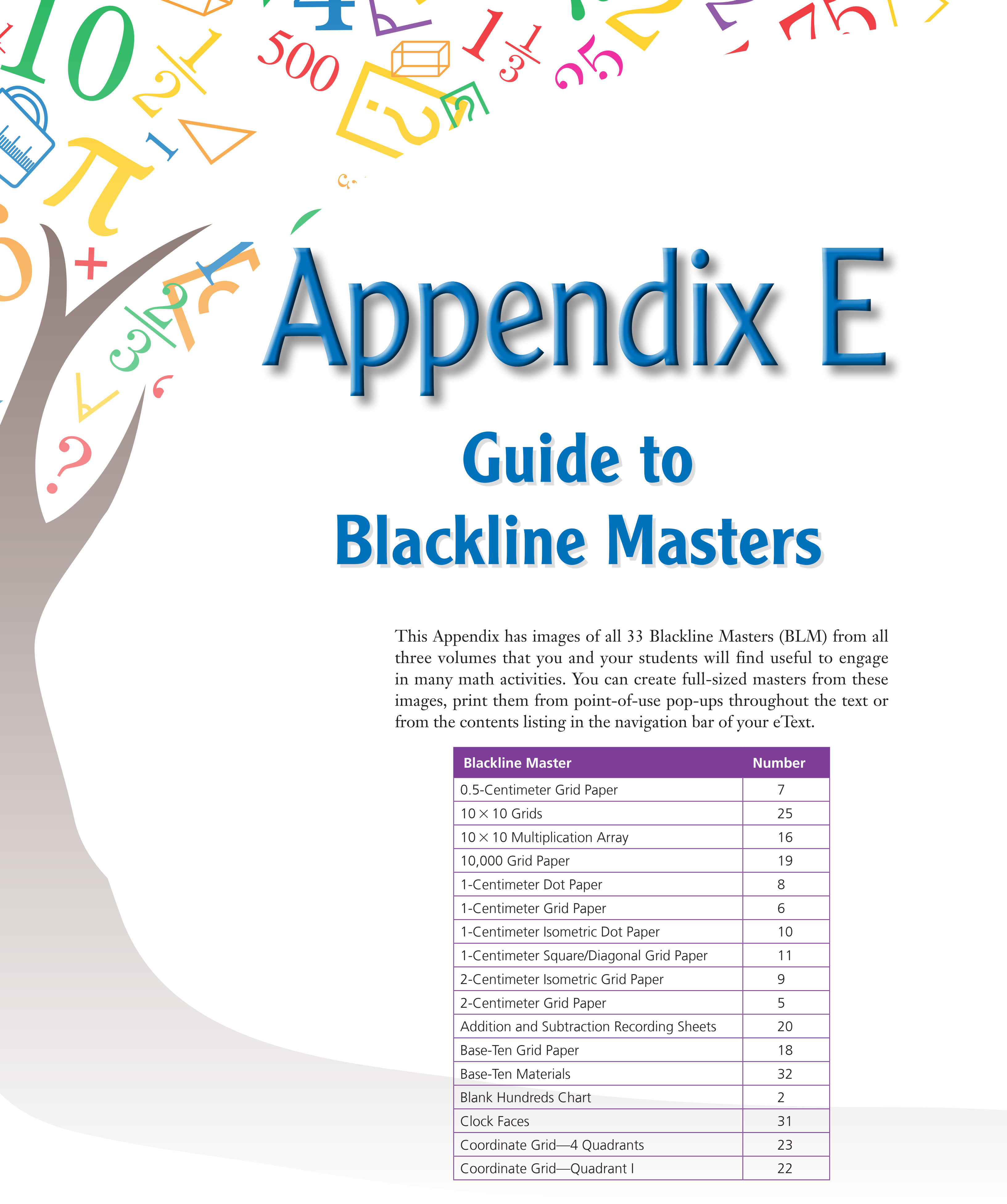
Chapter 16 Investigating concepts of probability.

Activity		Mathematical Content	CCSS-M	Page
16.1	Events in Lyrics: Possible or Not Possible?	Develop the concepts of impossible and possible	7.SP.C.5	400
16.2	Is It Likely?	Develop the concepts of impossible, possible, and certain	7.SP.C.5	400
16.3	1-2-3 How Likely?*	Explore unequal outcomes	7.SP.C.5; 7.SP.C.6	402
16.4	1-2-3 How Likely Are Sums?*	Explore the probability of two events	7.SP.C.5; 7.SP.C.6, 7.SP.C.8a	402
16.5	Race to Ten*	Develop the concepts of less-likely and more-likely events	7.SP.C.6; 7.SP.C.7a, b	402
16.6	Design and Test Bags*	Explore the probability of an event in a one-stage experiment	7.SP.C.6	405
16.7	Mystery Bags	Explore the probability of an event in a one-stage experiment	7.SP.C.6; 7.SP.C.7a, b	405
16.8	Fair or Unfair?	Explore theoretical analysis and experimental results	7.SP.C.6; 7.SP.C.7a	407
16.9	Chance of Hitting the Target?	Explore how likely possible outcomes are in an area model	7.G.B.4, 7.SP.C.6, 7.SP.C.7b	408
16.10	Drop It!	Explore how likely possible outcomes are	7.SP.C.7b	409
16.11	Get All 6	Develop the concept that the size of the trial matters	7.SP.C.6	410
16.12	What Are the Chances?*	Develop the concept of randomness	7.SP.C.6	410

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.

Activity		Mathematical Content	CCSS-M	Page
16.13	Lu-Lu	Explore the concept of sample space	7.SP.C.7a; 7.SP.C.8a	415
16.14	Are You a Spring Dog?	Explore the use of an area model to list all possible outcomes for independent events	7.SP.C.8a, b	416
16.15	Design a Winning Spinner	Explore the probability of independent events	7.SP.C.6; 7.SP.C.7a; 7.SP.C.8a, b	417
16.16	Keys to a New Car	Use an area representation to explore the probability of dependent events	7.SP.C.8a, b	417
16.17	Probability of Getting Water	Design a simulation to explore the probability of dependent events	7.SP.C.8b, c	419
16.18	Chance of Triplet Girls	Design a simulation to model independent events	7.SP.C.8b, c	421
16.19	Money in two Piggy Banks*	Use different representations to design a simulation	7.SP.C.8a, b, c	422

*Activity titles with an asterisk include a downloadable Activity Page, Expanded Lesson, or Blackline Master.



Appendix E

Guide to Blackline Masters

This Appendix has images of all 33 Blackline Masters (BLM) from all three volumes that you and your students will find useful to engage in many math activities. You can create full-sized masters from these images, print them from point-of-use pop-ups throughout the text or from the contents listing in the navigation bar of your eText.

Blackline Master	Number
0.5-Centimeter Grid Paper	7
10 × 10 Grids	25
10 × 10 Multiplication Array	16
10,000 Grid Paper	19
1-Centimeter Dot Paper	8
1-Centimeter Grid Paper	6
1-Centimeter Isometric Dot Paper	10
1-Centimeter Square/Diagonal Grid Paper	11
2-Centimeter Isometric Grid Paper	9
2-Centimeter Grid Paper	5
Addition and Subtraction Recording Sheets	20
Base-Ten Grid Paper	18
Base-Ten Materials	32
Blank Hundreds Chart	2
Clock Faces	31
Coordinate Grid—4 Quadrants	23
Coordinate Grid—Quadrant I	22

Blackline Master	Number
Degrees and Wedges	30
Double Ten-Frame	15
Five-Frame	12
Four Small Hundreds Charts	4
Geoboard Pattern (10 by 10)	28
Geoboard Pattern (5 by 5)	26
Geoboard Recording Sheets (10 by 10)	29
Geoboard Recording Sheets (5 by 5)	27
Hundreds Chart	3
Multiplication and Division Recording Sheets	21
Number Cards 0-10	1
Observation Checklist	33
Place-Value Mat (with Ten-Frames)	17
Rational Number Wheel	24
Ten-Frame (Horizontal)	14
Ten-Frame	13

Suggestions for Use and Construction of Materials Card Stock Materials for Volumes I-III

Card Stock Materials

A good way to have many materials made quickly and easily for students is to have them duplicated on card stock, laminated, and then cut into smaller pieces if desired. Once cut, materials are best kept in clear freezer bags with zip-type closures. Punch a hole near the top of the bag so that you do not store air.

The following list is a suggestion for materials that can be made from card stock using the masters in this section. Quantity suggestions are also given.

Five-Frames and Ten-Frames—12–14

Five-frames and ten-frames are best duplicated on light-colored card stock. Do not laminate; if you do, the mats will curl and counters will slide around.

10 × 10 Multiplication Array—16

Make one per student in any color. Lamination is suggested. Provide each student with an L-shaped piece of card stock to frame the array.

Base-Ten Materials—32

Run copies on white card stock. One sheet will make 4 hundreds and 10 tens or 4 hundreds and 100 ones. Cut into pieces with a paper cutter. It is recommended that you not laminate the base-ten pieces. A kit consisting of 10 hundreds, 30 tens, and 30 ones is adequate for each student or pair of students.

Place-Value Mat (with Ten-Frames)—17

Mats can be duplicated on any pastel card stock. It is recommended that you not laminate these because they tend to curl and counters slide around. Make one for every student.

Rational Number Wheel—24

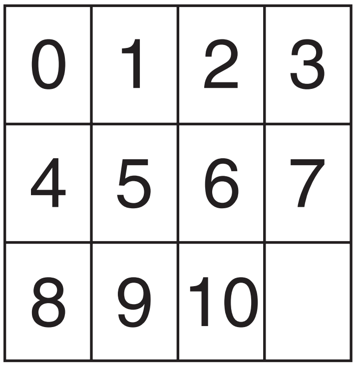
These disks should be made on card stock. Duplicate the master on two contrasting colors. Laminate and cut the circles and also the slot on the dotted line. Make a set for each student.

Many masters lend themselves to demonstration purposes. The 10 × 10 array, the blank hundreds board, and the large geoboard are examples. The place-value mat can be used with strips and squares or with counters and cups directly on the document camera. The missing-part blank and the record blanks for the four algorithms are pages that you may wish to write on as a demonstration.

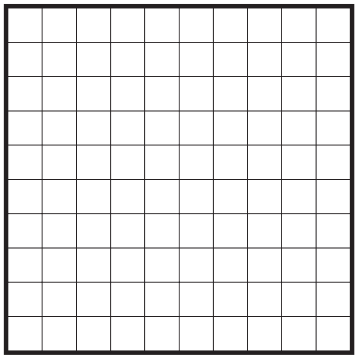
The 10,000 grid is the easiest way there is to show 10,000 or to model four-place decimal numbers.

The degrees and wedges page is the very best way to illustrate what a degree is and also to help explain protractors.

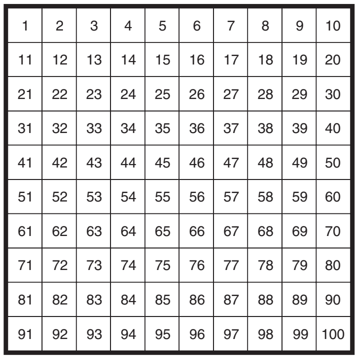
All of the line and dot grids are useful for modeling. You may find it a good idea to have several copies of each easily available.



Number Cards 0–10—1



Blank Hundreds Chart—2



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
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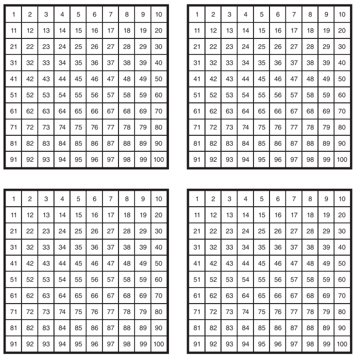
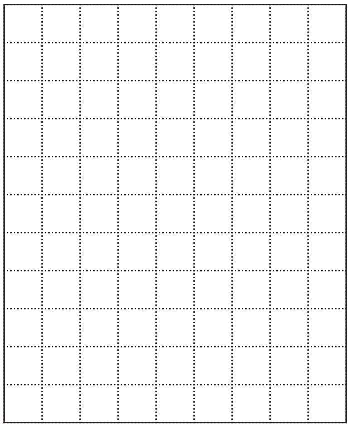
91	92	93	94	95	96	97	98	99	100

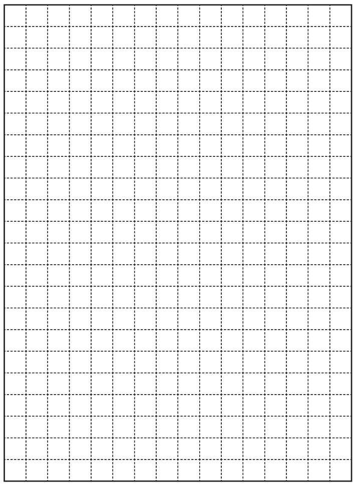
Hundreds Chart—3



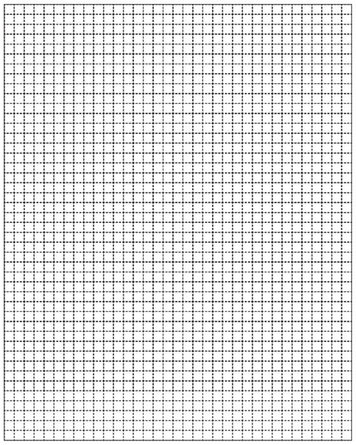
Four Small Hundreds Charts—4



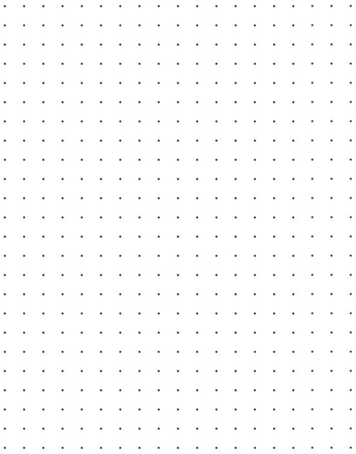
2-Centimeter Grid Paper—5



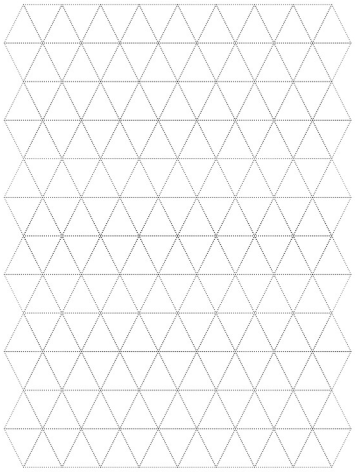
1-Centimeter Grid Paper—6



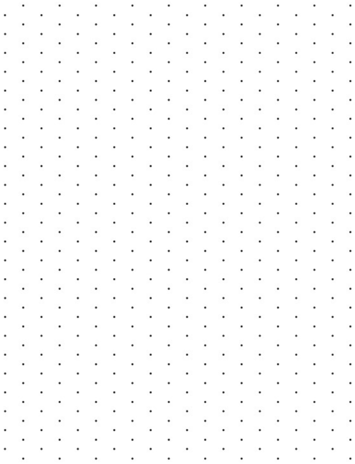
0.5-Centimeter Grid Paper—7



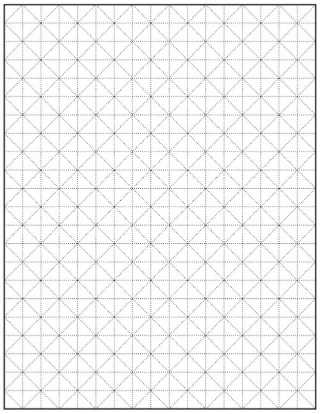
1-Centimeter Dot Paper—8



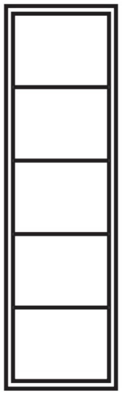
2-Centimeter Isometric Grid Paper—9



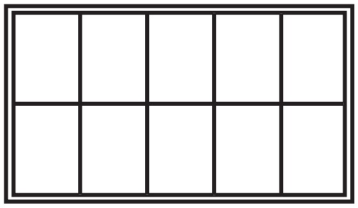
1-Centimeter Isometric Dot Paper—10



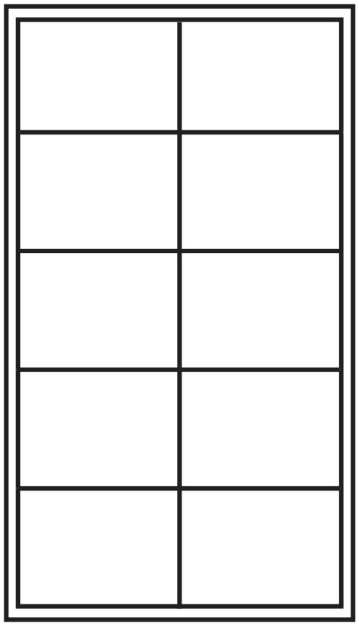
1-Centimeter Square/Diagonal Grid Paper—11



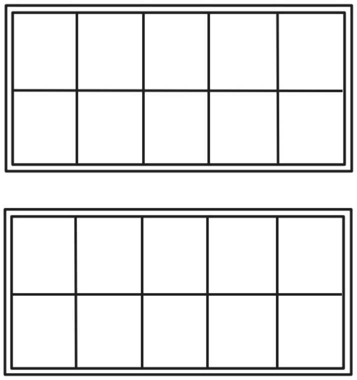
Five-Frame —12



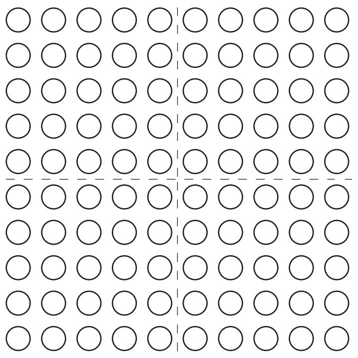
Ten-Frame—13



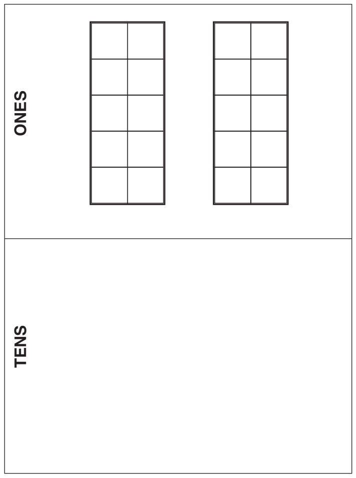
Ten-Frame Horizontal—14



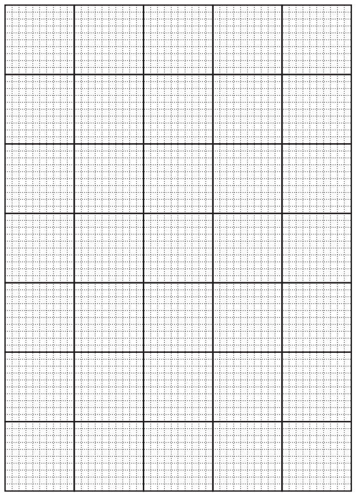
Double Ten-Frame—15



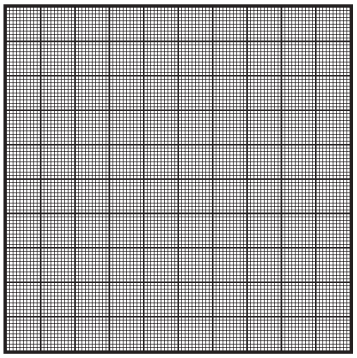
10 × 10 Multiplication Array—16



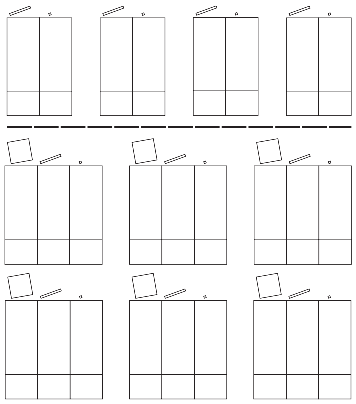
Place-Value Mat (with Ten-Frames)—17



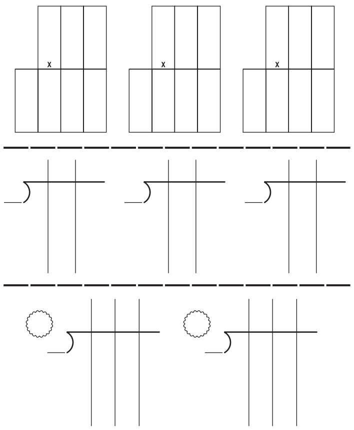
Base-Ten Grid Paper—18



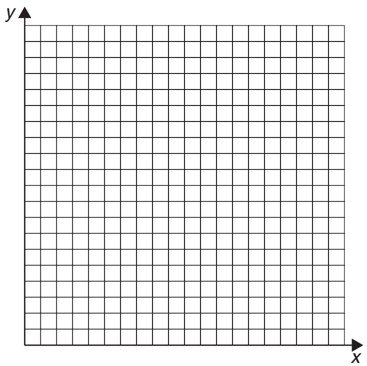
10,000 Grid Paper—19



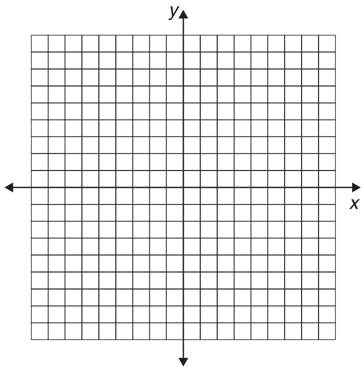
Addition and Subtraction Recording Sheets—20



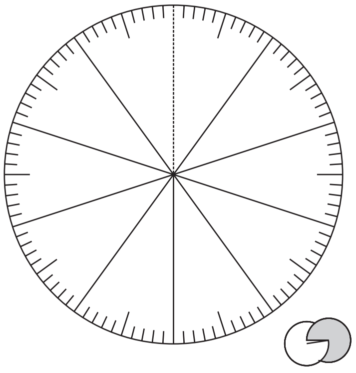
Multiplication and Division Recording Sheets—21



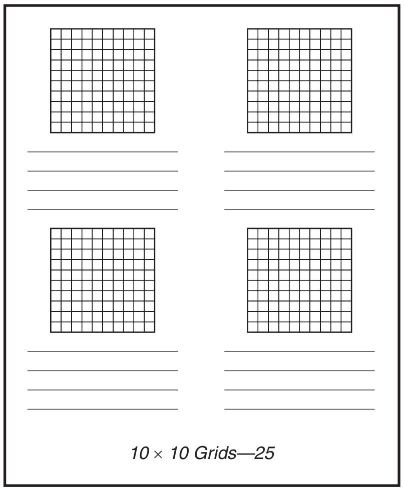
Coordinate Grid-Quadrant I—22



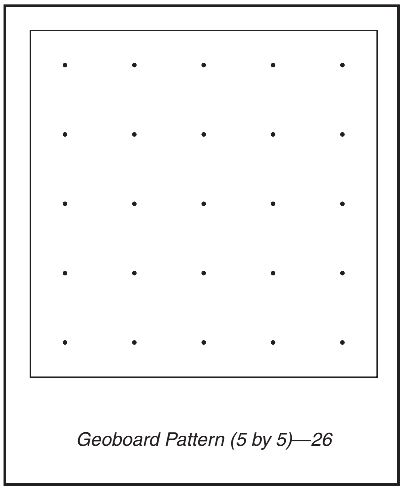
Coordinate Grid-4 Quadrants—23



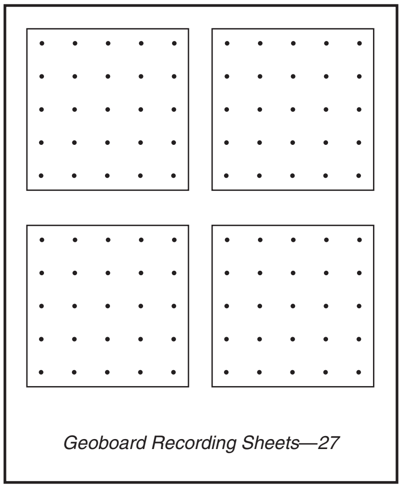
Rational Number Wheel—24



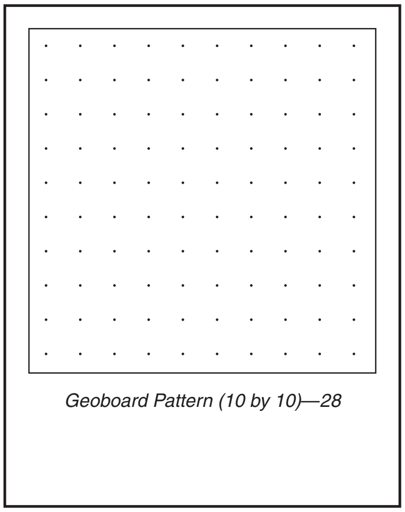
10 × 10 Grids—25



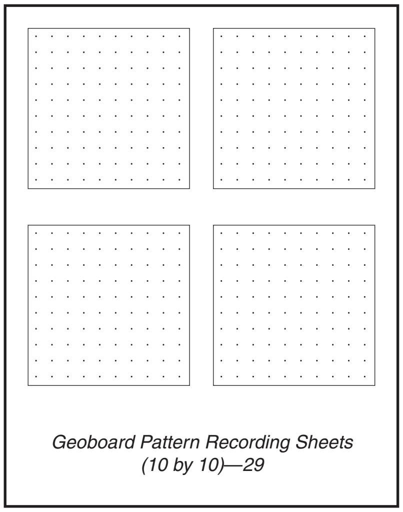
Geoboard Pattern (5 by 5)—26



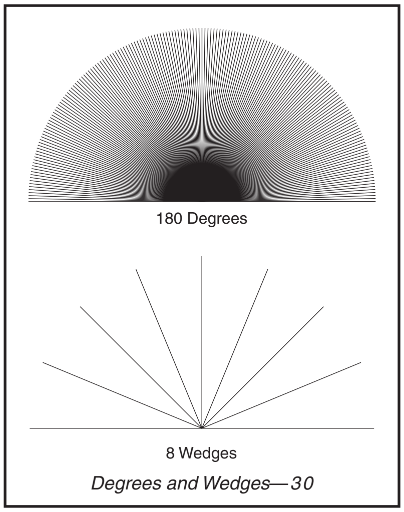
Geoboard Recording Sheets—27



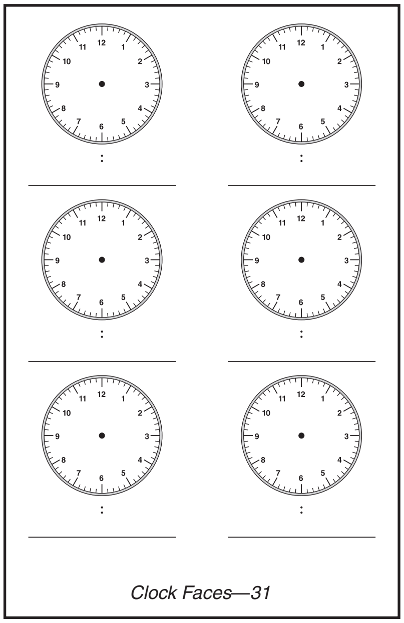
Geoboard Pattern (10 by 10)—28



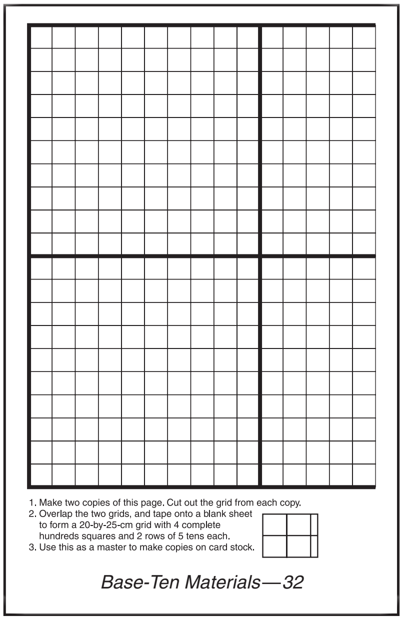
Geoboard Pattern Recording Sheets (10 by 10)—29



Degrees and Wedges—30

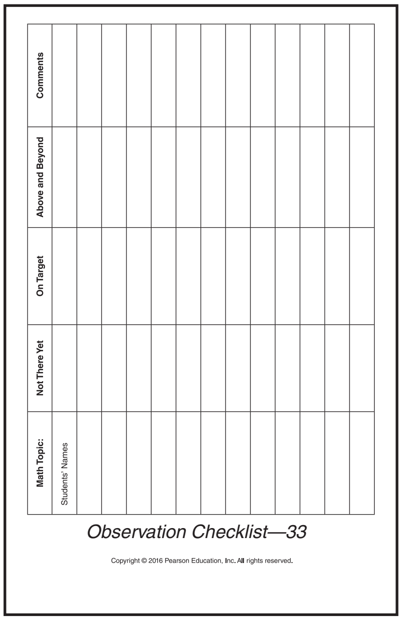


Clock Faces—31



1. Make two copies of this page. Cut out the grid from each copy.
2. Overlap the two grids, and tape onto a blank sheet to form a 20-by-25-cm grid with 4 complete hundreds squares and 2 rows of 5 tens each.
3. Use this as a master to make copies on card stock.

Base-Ten Materials—32



Math Topic:
Students' Names
Not There Yet
On Target
Above and Beyond
Comments

Observation Checklist—33

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