MARJORIE M. PETIT, ROBERT E. LAIRD, MATTHEW F. WYNEKEN, FRANCES R. HUNTOON, MARY D. ABELE-AUSTIN AND JEAN D. SEQUEIRA

A FOCUS ON RATIOS AND PROPORTIONS

Bringing Mathematics Education Research to the Classroom

Studies in Mathematical Thinking and Learning

eRESOURCES

ROUTLEDGE

A Focus on Ratios and Proportions

This resource offers a groundbreaking effort to make mathematics education research on ratios and proportions readily accessible and understandable to preservice and in-service teachers of grades 6 to 8. Using extensive annotated samples of student work and based on research gathered in the Ongoing Assessment Project (OGAP), *A Focus on Ratios and Proportions* teaches readers how students develop understanding and fluency involving ratio and proportion concepts.

Special features include:

- A close focus on student work, including 150+ annotated pieces of student work, to help teachers improve their ability to recognize, assess and monitor their students' errors and misconceptions, as well as their developing conceptual understanding.
- A focus on the OGAP Ratio and Proportions Progression, based on research conducted with hundreds of teachers and thousands of pieces of student work.
- Sections on how Common Core State Standards for Math (CCSSM) are supported by math education research.
- Student work samples and vignettes to illuminate the research, as well as end-of-chapter Looking Back questions and Instructional Links, which allow teachers to analyze evidence of student thinking and strategies and consider instructional responses.
- An accompanying eResource, available online, offers an answer key as well as extensive explanation of the Looking Back questions.

Like *A Focus on Multiplication and Division* and *A Focus on Fractions*, this book is designed to bridge the gap between what math education researchers know and what teachers need to know in order to better understand evidence in student work and make effective instructional decisions.

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A Focus on Ratios and Proportions

Bringing Mathematics Education Research to the Classroom

Marjorie M. Petit, Robert E. Laird, Matthew F. Wyneken, Frances R. Huntoon, Mary D. Abele-Austin and Jean D. Sequeira

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This book is dedicated to the thousands of teachers who have participated in Ongoing Assessment Project (OGAP) work during the past 16 years. The student solutions and suggestions they have shared impacted this book in countless ways. We thank them for their engagement and enthusiasm.

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Contents

Preface

Why Proportionality?

Proportionality is a mathematical topic that transcends lessons, units and grade levels. Like additive reasoning, multiplicative reasoning and fractions, student facility with proportionality develops over years and is comprised of a number of interconnected concepts, strategies and skills. On the one hand, proportionality can be thought of as a measure of one's understanding of elementary mathematical ideas (Lamon, 2012). In fact, Lesh, Post and Behr called proportionality "the capstone of elementary mathematics" (1988, pp. 93–94). Yet proportionality requires much more than an understanding of and facility with elementary mathematics topics, such as multiplication. In this way, proportionality serves as a vital foundation upon which more complex mathematical concepts are built; or the "cornerstone of high school mathematics" (Lesh et al., 1988, pp. 93–94). Essential proportional concepts play a central role in aspects of algebra and trigonometry and in science topics such as density, acceleration and force. Because of the central role proportionality plays in more complex mathematical concepts, it is a transcending topic in middle school mathematics. One simply has to peruse the Common Core Standards for Mathematics (CCSSM) for grades 6 through 8 or a typical grade 6 through 8 math textbook to see the amount of instructional emphasis placed on topics such as ratio, rates, proportions, scaling and similarity to name a few.

A Focus on Ratios and Proportion: Bringing Mathematics Education Research to the Classroom

A Focus on Ratios and Proportions: Bringing Mathematics Education Research to the Classroom is a product of the Ongoing Assessment Project (OGAP). OGAP started in 2003 to provide teachers with mathematics-specific formative assessment tools, strategies and knowledge in order to make formative assessment practices a central aspect of mathematics instruction. From its inception, OGAP worked closely with teachers. Classroom teachers piloted formative assessment items that ultimately became the OGAP Item Bank, reviewed and commented on the development of the OGAP Progressions and helped OGAP develop and refine the professional-development sessions that soon became an integral part of OGAP.

It was during this process that teachers clearly expressed the need for and interest in knowing more about the research that underpinned the OGAP formative assessment items, tools and professional-development sessions. These materials were developed based on mathematics education research focused on how students learn specific mathematical concepts related to important mathematical topics such as proportionality, fractions, multiplicative reasoning and additive reasoning, the common errors and misconceptions one can expect to accompany student learning and the types of instructional experiences that benefit all students as they become more fluent in these topics. Thus, *A Focus on Ratios and Proportions: Bringing Mathematics Education Research to the Classroom* is our attempt to communicate this research at a grain size that teachers can access to strengthen their use of formative assessment in middle-grades mathematics classrooms and to ultimately help all students gain a deep understanding of ratio and proportion concepts and skills.

This book joins *A Focus on Fractions: Bringing Research to the Classroom 2nd edition* (2016) and *A Focus on Multiplication and Division: Bringing Research to the Classroom* (2017). The overwhelming success of these books convinced us that math teachers across the country share our excitement for engaging with and understanding the mathematics education research about how students learn particular mathematical ideas and the concrete ways this knowledge can inform mathematics instruction. Through interactions with hundreds of mathematics educators, we continually see evidence that when teachers are provided important mathematics education research on how students learn, mathematics-specific formative assessment tools and strategies for analyzing evidence of student thinking and deep understanding of the mathematics content central to their grade level, they thoughtfully and intentionally change their instruction in ways that support deeper student learning.

With all this firmly in mind, the goals of this book are to provide the reader with the knowledge of:

- mathematics content central to effective proportionality instruction.
- strategies students use to solve ratio, rate and proportion problems.
- common errors and misconceptions that may interfere with learning new concepts or solving ratio and proportion problems.
- the OGAP Ratio and Proportion Progression to analyze student solutions.
- effective instructional strategies to help all students become strong and flexible proportional reasoners.

Unique Features of this Book

A Focus on Ratios and Proportions: Bringing Mathematics Education Research to the Classroom utilizes a number of important features that we believe help communicate specific aspects of the learning and teaching of proportionality. These features are:

- *The OGAP Proportionality Framework:* The *OGAP Proportionality Framework* clearly and concisely communicates important mathematics education research specific to ratios and proportions with the goal of providing teachers knowledge and strategies for incorporating this research into daily instruction. The *OGAP Ratio and Proportion Progression*, a key component of the framework, illustrates how students develop understanding of ratio and proportion concepts, strategies and procedures as well as the common errors students make and misconceptions that can impact new learning. The framework is introduced and explained in Chapter 3. Discussion of the progression through the lens of effective instruction and student learning is the focus of Chapter 4, which features specific examples of ways the progression can be used to impact instructional decisions. Both the framework and progression are returned to throughout the book to help teachers better understand and use activities and lessons in their math textbook, select or design formative assessment tasks, understand evidence in student work, make instructional decisions and provide actionable feedback to students.

- *Examples of Authentic Student Work:* This book uses more than 150 pieces of authentic student work samples to communicate specific research, provide the reader opportunities to analyze student solutions, emphasize particular mathematical concepts and practice using evidence to inform instructional decisions.
- *Discussion of Instructional Implications:* Throughout the book, you will find discussions of suggested instructional strategies to address evidence in student solutions. These sections are indicated by the following icon:

- *Looking Back: Looking Back* is included at the end of each chapter. There you will find a set of questions designed to help the reader more deeply examine particular ideas posed in the chapter or to contemplate related concepts. The answers to each Looking Back section are found at www.routledge.com.

Written for Teachers

Like its predecessors, *A Focus on Fractions* and *A Focus on Multiplication and Division*, this book was written first and foremost for classroom and preservice teachers. All the features of this book are specifically designed to help teachers learn and reflect on pertinent mathematics education research and ways to use this knowledge to analyze student thinking, take action based on the evidence in their students' solutions, select and create formative assessment problems and use textbook materials more effectively.

Teams of math teachers, particularly teachers participating in professional learning communities (PLC), can learn together by reading and discussing the chapters and answering and discussing questions posed in the *Looking Back* sections at the end of each chapter.

In addition, the abundant samples of authentic student work can be invaluable to instructors working with preservice teachers, as preservice teachers often do not have access to authentic student work. In addition, *A Focus on Ratios and Proportions: Bringing Mathematics Education Research to the Classroom* provides preservice teachers with an introduction to important educational research related to ratios and proportions, research that is vital yet sometimes lacking for teachers.

Final Thoughts

There are many important and thought provoking ideas in *A Focus on Ratio and Proportions: Bringing Mathematics Education Research to the Classroom*, all of which are closely linked to what we feel is the driving force of this book: the mathematics education research related to the teaching and learning of ratios and proportions and the important role that analyzing student work can play in providing all students effective and informed mathematics instruction. The ability to analyze student work to better understand how students understand mathematical concepts and skills as they are learning is paramount if we are to provide just-right instruction for all students.

Acknowledgments

We extend our deep appreciation to the thousands of Vermont, New Hampshire, Alabama, Nebraska, Michigan, Charleston and Philadelphia teachers who have been involved in the Ongoing Assessment Project (OGAP) over the past 16 years. We also thank the original OGAP design team members, who were instrumental in the early years of the project, as well as the members of the current OGAP National Professional Development Team.

Additionally, we extend our sincere thanks to Fritz Mosher, Caroline Ebby and Jon Supovitz from the Consortium for Policy and Research in Education, at Teachers College Columbia University and the University of Pennsylvania for their ongoing support and guidance as OGAP evolved over the past decade.

We also send out a heartfelt and deep thank-you to Cathy Stevens Pratt for her invaluable work with the line art that appears in this book.

The student work samples in this book were collected over the past decade through the Ongoing Assessment Project, which began in 2003 as part of the Vermont Mathematics Partnership.

1
Understanding Ratios, Rates and Proportions

Big Ideas

- A ratio is a multiplicative comparison between two or more quantities.
- There are two types of ratios: joining/composing and comparing.
- Every ratio has an associated rate.
- Ratio situations are communicated in different ways. This can be confusing for students and can impact their conceptualization of ratio and proportion situations and their solutions to problems.
- *A proportion* is a statement that two ratios are equivalent.
- To work effectively with a proportion, one must attend to two quantities.

Introduction to Understanding Ratios, Rates and Proportions

This chapter examines foundational mathematics concepts related to ratios and proportions. The study of ratios and proportions has long been acknowledged as "one of the most important goals in the school mathematics curriculum" (Dole, Hilton, Hilton, & Goos, 2015, p. 534). Lesh et al. (1988) identified proportionality as the "capstone of elementary mathematics and the cornerstone to all that follows" (pp. 93–94). Because ratios and proportions are inherently based on multiplicative relationships, student success with ratios and proportions is dependent on strong understanding and fluency with rational number multiplication and division developed in the elementary grades. At the same time, success in later related topics such as linear functions is dependent upon understanding the multiplicative relationships in ratio and proportional situations (Lamon, 2007). Figure 1.1 highlights how multiplicative concepts transition across the mathematics curriculum.

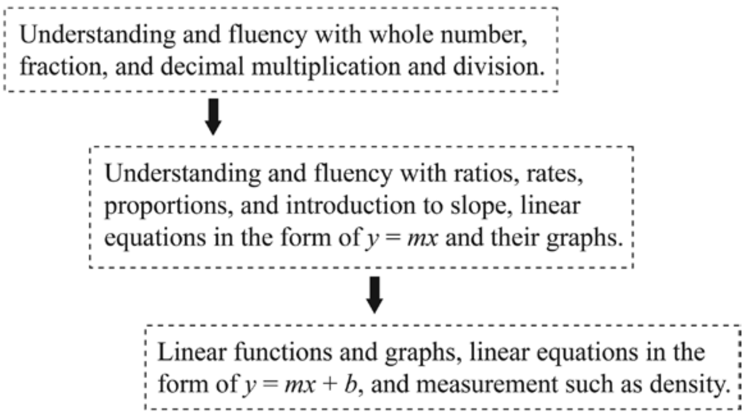
CCSSM and Ratios, Rates and Proportions

Instruction targeted on ratios and proportions in the CCSSM/NGA (2010) is first introduced in grade 6, focusing on understanding of ratios and rates and applying these concepts to familiar contexts like buying and selling and constant speed. That understanding is extended in grades 7 and 8 to a range of contexts and applications such as density, scale, slope, $y = kx$ and similarity.

This chapter includes discussions on:

- the meaning of a ratio,
- two different interpretations of ratios,
- understanding rates,
- the different language and notation used to communicate ratios and rates that may interfere with student understanding,
- the meaning of a proportion and what is meant by a proportional relationship.

Figure 1.1 How multiplicative concepts develop across mathematics curriculum



Ratios

A ratio is a multiplicative comparison of two or more quantities or measures. These comparisons can be part-to-part, such as 8 students to 2 adults or 2 cups of sugar to 5 cups of flour, or they can represent part-to-whole situations like 8 girls to 24 students in a class. Regardless of the situation, ratio comparisons are always related multiplicatively. For example, in the ratio 8 students to 2 adults, there are 4 times as many students as adults. One can describe this relationship in several ways such as:

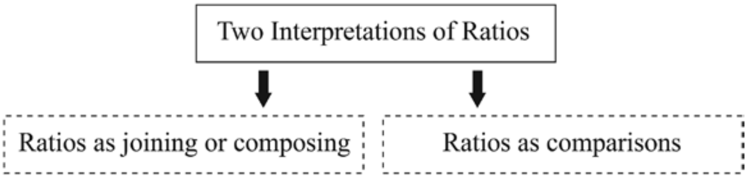
- 4 students for every 1 adult,
- 1 adult for every 4 students,
- the number of adults is $\frac{1}{4}$ the number of students.

Regardless of the way this relationship is communicated, there are 4 times as many students as adults.

Two Interpretations of a Ratio

Two related but different ways to form a ratio are shown in Figure 1.2. The first is by joining or composing two quantities in a way that "preserves a multiplicative relationship," and the second is multiplicatively comparing two quantities (Lobato, Ellis, Charles, & Zbiek, 2014, p. 18).

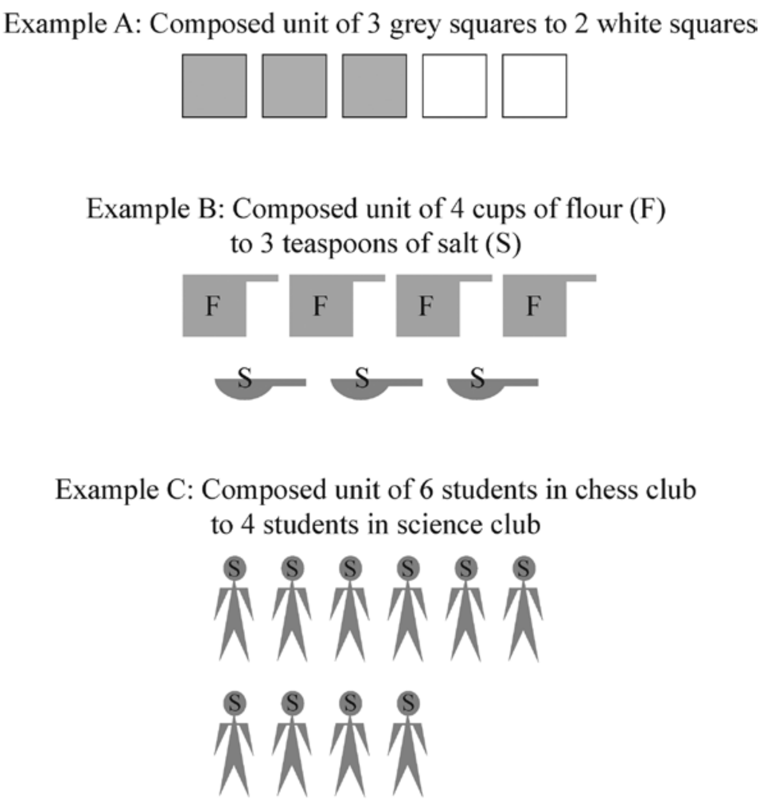
Figure 1.2 Two types of ratios



Ratios as Joining Two Quantities

As introduced already, one interpretation of a ratio is the joining of two quantities in a way that preserves a multiplicative relationship. In this interpretation, a certain number of one quantity together with a certain number of another quantity creates a *composed unit* (Beckmann, 2014). So we can think of this ratio interpretation as a *ratio as a composed Unit*. Figure 1.3 provides examples of composed units and the quantities used to create them.

Figure 1.3 Examples of part-to-part ratios as composed units



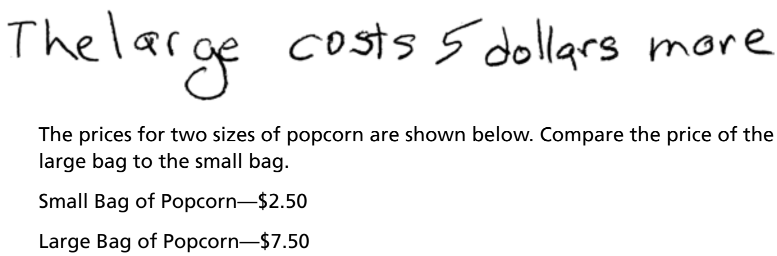
Notice that quantities in Example B, cups and teaspoons, are different while the quantities in Example C, are both students. Thus, the quantities that comprise a composed unit can be the same, or they can be different.

We will revisit this interpretation of a ratio as a composed unit later in the chapter when we examine concepts related to proportionality.

Ratios as a Multiplicative Comparison

In addition to a ratio as a composed unit as we discussed in the preceding pages, a ratio can also describe a multiplicative comparison between two quantities (Lobato et al., 2014). Tina's and Chad's solutions in Figures 1.4 and 1.5 introduce us to this interpretation of a ratio.

Figure 1.4 Tina's response



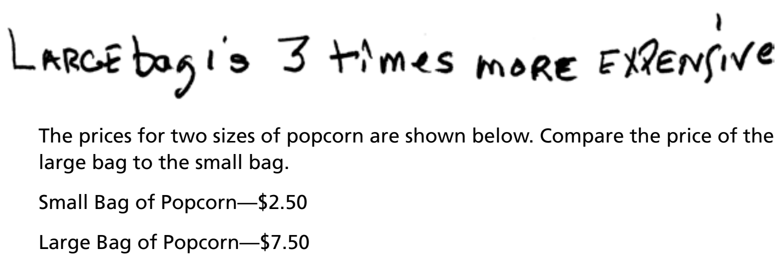
The large costs 5 dollars more

The prices for two sizes of popcorn are shown below. Compare the price of the large bag to the small bag.

Small Bag of Popcorn—$2.50

Large Bag of Popcorn—$7.50

Figure 1.5 Chad's response



LARGE bag is 3 times MORE EXPENSIVE

The prices for two sizes of popcorn are shown below. Compare the price of the large bag to the small bag.

Small Bag of Popcorn—$2.50

Large Bag of Popcorn—$7.50

Both responses accurately describe the relationship between the price of the small bag of popcorn and the price of the large bag of popcorn. Tina described this relationship in additive terms, while Chad compared the prices using multiplication. Chad's solution is an example of a ratio as a multiplicative comparison (Kaput & West, 1994; Thompson, 1994), and Tina's additive comparison is not. A ratio as a multiplicative comparison answers a question such as, "How many times greater or smaller is one quantity than another?" Tina found the additive difference between the two costs. This is a common response as students begin working with ratio and proportions and will be discussed later in this chapter and other chapters throughout the book.

Rates

Long-held definitions for rates and ratios were typically based on the nature of the quantities in each. Thus, a ratio was often interpreted as a comparison between two like quantities such as students to students, while a rate was defined as a comparison between different quantities such as gallons to dollars (Lamon, 2007). Despite disagreements over the essential similarities and differences between rates and ratios, these definitions remained and became common in math textbooks and ratio and proportion instruction (Lesh et al., 1988, p. 108).

The CCSSM Ratio and Proportion Progression (McCallum, Zimba, & Daro, 2011) and mathematics education researchers use the term *ratio* to describe both of these situations; those with like and those with unlike quantities. Furthermore, *every ratio has an associated rate*. For example, if a recipe calls for 6 cups of flour for every 2 cups of water, its associated rate is 3 cups of flour per 1cup of water. The rate is a special form of a ratio in which the compared quantity is a unit amount. A rate is often associated with the mental operations one brings to the situation and typically uses the word *per*. This means that a ratio becomes a rate when one understands that the ratio is applicable beyond one particular situation and a characteristic of a set of equivalent ratios (Thompson, 1994).

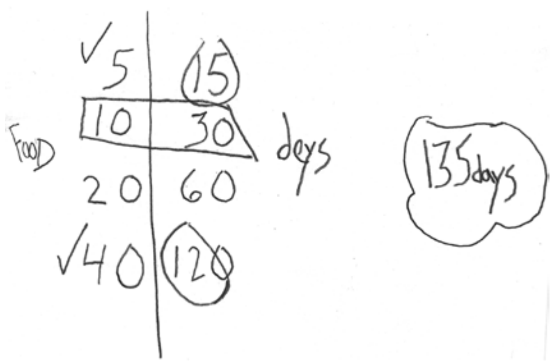
To better understand this interpretation of rates and their relationship to ratios, solve the Paul's Dog problem below.

Paul's Dog

Paul's dog eats 10 pounds of food in 30 days.
How long does it take Paul's dog to eat a 45-pound bag of dog food?

Tray's and Gina's solutions are shown in Figures 1.6 and 1.7. Examine each solution. Which solution has evidence of ratio reasoning? Which solution has evidence of rate reasoning? What is the evidence?

Figure 1.6 Tray's solution to the Paul's Dog problem

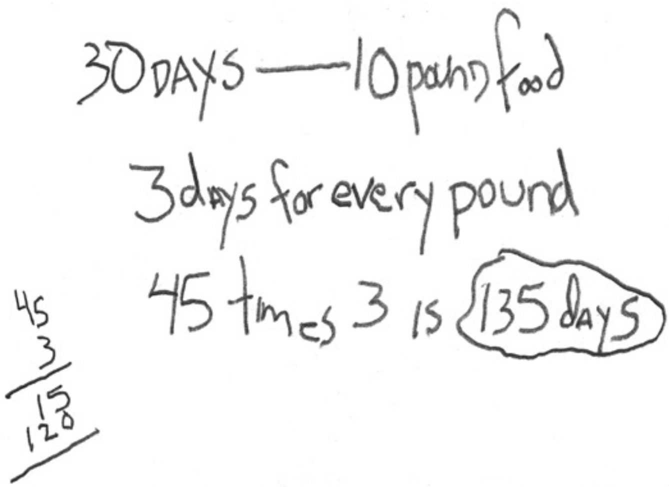


Paul's dog eats 10 pounds of food in 30 days.

How long does it take Paul's dog to eat a 45-pound bag of dog food?

Tray's solution is an example of using reasoning related to ratios. Students iterating or dividing a given ratio often characterizes this type of reasoning. Although we do not know how he created the ratios in his solutions, it appears that Tray's reasoning focused on four distinct ratios (Figure 1.8).

Figure 1.7 Gina's solution to the Paul's Dog problem



Tray's solution did not contain evidence of the special ratio, or rate, 3 days per pound of food. Instead, Tray found the number of days 45 pounds of food would last by combining 15 days + 120 days = 135 days.

Figure 1.8 Tray's four equivalent ratios

Pounds of Food	Number of Days
5	15
10	30
20	60
40	120

In contrast, Gina applied the rate of 3 days for every 1 pound of food. This allowed her to immediately multiply 45 pounds times 3 days per pound. Using this rate, Gina could find the number of days the food would last given any number of pounds of food. Notice that the rate *3 days for every pound of food* is true for every ratio in Tray's table and is true for any ratio equivalent to *10 pounds of food in 30 days*. This is an example of using a *unit rate* to solve a problem. That is, unit rates communicate the number of units of one quantity for every one unit of another quantity.

Other familiar unit rates include:

- \$2.57 for one gallon of gas or \$2.57 per gallon,
- 65 miles for one hour of driving or 65 miles per hour,
- 12 inches for every one foot or 12 inches per foot, and
- four quarters for each dollar or 4 quarters per dollar.

In addition, each ratio situation has associated with it two unit rates that could be used for solving problems. Let's consider bread that costs \$5.00 for 2 loaves. The two ratios that follow can be used to describe this relationship:

1. \$5.00 to 2 loaves of bread
2. 2 loaves of bread to \$5.00.

Each of these ratios has one related unit rate that is determined by dividing the first quantity by the second quantity. This process is illustrated in Table 1.1

Table 1.1 Two associated unit rates

Ratio	*Division to Determine a Unit Rate*	*Unit Rate*
\$5.00 to 2 loaves of bread	\$5.00 ÷ 2 loaves of bread =	$^5/_2$ dollars for one loaf of bread
2 loaves of bread to \$5.00	2 loaves of bread ÷ \$5.00 =	$^2/_5$ loaf of bread for one dollar

We can see this in Table 1.1. The ratio, 5 dollars to 2 loaves of bread is related to the fraction $\frac{5}{2}$, the number of dollars for one loaf of bread. And the ratio, 2 loaves of bread to 5 dollars, is related to the fraction $\frac{2}{5}$, the amount of bread per one dollar.

A unit rate is an important concept for students to understand and use because it generalizes across problem contexts such as density, buy/consume, scaling, similarity and distance/rate/time.

Ratio and Fraction Confusion

The relationship between fractions and ratios can be confusing for both teachers and students. Part of the confusion may stem from the fact that one of the notations for a ratio shares the same $\frac{a}{b}$ form with a fraction. Thus, the meaning of $\frac{a}{b}$ is highly dependent upon the context and the relationships between a and b. Consider the two contexts shown below. Why does each context require one to consider the meaning of the quantities differently?

Contexts:

1st In her first game, Jasmine had 3 hits in 4 at bats. In her second game, Jasmine had 2 hits in 5 at bats. What is the ratio of hits to at bats for Jasmine's first two games?

2nd Judi ran $\frac{3}{4}$ miles on Monday and $\frac{2}{5}$ miles on Tuesday. How many total miles did Judi run on Monday and Tuesday?

Study the solutions to the problems in Figures 1.9 and 1.10. How are they the same and how are they different? Why are the solutions different for these two problems? You probably noticed that even though each problem can be expressed using $\frac{a}{b}$ notation, each context requires a different interpretation. Study Figures 1.9 and 1.10, which provide examples of equations that represent the two contexts.

Figure 1.9 Jasmine's total number of hits and at bats

$$\frac{3\text{ hits}}{4\text{ at bats}} + \frac{2\text{ hits}}{5\text{ at bats}} = \frac{5\text{ total hits}}{9\text{ total at bats}}$$

Figure 1.10 The total number of miles traveled

$$\frac{3}{4}\text{mile} + \frac{2}{5}\text{mile} = \frac{23}{20}\text{miles}$$

The first context is an example of a ratio situation, and the second context is an example of a fraction situation. Notice that the first context involves 4 quantities: 3 hits to 4 at bats in her first game and 2 hits to 5 at bats in her second game. In contrast, the second context involves only two quantities: $\frac{3}{4}$ miles traveled on Monday and $\frac{2}{5}$ miles traveled on Tuesday. The discussion that follows provides examples of ways in which fractions and ratios behave differently and similarly.

A ratio describes a multiplicative relationship between two quantities, in this case "hits" to "at bats." Ratios can represent a variety of situations such as part to part and part to whole. An example of a part-to-part ratio is: $\frac{5\text{ chocolate chip cookies}}{4\text{ vanilla cookies}}$ in a cookie jar. An example of a part-to-whole ratio is $\frac{4\text{ red chairs}}{19\text{ total chars}}$. In general, each quantity in a ratio is defined by its own label and can be communicated in two different ways. We see this in the labels associated with ratios we examined earlier: $\frac{\text{hits}}{\text{at bats}}$, $\frac{\text{chocolate chip cookies}}{\text{vanilla cookies}}$ and, $\frac{\text{red chars}}{\text{total chairs}}$. The quantities can also be defined in other ways, for example, the ratio of hits to at bats is 3 : 4.

In addition, in many situations one can switch the position of the quantities in a ratio without changing the multiplicative relationship between the quantities. For example, the ratios $\frac{5\text{ chocolate chip cookies}}{4\text{ vanilla cookies}}$ and $\frac{4\text{ vanilla chip cookies}}{5\text{ chocolate cookies}}$ both communicate the fact that there are $1\frac{1}{4}$ times as many chocolate chip cookies as vanilla cookies in the cookie jar.

As described above, the second context is an example of a fraction, more specifically a fraction *as a number*. In this example, there is a fractional number of miles. The big idea here is that $\frac{3}{4}$ and $\frac{2}{5}$ are numbers that tell us the amount of miles Judi ran, whereas we saw above that each number in a ratio requires its own label. A fraction, on the other hand, describes one quantity. Examples include: $\frac{2}{5}$ miles, $\frac{7}{8}$ pizza, $\frac{4}{5}$ hour and, $\frac{1}{2}$ of the class.

Unlike with ratios, switching the position of the numerator and denominator in a fraction results in a different number. For example, $\frac{3}{4}\,mile \neq \frac{4}{3}\,mile$.

Sometimes Ratios and Fractions Act in Similar Ways

Despite these general differences between fractions and ratios, there are times when these two interpretations of $\frac{a}{b}$ act in similar ways and times when they act quite differently. This may be another reason students struggle with the differences between ratios and fractions. In some situations, for example, a part-to-whole ratio can meaningfully be interpreted as a fraction. One can restate the part-to-whole ratio example $\frac{4\ \textit{red chairs}}{19\ \textit{total chars}}$, as a fraction. That is, "$\frac{4}{19}$ of the chairs in the classroom are red." In addition, we can use the same procedure to identify equivalent fractions and equivalent ratios. Figure 1.11 provides an example.

Figure 1.11 Equivalent ratios and equivalent fractions

Ratio Example: $\frac{11\ \text{student} \times 2}{2\ \text{coaches} \times 2} = \frac{22\ \text{students}}{4\ \text{coaches}}$

Fraction Example: $\frac{3\times2}{4\times2}\ \text{hours} = \frac{6}{8}\ \text{hours}$

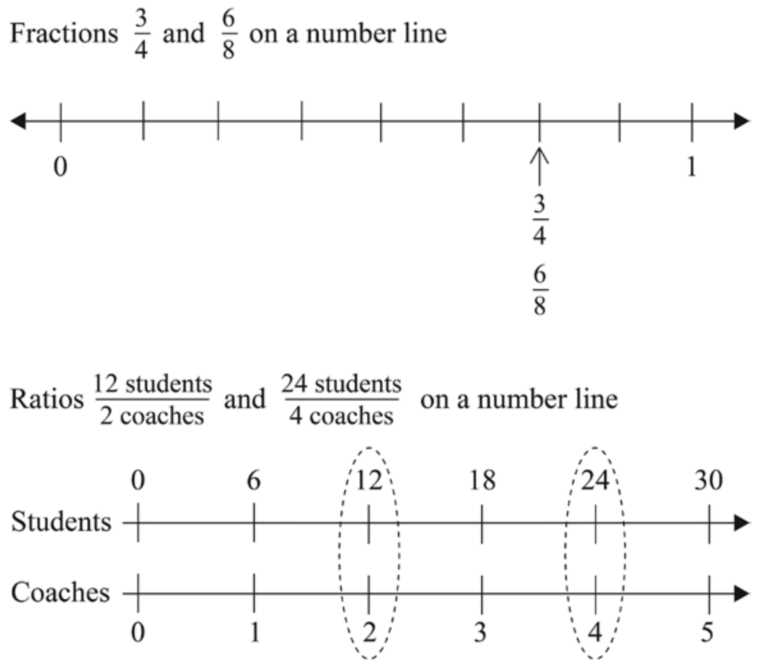
Although we use the same procedure, multiplying or dividing each quantity in a ratio or fraction by the same number, we interpret the meaning of these two situations differently. In the case of the ratio, $\frac{12\ \text{students}}{2\ \text{coaches}}$ is equivalent to $\frac{24\ \text{students}}{4\ \text{coaches}}$ because the multiplicative relationship between the quantities is the same in each ratio. That is, in each situation, there are 6 students for each coach. In the fraction example, $\frac{3}{4} = \frac{6}{8}$ means that the two fractions represent the same number. They share the same point on the number line. In contrast, ratios cannot be represented on a single number line because ratios are comparisons of quantities and therefore are often represented on a double number line.

Study Figure 1.12. Notice that $\frac{3}{4}$ and $\frac{6}{8}$ are at the same location on the number line because they represent the same numerical value. In contrast, the equivalent ratios $\frac{12\ \text{students}}{2\ \text{coaches}}$ and $\frac{24\ \text{students}}{4\ \text{coaches}}$ are represented by a double number line each representing different quantities. Also notice the ratios are in different places on the double number line because they do not represent the same numbers of students and coaches. Rather, they are related multiplicatively.

Last Words on Ratios and Fractions

The topic of the relationship between ratios and fractions is broad, deep and nuanced, and we have addressed a small but important part of this domain in this chapter. A deeper, more comprehensive discussion would certainly include concepts related to the different interpretations of fractions such as fractions as part to whole, fractions as operators and fractions as a quotient. In light of the complexity of this topic, it is most important that students are provided regular opportunities to interpret $\frac{a}{b}$ in different

Figure 1.12 Equivalent fractions and ratios on number lines



contexts and situations, with differing complexity of numbers and number relationships and through different visual models. These opportunities should provide for plenty of student discussion about the different ways fractions and ratios interact with each other and how different contexts require different ways of interpreting $\frac{a}{b}$.

The Impact of Language Used With Ratios and Rates

It is important to realize that ratios are communicated in a variety of ways, and this can be confusing for students, can impact their conceptualization of ratio and proportion situations and can impact their solutions to problems.

These examples highlight different ways one can say or write a ratio:

- 8 student to 2 adults
- 4 cups for every 1.5 gallons
- 5 out of every 6 households
- 3 parts juice to 2 parts soda
- 3 miles in 45 minutes
- 20 miles per gallon
- the ratio of 3 pounds to 2 dollars

In addition, the word "rate" is sometimes applied to situations that do not involve unit rates and can also be used without identifying both quantities that comprise the rate.

Example of rate where the quantities that comprise the rate are not identified.

- a heart rate of 72 (72 beats per minute)
- the speed limit is 65 (65 miles per hour)

Examples where the term "rate" is used where the rate is not a unit rate.

- the unemployment rate is 4.2% (the number of unemployed per 100 people in the workforce)
- a country's birthrate is 15.2 (number of live births per 1,000 in a population in a year)

Notice that all of the examples imply but do not specifically state an associated second quantity (Lamon, 2012).

And finally, there are also different mathematical notations used to communicate a ratio. These can include:

- the use of a colon as in 3 : 2
- the fraction bar or vinculum as in $\frac{24\text{ inches}}{2\text{ feet}}$ or $\frac{24\text{ inches}}{2\text{ feet}}$.

(McCallum et al., 2011; Lamon, 2012)

The type of notation, the context and the way the ratio is described in a problem can impact both the difficulty of the problem and students' solution strategies. Ultimately, to prepare our students to understand ratio and rate concepts, interpret ratios and rates in their lives and solve problems involving rates and ratios, it is important that instruction provides students a variety of ratio contexts, various written and verbal references to ratios and different mathematical notations. Instruction must support students in analyzing each ratio situation and interpreting the nuanced differences that accompany ratio use in everyday life. Susan Lamon states,

> *Part of the difficulty is that everyday language and usage of rates and ratios is out of control . . . Students are exposed to less-than-correct usage and terminology, and it is no easy task to reconcile precise mathematical ideas with informal colloquial usage.*
>
> (Lamon, 2012, pp. 226–227)

Proportions

As we have seen so far, ratios and rates are important pieces of the bigger domain of proportional reasoning. We use ratios to compare two quantities multiplicatively or to join two quantities in a way that preserves a multiplicative relationship. Although multiplicative processes are at the heart of ratio reasoning, understanding and using rates and ratios to solve problems requires more than multiplication. For one thing, reasoning with ratios requires students to consider two quantities simultaneously.

We sometimes use a single ratio to describe a situation. Imagine you count 21 vehicles in the school parking lot and notice that 16 of the vehicles are cars and 5 are trucks. The ratio, 16 cars to 5 trucks, communicates that there are about 3 times as many cars in the parking lot as trucks. In this case, we used one ratio to make sense of a situation. Other times we consider two (or more) ratios simultaneously. Consider the two ratios that describe the relationship between the number of flower plants in two flower beds.

$$\frac{4\text{ plants}}{2\text{ square yards}} = \frac{12\text{ plants}}{6\text{ square yards}}$$

You might consider one of the ratios shown earlier, independent of the other, and it would tell you something about the flowerbed. Four plants to 2 square yards, for example, provides information about how *crowded* the flower bed is with flower plants. You might deduce that there are 2 plants in each square yard, and this might even provide you with a mental picture of an area, 1 yard by 1 yard, that contains 2 plants.

However, the example asks us to consider two ratios in the context of equality. This is an example of a proportion. Thus, a *proportion is a statement that two ratios, or two pairs of quantities, are equivalent*. In this case, one might say 4 plants in 2 square yards is in the same proportion as 12 flower plants in 6 square yards. Perhaps one wants to extend this relationship to find other numbers of plants and their corresponding size flowerbeds that are also in the same proportion as the two flowerbeds we examined. The ratio table in Figure 1.13 identifies other pairs of flower plants and flowerbeds that are related in the same way that 4 plants is related to 2 square yards.

Figure 1.13 Ratio table

Number of Plants	*Square Yards*
4	2
8	4
12	6
16	8
20	10

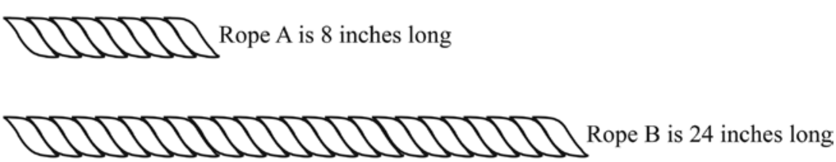
The term *proportional relationship* is used to describe ratios that share the same multiplicative relationship between the two quantities in each ratio. So the five ratios in the ratio table are in a proportional relationship.

Making Sense of Proportions Requires Relative Reasoning

Tina's solution to the Popcorn problem in Figure 1.4 where she found the additive difference in the cost of a small and a large popcorn is an example of a student bringing an additive strategy to a situation that required a multiplicative strategy. The transition from additive or *absolute* thinking between two quantities to multiplicative or *relative* thinking can be a challenge for some students. Next, we examine some of the issues related to this transition.

Read the problem in Figure 1.14. What are the different ways one might interpret this situation?

Figure 1.14 Absolute versus relative reasoning—How much longer is Rope B than Rope A? (Hulbert, Petit, Ebby, Cunningham, & Laird, 2017)

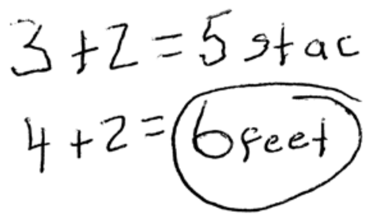


The question can be answered using the absolute or additive difference between the two ropes. This reasoning focuses on the fact that Rope B is 16 inches longer than Rope A. It can also be answered using relative or multiplicative reasoning. This interpretation is based on the realization that Rope B is 3 times longer than Rope A. Thus, the statement "Rope B is 16 inches longer than Rope A" represents an absolute interpretation of this situation, and "Rope B is 3 times longer than Rope A" represents a relative interpretation.

A strong body of research suggests that some students struggle to acquire relative thinking concepts, which are essential in the development of proportional reasoning. Susan Lamon states that children can have "great difficulty . . . in describing a relative perspective even when they recognize it as an alternative to thinking in absolute terms" (Lamon, 2012, p. 43). The ability to reason relatively is at the heart of interpreting ratio and proportional situations.

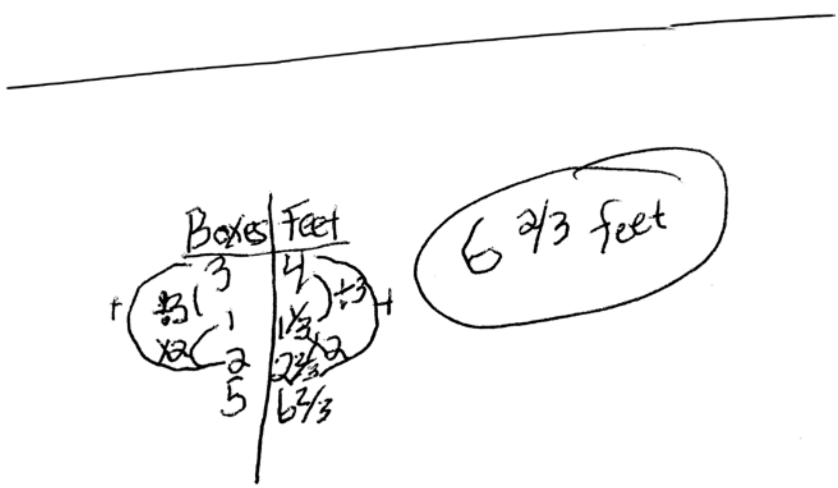
Consider Stacy's and Ra's solutions to the Stacking Boxes problem in Figures 1.15 and 1.16. Which solution contains evidence of absolute reasoning? Which solution has evidence of relative reasoning? What is the evidence?

Figure 1.15 Stacy's solution to the Stacking Boxes problem



A company stacks same-size boxes on top of each other. The height of three stacked boxes is 4 feet. What is the height of five stacked boxes?

Figure 1.16 Ra's solution to the Stacking Boxes problem



Stacy appeared to apply absolute reasoning to the relationship between the number of boxes and the number of feet. Evidence suggests she determined 3 boxes measure 4 feet and an additional 2 boxes measure 2 feet. She used only addition to communicate her thinking. Ra used a ratio table to determine the height of 5 boxes. He appeared to apply division and multiplication to scale the given ratio, 3 boxes to 4 feet, up and down to create equivalent ratios. Thus, the evidence suggests that Ra interpreted this relationship using relative thinking.

Again, ratios, rates and proportions are first and foremost notions grounded in multiplicative processes and reasoning. Yet, as Stacy's solution suggests, it is common for some students to inappropriately describe this relationship in absolute terms, using additive or subtractive language. Thus, a foundational understanding and a feature of proportional reasoning is the ability to differentiate between multiplicative and additive situations and to understand and flexibly use the multiplicative relationships in ratios and proportions students need to solve problems. For many students, this is a new idea and represents a change in thinking about number relationships. You will learn in subsequent chapters that this transition from additive to relative thinking can be enhanced by understanding and purposefully choosing appropriate problem structures, number relationships and contexts in your proportional reasoning instruction.

Go To Chapters 5, 6 and 7 have detailed discussions about the impact of different problem structures on student learning and the strategies students use to solve ratio and proportion problems.

Although ratios and proportions are grounded in multiplicative processes, it is important to understand that these concepts differ from the multiplication concepts in the elementary grades in significant ways. The next section examines these differences.

Proportional Relationships and Multiplication

While the development of ratio and proportion concepts is dependent on strong multiplicative reasoning and fluency in the elementary grades, multiplication concepts in the elementary grades and ratio and proportion concepts in the middle grades are not the same. Working with ratios requires attention to the multiplicative *relationship* between two quantities, not a focus just on the product, which is generally the case in elementary multiplication. In addition, working with ratios often involves scaling in such a way that a multiplicative relationship between equivalent ratios is maintained. Discussion of both of these important differences between multiplication and division in the elementary grades and proportional concepts in the middle grades is contained in this section.

How Multiplication With Ratios Differs From Elementary Experience With Multiplication

Read and solve the problems in Figure 1.17. What do you notice about the two quantities given in each problem and how these quantities are related?

Figure 1.17 How are the quantities in these problems related?

Problem A

A marathon is approximately 26.2 miles. My mother ran 6 marathons last year.

About how many total miles did she run in the 6 marathons?

Figure 1.17 Continued.

Problem B

The ratio of votes for Bob to votes for Marge is 3 to 6.

How many times more votes did Marge get than Bob?

How many times fewer votes did Bob get than Marge?

As you probably noticed, the two problems are mathematically different even though Problems A and B both involve multiplicative relationships. To solve Problem A, one has to reason that the total number of miles run in 6 marathons is described by: 6 marathons × 26 miles per marathon = 156 total miles. Importantly, the whole number of miles in a marathon will always be 26.2 miles. The only way that the total miles run can change is if the number of marathons that someone participates in changes. In contrast, in Problem B, the two quantities are considered simultaneously: the number of votes for Bob and the number of votes for Marge. That is, Marge received twice as many votes as Bob, or Bob received half as many votes as Marge. This multiplicative relationship does not change regardless of the total number of votes cast.

Figure 1.18 illustrates the difference between the two problems multiplicatively.

Figure 1.18 Problem A and B multiplicative relationships

Problem A

26.2 miles per marathon × 4 marathons = 104.8 miles

26.2 miles per marathon × 5 marathons = 131 miles

Problem B

$\times 2$

3 votes for Bob: 6 votes for Marge

$\times 3$ $\qquad$ $\times \frac{1}{3}$

9 votes for Bob: 18 votes for Marge

$\times \frac{1}{2}$

Reasoning with proportions, therefore, requires the ability to consider and coordinate two different quantities simultaneously (Lobato et al., 2014), not just the product as in the marathon problem. The transition to considering two quantities at the same time can be difficult for students.

The Ranch Problem in Figure 1.19 and Figure 1.20 involves the ratio of horses to acres and can help illustrate this point. Solve the Ranch problem and then analyze Antonia's and

Derek's solutions. Which solution shows evidence of using the relationship between both quantities in each ratio to determine which ranch has more horses per acre? In which solution is there evidence of using only one quantity in each ratio? What is the evidence?

Figure 1.19 Antonia's solution to the Ranch problem

Big Horn Ranch raises 120 horses on 40 acres.

Jefferson Ranch raises 70 horses on 35 acres.

Which has more horses per acre?

Big H. ranch – it has
50 more horses

$$\begin{array}{r} 120 \\ -\ 70 \\ \hline 50 \end{array}$$

Figure 1.20 Derek's solution to the Ranch problem

Big Horn Ranch raises 120 horses on 40 acres.

Jefferson Ranch raises 70 horses on 35 acres.

Which has more horses per acre?

+ 3 times more horses than acrs

2 times more horses than acres

BhR – has 1 more horrs on each Acres

Although the problem involved two different quantities, Antonia's solution incorrectly focuses on only the number of horses *without considering* the number of acres or the multiplicative relationships between the number of acres and the number of horses. Her solution represents a common error evidenced as students begin to make sense of ratio and proportion concepts. Derek's solution attended to both the number of horses and the number of acres in both ranches as well as the multiplicative relationships between acres and horses. That is, Derek used multiplication to determine the relationship between the number of horses and the number of acres, while Antonia's solution was inappropriately based on an additive interpretation.

Iterating a Composed Unit to Solve Proportional Problems

Earlier in the chapter, we used the context of a flower bed to understand the meaning of a proportion and what it means for ratios to be equivalent. But why might one want to find equivalent ratios in the first place? We often use proportions to predict an unknown quantity based on given quantities. Four examples are shown below.

A. Gasoline costs \$2.75 per gallon. How much will I pay for 10.5 gallons of gasoline?
B. The weather forecast calls for snow to fall at the rate of 2 inches per hour. How many inches of snow should I expect after 2 hours?
C. It takes me approximately 15 minutes to jog 1 mile. About how many miles can I jog in 1 hour?
D. Copy paper is packaged 500 sheets per pack. How many sheets in 8 packs of copy paper?

In each of these examples, we can extend the rate provided in the situation to find specific equivalent ratios. For example, in situation A, we are provided the rate:

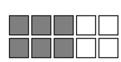
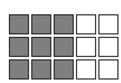
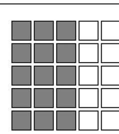
$$\frac{\$2.75}{1\,\text{gallon}}$$

We can use that rate to determine the cost of 10.5 gallons of gas, which can be represented with the proportion:

$$\frac{\$2.75}{1\,\text{gallon}} = \frac{?\,\text{dollars}}{10.5\,\text{gallons}}$$

There are a number of strategies one can use to accomplish this. We will examine a strategy that involves using a composed unit. This is done by either iterating a composed unit any number of times or by dividing the composed unit. Table 1.2 provides a visual example.

Table 1.2 Iterating a composed unit to create equivalent ratios

Number of Sets of Composed Units	Number of Grey Squares	Number of White Squares	Visual Model of Iterations 3 grey for every 2 white
1	3	2	
2	6	4	
3	9	6	
4	12	8	

Notice that Table 1.2 shows three other ratios that are equivalent to 3 gray squares to 2 white squares. Each of these equivalent ratios was determined by simply iterating or copying the 3 : 2 ratio a certain number of times. Iterating the 3 : 2 ratio twice results in the equivalent ratio 6 gray squares to 4 white squares, iterating it three times results in the equivalent ratio 9 gray squares to 6 white squares and iterating it four times results in the equivalent ratio, 12 gray squares to 8 white squares. One could continue iterating the 3 : 2 ratio to identify countless equivalent ratios.

What is the visual cue that each ratio in the table is indeed equivalent? Notice that for each equivalent ratio, the number of gray squares is $1\frac{1}{2}$ × the number of white squares. For example, 9 gray squares is $1\frac{1}{2}$ times greater than 6 white squares. Thus, each ratio in Table 1.2 preserves this multiplicative relationship ($1\frac{1}{2}$) between white and gray squares.

- There are $1\frac{1}{2}$ times as many gray squares as white squares.
- The number of white squares is $\frac{2}{3}$ the number of gray squares.
- There are $1\frac{1}{2}$ gray squares per 1 white square.
- There are $\frac{2}{3}$ white squares per 1 gray square.
- 2 white squares for every 3 gray squares.
- 3 gray squares for every 2 white squares.

From this example, we can see that the two quantities, the number of white squares and the number of gray squares, change predictably, while the multiplicative relationship between the equivalent ratios stays constant.

As indicated in the example, one can also divide a composed unit to create equivalent ratios. Consider a large bread recipe that calls for 24 cups of flour for every 8 cups of water. We can use this composed unit to consider other bread recipes that maintain the same ratio of flour to water. For example, we can divide this composed unit by 2, resulting in 12 cups of flour for every 4 cups of water. This new ratio is equivalent to the original given composed unit because it shares the same multiplicative relationship between cups of flour and cups of water. Using this process of division, we could identify other equivalent ratios of flour to water.

As with all equivalent ratios, the multiplicative relationship is the same between the quantities; in this case the scale factor between the cups of water and the cups of flour is 3. See Table 1.3.

Table 1.3 Equivalent ratios

Bread recipes in the same ratio as the original composed unit	
Cups of Flour	*Cups of Water*
24	8
18	6
12	4

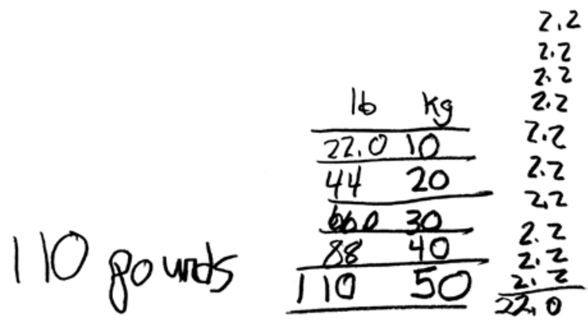
That is, in each ratio, the number of cups of flour is always 3 times greater than the number of cups of water even though the amounts of flour and water change. Another way to think about this situation is the number of cups of flour and the number of cups of water change such that the multiplicative relationship is preserved.

Pre-Ratio Reasoning

As students begin working with composed units to form equivalent ratios, they often iterate a composed unit by repeatedly adding. Thus, a student can create equivalent ratios without considering the multiplicative relationships between the quantities in the equivalent ratios. In this case, the quantities are pounds to days. Some researchers refer to this reasoning as pre-ratio reasoning (Lesh et al., 1988). Charles's solution in Figure 1.21 provides an example of this pre-ratio reasoning.

Figure 1.21 Charles's solution to the Suitcase problem uses pre-ratio reasoning characterized by iterating a unit and adding up.

There are 2.2 pounds in 1 kilogram. A suitcase weighs 50 kilograms. How many pounds does the suitcase weigh?



Ultimately, the goal for all students is to understand and use the multiplicative relationships inherent in ratios and proportions to solve problems. Yet we should not be surprised to see students creating equivalent ratios through repeated addition. "Initially, when students make tables of quantities in equivalent ratios, they may focus only on iterating the related quantities by repeated addition to generate equivalent ratios" (McCallum et al., 2011, p. 5).

The OGAP Proportion Progression and Ratios and Rates

Chapter 3 and 4 will provide strategies to use the OGAP Progression to analyze evidence in student work and make instructional decisions related to ratios and rates.

Later chapters will build on foundational ratio, rate and proportion concepts examined in this chapter. They will detail topics such as: ways ratio and proportion concepts can be developed using visual representations, the OGAP Ratio and Proportion Progression and using evidence in student solutions to ratio and proportion problems to inform instructional decisions.

Chapter Summary

This chapter focused on some foundational concepts related to ratios and rates such as:

- Ratios are multiplicative comparisons of two or more quantities that require students to pay attention to the multiplicative relationship between two quantities simultaneously. This feature of a ratio can be difficult for some students and must be a focus of ratio instruction.

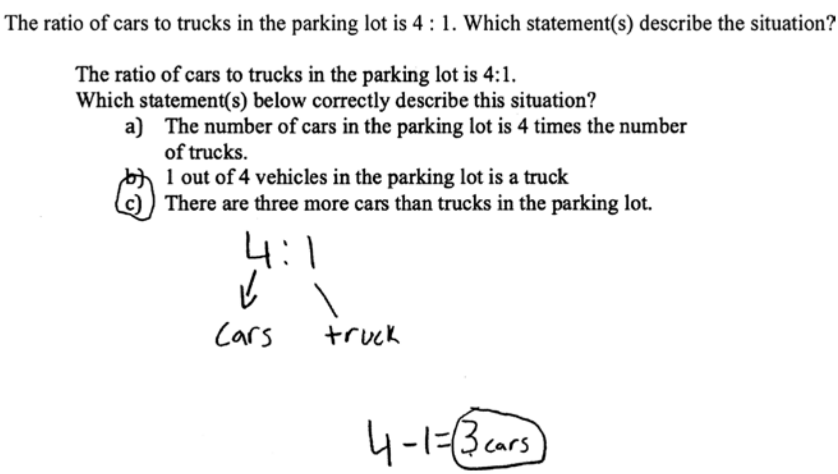
- Ratios can be interpreted as joining/composing or as comparisons.
- Equivalent ratios are comprised of two quantities that vary and one multiplicative relationship that does not vary.
- A rate is a special ratio that applies beyond a particular situation and describes a characteristic of a whole class of equivalent ratios.
- Depending on a student's understanding of ratio and various structures of ratio problems, solutions may be based on iterating or dividing ratios or on application of a rate.
- Ratio situations can be communicated in a variety of different ways, and the type of notation, the context in which the ratio is used and the way the ratio is described in a problem can impact both the difficulty of the problem and the student's solution strategy.
- A unit rate communicates the number of units of one quantity for every one unit of another quantity.

Looking Back

1. Study Carrie's and Parker's solutions to the Parking Lot problem shown in Figures 1.22 and 1.23.
   a. What understanding(s) or misunderstanding(s) of the meaning of a ratio are evidenced in each solution?
   b. In what ways can you use the idea of a composed unit and equivalent ratios in the context of this problem to help strengthen the understandings of the relationships in ratios?

Figure 1.22 Carrie's solution to the Parking Lot problem

The ratio of cars to trucks in the parking lot is 4 : 1. Which statement(s) describe the situation?


The ratio of cars to trucks in the parking lot is 4 : 1. Which statement(s) describe the situation?

The ratio of cars to trucks in the parking lot is 4:1.
Which statement(s) below correctly describe this situation?
a) The number of cars in the parking lot is 4 times the number of trucks.
~~b)~~ 1 out of 4 vehicles in the parking lot is a truck
c) There are three more cars than trucks in the parking lot.

Figure 1.23 Parker's solution to the Parking Lot problem

The ratio of cars to trucks in the parking lot is 4:1.
Which statement(s) below correctly describe this situation?

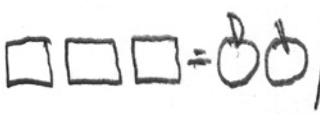
1) The number of cars in the parking lot is 4 times the number of trucks.
2) 1 out of 4 vehicles in the parking lot is a truck
3) There are three more cars than trucks in the parking lot.

Because it's a ratio of 4:1, for every 1 truck there are 4 cars. So there will always be 4 more cars then trucks.

2. Study Edith's solution to the Packing Apples problem shown in Figure 1.24.
   a. Describe Edith's solution to this problem. Which parts of her solution are correct? Which parts are incorrect?
   b. What is a feature of this problem that may have led to Edith's incorrect solution?

Figure 1.24 Edith's solution to the Packing Apples problem

Carrie is packing apples. It takes 3 boxes to pack 2 bushels of apples. How many boxes does Carrie need to pack 7 bushels of apples?



3:2

6 boxes = 4 bushels
12 boxes = 8 bushels
11 boxes = 7 bushels

11 boxes

3. Study Sandi's solution to the same problem shown in Figure 1.25.
   a. How does Sandi's solution differ from Edith's solution in Figure 1.24?
   b. Explain why Sandi's solution is more efficient and generalizable than Edith's.

Figure 1.25 Sandi's solution to the Packing Apples problem

Carrie is packing apples. It takes 3 boxes to pack 2 bushels of apples. How many boxes does Carrie need to pack 7 bushels of apples?

$7 \div 2 = 3.5$

$3.5 \times 3 = 10.5$

4. Letters a–d show common ratios that are described with one word, thus obscuring the quantities. Identify the two quantities that comprise each ratio below.
   a. Population density
   b. Speed
   c. π (pi)
   d. Percent

5. Each student solution shown in Figure 1.26 contains the use of a unit rate, but the rates do not include quantities. In each solution, identify the unit rate and provide the correct quantities.

Figure 1.26 Solutions that contain a unit rate

Solution A

Mrs. White bought 4 pizzas for her 18 students. Mr. Green bought 5 pizzas for his 24 students. All the pizzas were the same size. If the students in each class got equal shares of the pizza, who got more pizza—a student in Mrs. White's class or a student in Mr. Green's class? Show your work.

$$\begin{array}{r} 4.5 \\ 4\,\overline{)\,18} \\ -16 \\ \hline 20 \end{array}$$

$$\begin{array}{r} 4.8 \\ 5\,\overline{)\,24} \\ -20 \\ \hline 40 \end{array}$$

Figure 1.26 Continued.

Solution B

A company stacks same-size boxes on top of each other. The height of three stacked boxes is 4 feet. What is the height of five stacked boxes? Show your work.

1	2	3	6	12
$1.\overline{33}$	$2.\overline{66}$	4	8	16

Solution C

A 20-ounce box of Toasty Oats cereal costs $3.00. A 15-ounce box of Toasty Oats costs $2.10. Which box costs less per ounce?

$$\frac{\$3}{20} = 15¢$$

$$\frac{\$2.1}{15} = 14c$$

6. Figure 1.27 shows Drew's and Alyssa's solutions to the Pizza problem. Read the problem, study the solutions and answer the questions that follow.
   a. Both students used unit rates in their solutions. What are the quantities associated with each quantity in their solutions?
   b. Use the unit rates in each solution to solve the problem. Explain your reasoning.

Figure 1.27 The Pizza problem and two student solutions

Mrs. White bought 4 pizzas for her 18 students.

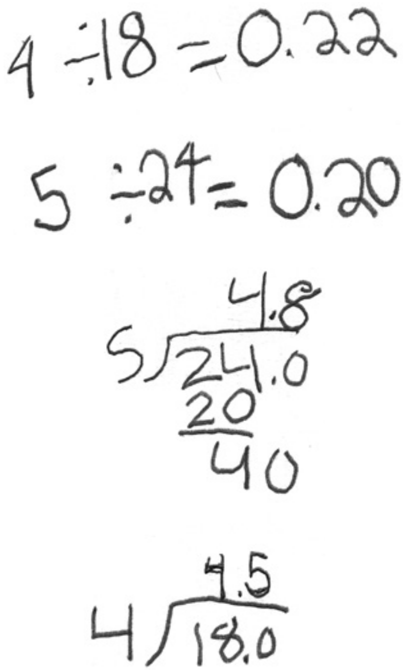
Mr. Green bought 5 pizzas for his 24 students.

All pizzas were the same size.

Who got more pizza—a student in Mrs. White's class or a student in Mr. Green's class?

Drew's response Alyssa's response

Figure 1.27 Continued.



Instructional Link

1. To what degree do you or your instructional materials provide opportunities for students to solve problems:
   - involving ratios as a composed unit?
   - involving ratios as a multiplicative comparison?
   - that require reasoning with rates to solve ratio problems?
2. Based on this analysis, what adjustments do you need to make to your math instruction to assure students are provided opportunities to reason with the different interpretations of ratios?

2
Deepening Understanding of Ratio and Proportion Concepts

Big Ideas

- Instruction should include the intentional use of representations to deepen ratio and proportion concepts.
- Ratio tables, tape diagrams, double number lines, coordinate planes and equations are important representations of proportional situations.
- Students should have the opportunity to solve qualitative reasoning problems as well as problems in which they have to distinguish between proportional and non-proportional situations in order to strengthen their proportional reasoning.

The focus in this chapter is on three research-based strategies to help deepen understanding of ratio and proportion concepts.

1. Using multiple representations
2. Discriminating between examples and nonexamples of proportional situations
3. Interpreting the effects of change on variables without numerical values

Each of these strategies plays a critical role in helping students to examine the multiplicative relationships in ratio and proportion situations and is discussed in detail in this chapter.

Using Multiple Representations

In this section, the role representations play in developing understanding of ratio and proportion concepts is examined. Specifically, how representations (e.g., tape diagrams, double number lines, ratio tables, coordinate planes and equations) can be used to deepen understanding of ratio and proportion concepts. Researchers indicate because each representation characterizes a given concept from a different perspective, the use of multiple representations in mathematics instruction provides a more complete and deeper look at the concept. It is as if each representation examines the concept through a unique and important lens linking the abstract to the concrete (Tripathi, 2008).

Effective use of representations involves explicitly examining ratio and proportion concepts both within individual representations and across representations.

Using Ratio Tables

Read the Healthy Bread Company situation that follows to begin developing these concepts. What are the quantities in this situation? What is the unit rate in this problem situation? How can this rate be used to keep track of the amount of flour to order?

> ***Healthy Bread Company***
>
> Healthy Bread Company uses flour that comes in 25-pound bags. It takes about one pound of flour to make one loaf of bread. The manager has to keep track of the amount of flour to order based on the number of loaves of bread they plan to make.

Study the ratio table (Table 2.1) that represents this situation and consider the following two prompts:

- What are two quantities that make up this proportional situation?
- What is the associated rate connected to each ratio?

Table 2.1 Bags of flour and loaves of bread

Healthy Bread Company Flour and Loaves of Bread	
Number of Bags of Flour	*Number of Loaves of Bread*
1	25
2	50
3	75
4	100
5	125

You probably identified the two quantities in this proportional situation. They are the number of bags of flour and the number of loaves of bread. One feature of this representation is that it connects the number of bags of flour with the number of loaves of bread to form a ratio. There are six ratios displayed in this representation. They are:

$$\frac{25\,loaves\,bread}{1\,bag\ flour}, \frac{50\,loaves\,bread}{2\,bags\ flour}, \frac{75\,loaves\,bread}{3\,bags\ flour}, \frac{100\,loaves\,bread}{4\,bags\ flour}, \frac{125\,loaves\,bread}{5\,bags\ flour}\ldots$$

Although the associated rate appears in the pair of quantities where one of the quantities equals 1, in this case $\frac{25\ loaves\ bread}{1\ bag\ \ flour}$, the rate is a bit less apparent in a table than are the quantities. Table 2.2 shows this situation in a slightly different way, which might make the associated rate more obvious.

Study Table 2.2. How does Table 2.2 differ from Table 2.1?

Table 2.2 Bags of flour, loaves of bread and calculation

Healthy Bread Company Flour and Loaves of Bread		
Number of Bags of Flour	*Calculation Applying Unit Rate 25 loaves of bread per bag of flour*	*Number of Loaves of Bread*
1	1 bag × 25	25
2	2 bags × 25	50
3	3 bags × 25	75
4	4 bags × 25	100
5	5 bags × 25	125

The column titled "Calculation Applying the Unit Rate" simply records the calculation one could perform to determine the number of loaves of bread the Healthy Bread Company could make with a given number of bags of flour. A table that illuminates the functional relationship between quantities by including a calculation column is sometimes called a pattern-sensitive table (Gross, 2017). There are a few features to notice in the calculation column that help shed light on the proportional situation in this case. As you look down the columns, notice that both the number of bags of flour and the number of loaves of bread change, but the "× 25" does not change. This describes the multiplicative relationship between these two quantities that is hidden in Table 2.1. This multiplicative relationship is the rate that is associated with each of the ratios in the table. This rate stated in words is "twenty-five loaves of bread per bag."

Thus, in a proportional relationship there are two quantities that change predictably (e.g., bags of flour and loaves of bread) and are related multiplicatively, and one rate, in this case, × 25 loaves per bag of flour, that does not change. In fact, one reason we know that all these ratios are equivalent is because the same rate connects the number of bags of flour to the correct number of loaves of bread.

Notice that one could generate equivalent ratios of number of bags of flour and loaves of bread by continuing the counting pattern in the Number of Bags of Flour column; 0, 1, 2, 3, 4, 5, 6, 7 . . . and in the Number of Loaves of Bread column; 0, 25, 50, 75, 100, 125, 150, 175 . . . This is an example of iterating in a table. Proportional strategies, however, are often characterized by applying the associated rate to one quantity (e.g., one value in one of the columns) to determine the related quantity that is proportional to it (e.g., the proportional value in the other column). For example, one can use the following calculation to determine the number of loaves of bread Healthy Bread Company could bake with 6 bags of flour:

$$6 \text{ bags flour} \times 25 \text{ loaves per bag} = 150 \text{ loaves of bread}$$

Thus, one can interpret a proportional situation represented in a ratio table by iterating quantities "down the columns" or by using the associated rate that links quantities across a row.

Each representation for ratios and proportions deals with the two changing quantities and the associated rate in a unique way. We already saw in Table 2.1 that a typical ratio table represents the two changing quantities more obviously and clearly than it represents the associated rate.

The Tape Diagram—a Visual Model

A tape diagram, sometimes called a bar model, is another representation commonly used in ratio and proportion instruction. Tape diagrams are referred to in the CCSSM for Ratio and Proportion in grade 6 and are contained in many middle school math programs. In this section, we provide an example of a tape diagram, how it represents the two quantities in a ratio and the associated rate and contexts in which it is most appropriate.

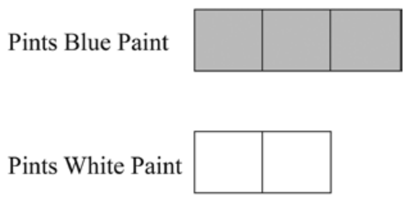
Consider the following situation.

Mixing Paint

The light-blue paint that Maggie is using to paint her bedroom is made by mixing 3 pints of royal blue paint with 2 pints of pure white paint.

Study the tape diagram in Figure 2.1 that represents this situation.

Figure 2.1 Tape diagram that represents Maggie's paint

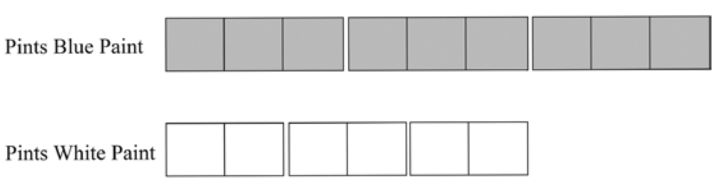


As discussed in the previous section, each visual representation used with ratios and proportions characterizes the quantities and the associated rate in a unique way. Where are these in the tape diagram?

The quantities in the situation are the number of pints of blue paint and the number of pints of white paint. Each pint of blue paint is represented by a gray rectangle, and each pint of white paint is represented by a white rectangle. This means that the entire gray bar stands for 3 pints of blue paint and the entire white bar stands for 2 pints of white paint. Thus, the gray and white bars together represent the ratio $\frac{3 \textit{ pints blue paint}}{2 \textit{ pints white paint}}$. As with the table representation we examined previously, the quantities in the situation are obvious in a tape diagram.

The tape diagram is an example of a representation of a ratio as a composed unit, an idea we examined in Chapter 1. We can see this interpretation at work if we want to extend this $\frac{3 \textit{ pints blue paint}}{2 \textit{ pints white paint}}$ ratio to identify other combinations of blue and white paint that create the same shade of light blue paint that Maggie wants. Table 2.2 illustrates this point.

Figure 2.2 Tape diagram that extends the given 3 : 2 ratio to 9 : 6



As you probably noticed, this tape diagram was created by visually iterating the original tape diagram in Figure 2.2. What other ratios can you see in this tape diagram that represent amounts of blue and white paint that, when mixed, create the same light-blue paint Maggie wants?

You probably saw the ratios $\frac{6\ pints\ blue\ paint}{4\ pints\ white\ paint}$ and $\frac{9\ pints\ blue\ paint}{6\ pints\ white\ paint}$. One could continue iterating the given $\frac{3\ pints\ blue\ paint}{2\ pints\ white\ paint}$ ratio to create an unlimited number of equivalent ratios. So one might see a tape diagram as a representation that supports ratio understanding based on iterating a composed unit to create equivalent ratios.

As with the table representation, the associated rate is less prominent in a tape diagram. How might one use the original tape diagram in Figure 2.1 to determine rate associated with these particular quantities?

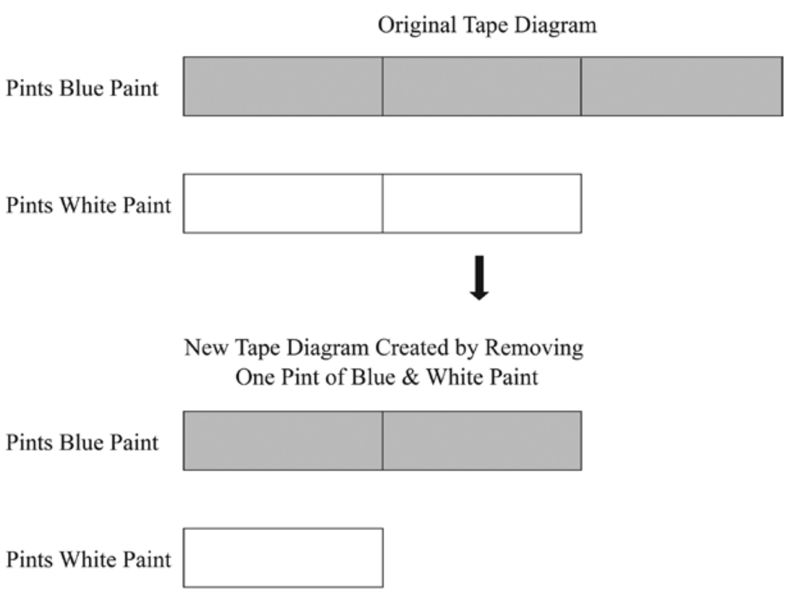
The Associated rate in a Tape Diagram—a Common Misconception

Before considering the associated rate through the context of a tape diagram, it is helpful to recall that the rate we are looking for is the number we can multiply one of the quantities by to determine the other. In the context of this situation, this could be stated as:

- What number do we multiply a particular number of pints of blue paint by to determine the correct number of pints of white paint?
- What number do we multiply a particular number of pints of white paint by to determine the correct number of pints of blue paint?

First, let us consider a common error one might see regarding this question. This involves removing one pint of white paint and one pint of blue paint from the original tape diagram in Figure 2.1. Figure 2.3 models this thinking.

Figure 2.3 Common error—finding the associated rate by removing one pint of blue and one pint of white paint from the tape diagram



The resulting tape diagram suggests that the rate associated with this situation is 2 pints of blue paint for every 1 pint of white paint, or the number of pints of blue paint is 2 times the number of pints of white paint. Why is this an incorrect interpretation?

Although it might be tempting to remove one pint of each paint, this is an example of bringing absolute or additive reasoning to a situation that requires a relative or multiplicative strategy. Subtracting one pint of each color of paint does not create an equivalent ratio. In this case:

$$\frac{2\ \textit{pints blue paint}}{1\ \textit{pint white paint}} \neq \frac{3\ \textit{pints blue paint}}{2\ \textit{pints white paint}}$$

We provide this example to emphasize an important idea we introduced in Chapter 1. That is, ratios and proportions are built on multiplicative processes, and it is common for students to inappropriately bring their additive understanding to proportional situations. This means an important focus of ratio and proportion instruction is helping students develop relative reasoning understandings and differentiate between situations that require absolute reasoning (additive) from situations that require relative reasoning (multiplicative).

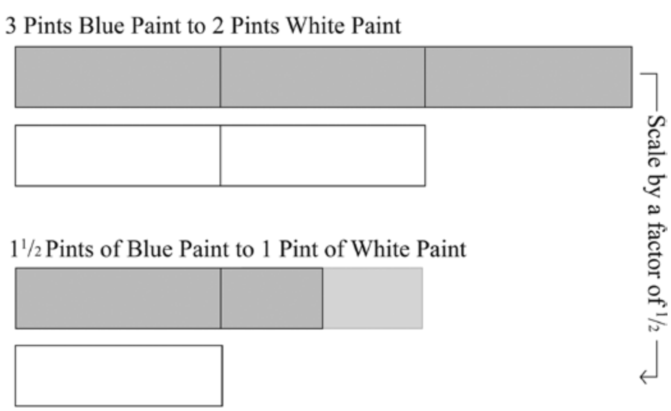
Student solutions involving the inappropriate use of absolute strategies will be discussed in more detail in Chapter 4: "Using the OGAP Progression to Inform Instruction and Student Learning." There you will learn about using the OGAP Ratio and Proportion Progression to analyze strategies students apply to ratio and proportion problems and ways instruction can help move student understanding to more generalizable and efficient proportional solutions and understandings.

Tape Diagram—Using Multiplicative Reasoning to Determine the Associated Rate

One must use relative or multiplicative reasoning to identify the associated rate. We saw in the previous section that application of absolute or additive strategies does not result in maintaining a proportional relationship.

So rather than applying subtraction to the quantities in the situation, one can think about *scaling* them. Figure 2.4 applies scaling to the original tape diagram in Figure 2.1

Figure 2.4 Scaling tape diagram to determine the associated rate



One way to think about scaling is multiplying the two quantities in a ratio by the same factor. This results in a new ratio that is equivalent to the original. The diagram in Figure 2.5 applies this concept to the original tape diagram representing the ratio $\frac{3\ \textit{pints blue paint}}{2\ \textit{pints white paint}}$. Notice that scaling the quantities in the original tape diagram by the factor $\frac{1}{2}$ results in the equivalent ratio $\frac{1\frac{1}{2}\ \textit{pints blue paint}}{1\ \textit{pint white paint}}$. One and one-half pints of blue paint per pint of white paint is the rate associated with this situation and can be communicated in a number of ways. These include:

- There are $1\frac{1}{2}$ pints of blue paint for every 1 pint of white paint in Maggie's paint mixture.
- There are $1\frac{1}{2}$ times more pints of blue paint than white paint in Maggie's paint mixture.
- The ratio of pints of blue paint to white paint in Maggie's mixture is $1\frac{1}{2} : 1$.
- To determine the number of pints of blue paint in Maggie's paint mixture, multiply the number of pints of white paint by $1\frac{1}{2}$.

If you consider this representation of $1\frac{1}{2}$ pints in the tape diagram to be somewhat confusing, you are not alone. The tape diagram does not represent rational number quantities clearly. We discuss this idea later in the chapter.

In this section, we examined features of a tape diagram and how it can be used to find equivalent ratios by iterating a composed unit. We saw that like a ratio table, it *makes the quantities in the proportional situation obvious* and somewhat *hides the associated rate*. Finally, we applied the concept of scaling to a tape diagram to determine the associated rate. The next section takes a close look at another visual representation commonly used in ratio and proportion instruction, namely the double number line.

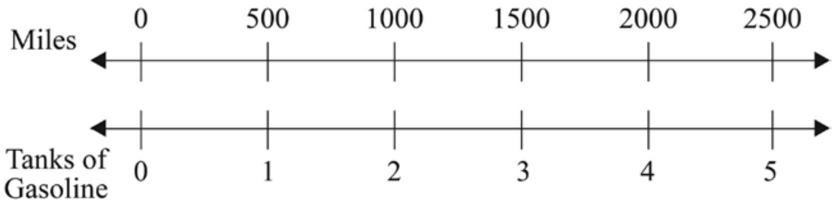
The Double Number Line

The double number line, like a ratio table and the tape diagram, is a representation often used to explore concepts related to ratios and proportions. As you will see, the double number line shares some attributes with the other representations we have examined and differs in some important ways. To begin to examine double number lines, first study the Fran's Trip scenario that follows.

> ***Fran's Trip***
>
> Fran is taking a two-week trip to visit several states across the country. Her car travels about 500 miles per tank of gas.

Study the double number line in Figure 2.5 that represents this situation.

Figure 2.5 Double number line for Fran's Trip



As with the other two representations examined earlier in the chapter, the double number line makes obvious the quantities in the situation. In this case, they are the number of miles and the number of tanks of gas. What is the visual cue on a double number line that a pair of quantities is in a proportional relationship?

The two quantities, 2000 miles and 4 tanks of gas, for example, are in a proportional relationship because they are the same distance from zero. One can use this concept to find other pairs of quantities that are in the same proportion. One way to do this is to multiply or divide each quantity in a given ratio by the same number. For example, one could divide the two proportional quantities, 2000 miles and 4 tanks of gasoline, by 2 to find two other quantities, namely 1000 miles and 2 tanks of gasoline, that are the same distance from zero and equivalent to 2000 miles per 4 tanks of gasoline. One can use this strategy to identify an unlimited number of equivalent ratios.

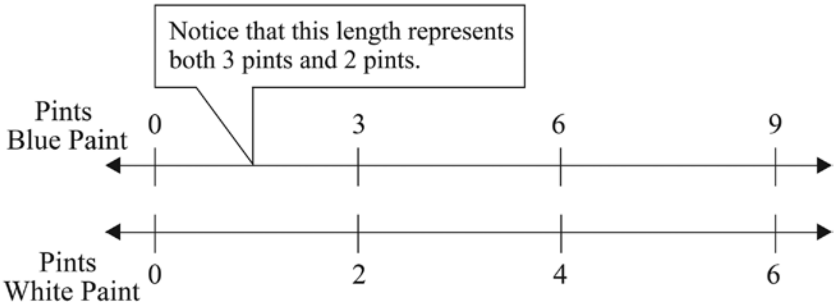
The associated rate is also shown on the double number line. Where is it?

As you probably noticed, the associated rate is the ratio in which one of the quantities is 1. Thus, the ratio $\frac{500\,miles}{1\,tank\ of\ gasoline}$ is the rate. As discussed in the tape diagram section, this rate can be described in a variety of ways. Regardless of how one explains it, this rate $\frac{500\,miles}{1\,tank\ of\ gasoline}$ applies to all the equivalent ratios applicable to this situation. In all cases, the number of miles Fran's car can travel is 500 times greater than the number of tanks of gasoline.

Final Thoughts on Tape Diagrams and Double Number Lines

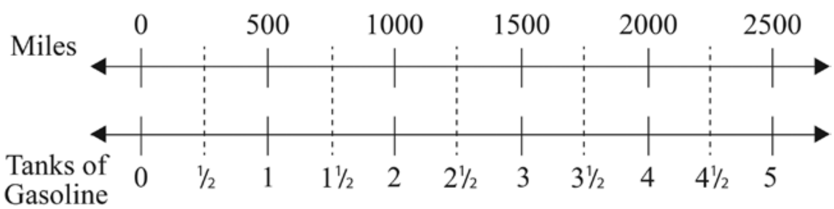
Notice that the quantities we examined in the tape diagram discussion were pints of blue paint and pints of white paint. In the double number line example, the units we used were miles and gallons. This is because each of these representations are more suited to a particular type of quantity relationship. The tape diagram is best used when the quantities are the same—thus pints and pints—while the double number line is more appropriate for representing situations in which the quantities are different. In this example, they were miles and gallons. One reason for this is that if one used a double number line to represent a situation involving the same quantities, one length would represent two different amounts (McCallum et al., 2011). Figure 2.6 provides an example using the ratio $\frac{3\ pints\ blue\ paint}{2\ pints\ white\ paint}$.

Figure 2.6 A double number line representing the same quantities is problematic



Unlike the tape diagram, a double number line more coherently represents ratios in which one or more of the quantities is a rational number. The double number line in Figure 2.7 is an example of one that represents quantities that are rational numbers.

Figure 2.7 Double number line with rational number quantities



Notice that the dotted vertical coordinates on the double number line mark the half tanks of gasoline; $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, etc. Each dotted coordinate also extends to intersect the upper number line that indicates the number of miles associated with each number of tanks of gasoline. Thus, with relatively logical partitioning, we can see that Fran's car travels 250 miles per $\frac{1}{2}$ tank of gasoline, 750 miles per $1\frac{1}{2}$, tanks of gasoline and so on. Again, the double number line allows for a clearer representation of rational number quantities than does the tape diagram.

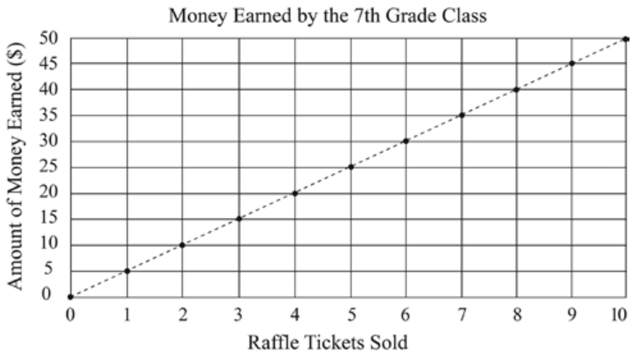
Coordinate Plane

A representation that is vitally important in the teaching and learning of ratios and proportions is the coordinate plane. The coordinate plane in Figure 2.8 represents a proportional relationship between the number of raffle tickets sold and the amount of money the seventh-grade class earned.

Study the coordinate plane in Figure 2.8 and consider the following prompts:

- Where are the quantities represented in a coordinate plane?
- What visual element on the graph represents the associated rate?

Figure 2.8 Coordinate plane



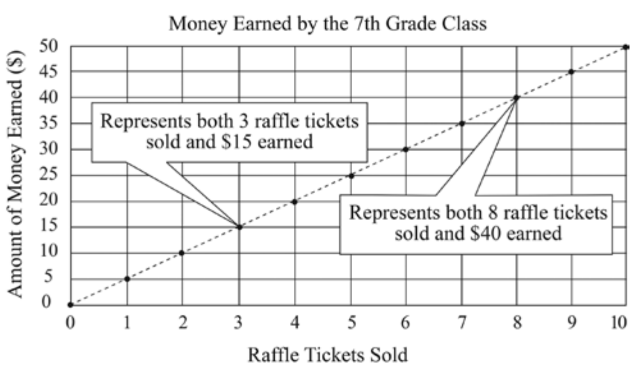
Perhaps the most obvious visual element in a coordinate plane is the diagonal line that connects the various points on the coordinate grid. The slope of this line represents the associated rate that connects each number of tickets sold to the corresponding amount of money earned. Unlike in the ratio table, the tape diagram and the double number line in which the most prominent visual cues were related to the quantities, the most obvious visual element in a coordinate plane communicates the rate. Thus, the coordinate plane is the visual representation most focused on the associated rate in a proportional situation.

The quantities are represented by points on the coordinate plane. Eleven distinct points are plotted on this graph, and each point on the line represents two quantities, one that specifies a number of tickets sold and one that indicates the corresponding amount of money earned. These two related quantities are in the same proportional relationship as any other two related quantities that lie on this diagonal line.

Another important visual feature of the line is that it passes through the origin, i.e., the point (0,0). Two features characterize a proportional relationship on a coordinate plane:

a. a straight line
b. and the straight line passes through the origin

Figure 2.9 In a coordinate plane, two quantities are represented by the plotted points on a line



In the other representations we examined, each quantity was represented by its own number or visual element. In a ratio table, each quantity was represented by a number; in the tape diagram, each quantity was represented by a number of rectangles; and in a double number line, each quantity was represented by a distinct location on one of the number lines. Thus, the fact that two quantities are denoted by one visual feature, one point on a line, makes the coordinate plane different than the other representations.

The Line Representing a Proportional Relationship Passes Through the Origin

Notice the line in Figure 2.9 is a straight line and that it passes through the point identified with coordinates (0,0)—the origin. Proportional relationships graphed on a coordinate plane share these attributes. The straight line is a function of both quantities changing at the same rate. The line passes through the origin because a proportional situation is one in which both quantities begin at zero. In this example, zero tickets sold results in zero money earned. In the context used in Figure 2.7, zero miles traveled uses zero tanks of gas. As we have seen throughout this chapter, the relationship between the two quantities in a proportional relationship is based solely on multiplicative processes.

Some contexts will produce a straight line on a coordinate plane that does not pass through the origin. Let's change the raffle ticket context slightly to illustrate this type of relationship. Imagine the seventh-grade class has $10.00 in its account prior to selling the raffle tickets, and they were keeping track of the total amount of money in this account. Table 2.3 records this situation.

Table 2.3 Seventh-grade raffle—class purchases a roll of tickets

Tickets Sold	*Calculation*	*Total Amount of Money in Account*
0	$(0 \times 5) + 10$	$10.00
1	$(1 \times 5) + 10$	$15.00
2	$(2 \times 5) + 10$	$20.00
3	$(3 \times 5) + 10$	$25.00
4	$(4 \times 5) + 10$	$30.00
5	$(5 \times 5) + 10$	$35.00
6	$(6 \times 5) + 10$	$40.00

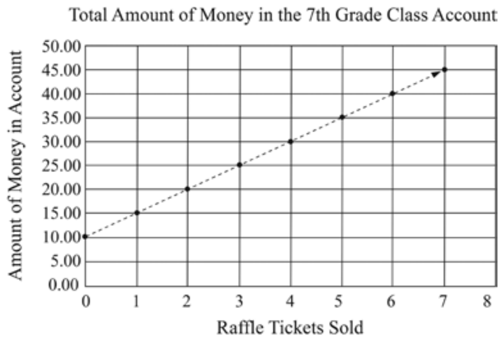
Figure 2.10 shows a coordinate plane representation of this situation.

Notice that in this context, the quantities, tickets sold and total amount of money in the account are related both multiplicatively and additively. In other words, one has to apply both multiplication and addition to calculate the amount of money in the seventh-grade account given the number of tickets sold. Because of this, this situation results in a straight line that does not pass through the origin and is an example of one type of non-proportional situation.

In a Coordinate Plane, the Slope of the Line Communicates the Rate

The amount of *steepness* of the line, called the *slope*, is the visual element that defines the associated rate. The slope of the line passing through the origin on a coordinate plane that represents a proportional situation can be interpreted in a number of ways. One way to

Figure 2.10 Coordinate plane with a straight line that does not pass through the origin



conceptualize the slope is to consider the amount the vertical coordinate *rises* for every one unit of increase on the horizontal axis. In this situation, the coordinate rises $5.00 for every one additional ticket sold. This rate of $5.00 for each ticket sold is true for every pair of quantities on this graph. Each time the class sells one more ticket, the line rises $5.00.

Another way to determine the slope of the line that represents a proportional situation is to divide each quantity on the vertical axis, in this case a certain amount of money earned, by the related number of tickets sold. See Figure 2.11.

Figure 2.11 Using division to determine the slope of a line in a proportional situation

$5.00 ÷ 1 ticket sold = $5.00 per ticket

$10.00 ÷ 2 tickets sold = $5.00 per ticket

$15.00 ÷ 3 tickets sold = $5.00 per ticket

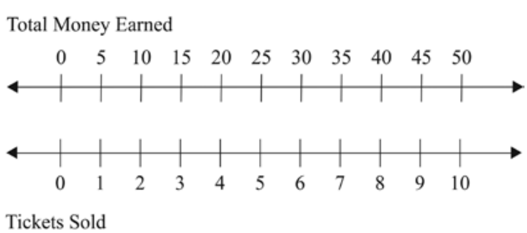
$50.00 ÷ 10 tickets sold = $5.00 per ticket

Notice that the quotient for any pair of related quantities will be $5.00 per ticket, which is both the slope of the line and the associated rate in this proportional situation.

Connecting the Double Number Line and the Coordinate Plane

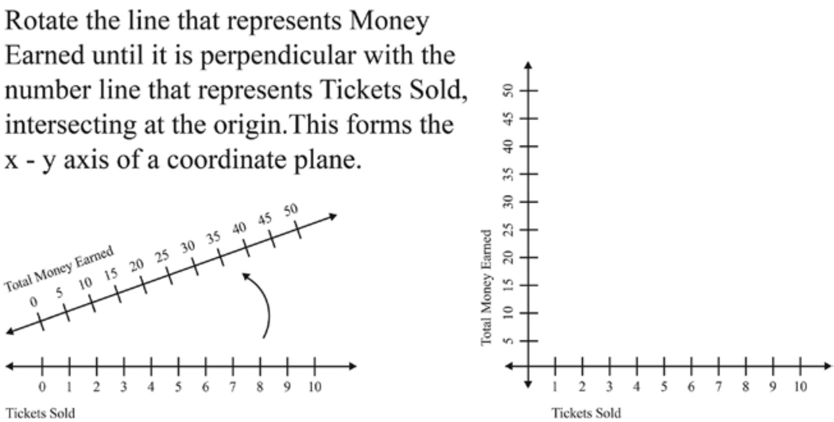
Figure 2.12 is a double number that represents the same situation as the coordinate plane in Figure 2.8. Compare this double number line representation to the coordinate plane in Figure 2.8.

Figure 2.12 A double number line representing the number of raffle tickets sold and the total money earned



These two representations communicate the same 10 equivalent ratios of total money earned to tickets sold. In addition, the coordinate plane and the double number line both use the number line as a foundational aspect of the representation. In fact, one can imagine separating and rearranging the two number lines in the double number line to create the coordinate plane. Figure 2.12 illustrates this idea.

Figure 2.13 Relationship between a double number line and a coordinate plane



In this way, one can think of a coordinate plane and a double number line as related in certain ways. Certainly, they are not the same, and one might want to alter the scale on the coordinate plane to more accurately reflect the situation. In addition, the pairs of related quantities that were obvious in the double number line are missing in the newly constructed coordinate plane. We saw earlier that each pair of related quantities in a coordinate plane is represented with a point. Thus, the coordinate plane needs points to communicate the same information meaningfully. Despite the obvious differences, these two visual representations are related. Teachers can help build student understanding of a coordinate plane from students' prior experiences with the double number line while deepening understanding of quantities and rates reflected in each representation.

The Equation That Represents Proportional Relationships

The equation used to communicate proportional relationships is an important representation, as it generalizes many of the ideas presented in this chapter and links proportions to expressions and equations, both foundational algebra concepts. This equation is also commonly linked to both ratio tables and the coordinate plane.

The equation for a proportional relationship is $y = kx$. One can describe this in a number of ways. Two examples are listed here.

- y is proportional to x, or y varies proportionally with x.
- To determine the value of y, one multiplies x by k.

Note: The CCSSM uses the equation $y = cx$. One can use k or c or other letters to indicate a *constant of proportionality*.

Perhaps the terms of this equation—*y*, *k* and *x*—are more clearly understood through examples. Figure 2.14 provides some examples of where these terms are found in a ratio table and in a coordinate plane.

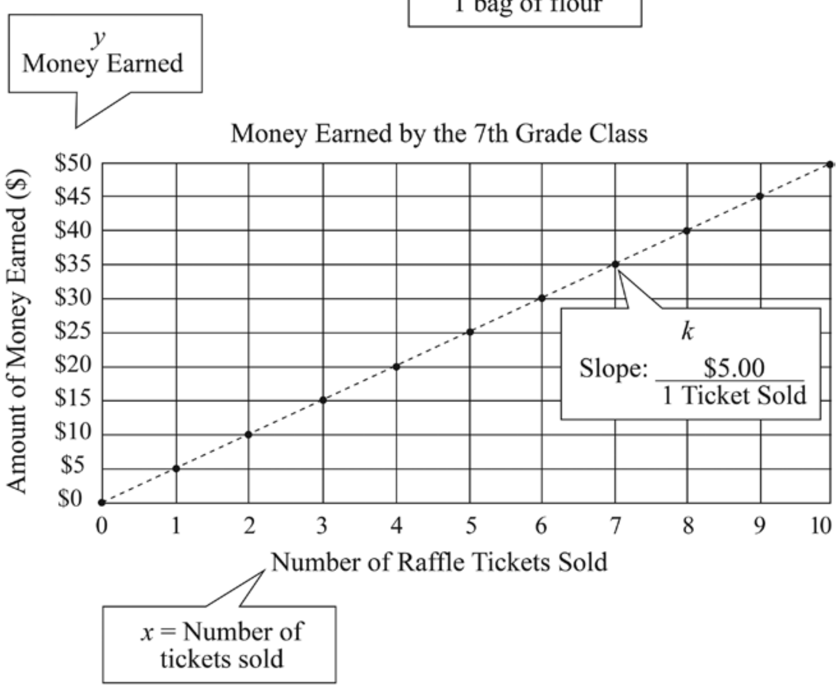
Figure 2.14 *y*, *k* and *x* in a ratio table and a coordinate plane

$x =$ Number of Bags of Flour

$y =$ Number of Loaves of Bread

Healthy Bread Company Flour and Loaves of Bread		
Number of Bags of Flour	Calculation Applying Unit Rate 25 Loaves of Bread per Bag of Flour	Number of Loaves of Bread
1	1 bag x 25	25
2	2 bags x 25	50
3	3 bags x 25	75
4	4 bags x 25	100
5	5 bags x 25	125

$k = 25$ loaves per 1 bag of flour



Each of these terms of the equation for a proportion y, k and x is referred to in different ways, resulting in a variety of different names that are impacted by the representation or the specific context. Table 2.4 provides a few examples of the different ways that each of these terms are named, represented or referred to. This is certainly not an exhaustive list, but it provides a picture of the number of different ways these terms are interpreted.

Table 2.4 Some ways y, k and x can be represented in a coordinate plane and a ratio table

Common Names for k and v	
Terms	*Names/Ways It Is Referred To/Represented*
y	• Quantity associated with the vertical axis on a line graph • Quantity in the rightmost column in a ratio table • Quantity in the bottom row of a horizontal table • Dependent variable • Output
k	• Unit rate • Slope of the line on a line graph • Constant of proportionality • Constant rate • Rate of change
$\boldsymbol{x}$	• Quantity associated with the horizontal axis on a line graph • Quantity in the leftmost column in a ratio table • Quantity in the top row of a horizontal table • Independent variable • Input

From this table, we begin to see the complexity of these terms, as each is interpreted differently depending on the context. This is one reason students need to have experiences engaging in ratio and proportion concepts in different contexts and through different representations. Students need plenty of experience interpreting and reinterpreting these foundational concepts related to ratios and proportions. In particular, k can be interpreted in a multitude of ways and deserves a bit more discussion.

The Constant of Proportionality k

Throughout this chapter, we have used the term *associated rate* when describing the rate that links two quantities in a proportion. This rate is most commonly called the *constant of proportionality*. As we briefly described in the previous section, k can be interpreted as a unit rate and as the slope of the line in a coordinate plane. Yet these are two of many ways k is interpreted in proportional contexts one engages in regularly. Consider each context that follows. What do we call the k in each one?

Interpret k *in Each Context*

- Reading a highway map
- Double-checking a restaurant receipt
- A highways sign indicating a steep hill

- Comparing the prices of different sizes of boxes of cereal
- Making sense of a paycheck from a summer job

Although each of these situations requires one to interpret a constant of proportionality, the context of the situation determines the vocabulary one uses and how one understands the situation.

Table 2.5 provides some possible ways the constant of proportionality is interpreted in each situation.

Table 2.5 Various interpretations of the constant of proportionality

Context	*Interpretation of the Constant of Proportionality*
A highway map	Scale—The constant of proportionality might be written as: 1 inch = 10 miles.
A restaurant receipt	Tax—The constant of proportionality might be written as: 5% meal tax.
A highway sign indicating a steep hill	Grade—This term is used to explain the steepness of a portion of highway and written as a percent. Thus a *6%* grade means the highway climbs 6 units for every 100 units traveled horizontally.
Comparing prices of different sizes of boxes of cereal	Unit Price—The constant of proportionality might be written as: $0.22 per ounce.
Paycheck from a summer job	Hourly Wage—The constant of proportionality might be $10.50 per hour.

These examples suggest the complexity of these terms, as each is interpreted differently depending on the context. The next section addresses instructional implications for the concepts discussed in this chapter.

Instructional Considerations

This chapter has examined five different representations associated with ratios and proportions: ratio tables, tape diagrams, double number lines, coordinate planes and equations. Through discussion and examples, we saw ways in which these representations are similar and ways in which they differ. As explained at the beginning of the chapter, learning is best supported through the intentional and thoughtful use of multiple representations that differ, in this case in the ways they interpret and represent ratios and proportions. The use of multiple representations provides students important opportunities to reinterpret and transfer prior learning to related but conceptually different contexts. Teacher understanding of the various visual representations associated with ratio and proportions supports intentional use of representations in ratio and proportion instruction.

Although there are countless ways to integrate the use of multiple representations in ratio and proportion instruction, instruction should provide students ongoing opportunities to discuss, create, compare and contrast representations and contexts. Rich instructional opportunities can be created around prompts such as:

- Compare the ratio table, the double number line and the coordinate plane we created. Where is the constant of proportionality in each? Where in the coordinate plane are the quantities in the columns in the table?
- Create a ratio table from the tape diagram you created.
- Create a ratio table from the proportional equation $y = 4x$.
- Create a double number line from a coordinate plane.
- Represent a proportional situation using two different representations. Which representation do you find more effective? Why?

The next two sections of this chapter build from Chapter 1 and the previous sections in this chapter on representations, with an emphasis on two strategies that help build sense making in ratio and proportion situations (Cramer, 2017) and thus strengthen ratio and proportion understanding and fluency. These are listed and then more fully discussed throughout this chapter.

Sense-making strategies (Cramer, 2017, p. 27):

1. Discriminating between examples and nonexamples of proportional situations.
2. Interpreting the effects of changes on variables without numerical values.

Distinguishing Between Proportional and Non-Proportional Situations

As has been described in Chapter 1 and again in the examples in this chapter, evidence of a deep understanding of proportionality is the ability to recognize the multiplicative relationship in proportional situations (Cramer, 2017). That is, "Students should be able to analyze the problem task and conclude whether or not it is proportional and to justify any claims to proportionality by making connections to the multiplicative relationships" (Cramer, 2017, p. 27). Researchers suggest that one way to help students analyze situations is to provide students with tasks that require discriminating between proportional and non-proportional situations so they do not fall into a pattern of making that decision based on repeated experience with just proportional problems (Cramer, 2017; Cramer, Post, & Currier, 1993).

Solve the problem in Figure 2.15 and then study the student solutions. Is this a proportional or non-proportional situation? What is the evidence? Which student(s) interpreted the relationship appropriately? What is the evidence?

Figure 2.15 Kim and Bob were running equally fast around a track. Kim started first. When she had run 9 laps, Bob had run 3 laps. When Bob had run 15 laps, how many laps had Kim run?

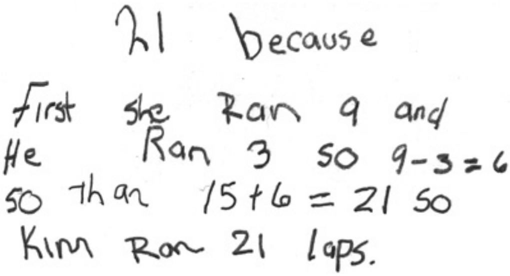
Kim's response

$$\frac{\text{Kim}}{\text{Bob}} = \frac{9\text{ laps}}{3\text{ laps}} \underset{\times 5}{\overset{\times 5}{=}} \frac{45\text{ laps}}{15\text{ laps}}$$

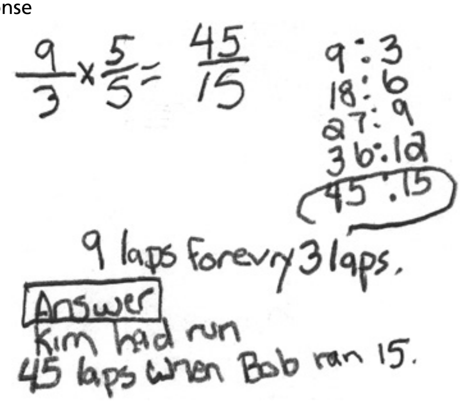
Kim had run 45 laps.

Figure 2.15 Continued.

Tabor's response



Patty's response



What is interesting about these solutions is the confidence with which Kim's and Patty's responses show a multiplicative relationship even though the situation is additive, as shown in Tabor's response. Any teacher would be glad to see the strategies that Kim and Patty used if the situation was proportional. The student work in Figure 2.15 was a part of an OGAP study in which 82 sixth-grade students were given the track problem. Of these students, 48% (n = 39) solved this problem as a proportion, 12% (n = 10) did not attempt the problem and 40% (n = 33) solved the problem reflecting the additive situation. These data reinforce the recommendation by researchers that it is important to provide students with problem situations that are both proportional and non-proportional during unit(s) of instruction on ratio and proportions to develop an analytical approach to determine the relationships in the problem situations.

Other ways to engage students in problems that demand they discriminate between proportion and non-proportion involve studying relationships in ratio tables and on coordinate graphs or solving a problem that requires students to recognize a non-proportional situation as in Thrill Amusement Park in Figure 2.16. Solve the Thrill Amusement Park problem in Figure 2.16. Is this a proportional or non-proportional situation? What is the evidence?

Figure 2.16 Sample problems designed to have students discriminate between proportional and non-proportional situations

Thrill Amusement Park

Thrill Amusement Park charges a $10 admission fee plus $1 per ride. Maria determined that the cost of going on three rides at Thrill Amusement Park is $33. Her reasoning is shown below. Is Maria's reasoning correct. Explain your answer.

$$\frac{\textbf{11 dollars}}{\textbf{1 ride}} = \frac{\textbf{33 dollars}}{\textbf{3 rides}}$$

The Thrill Amusement Park problem is a non-proportional situation. There is an initial entry fee, and then each ride costs $1. Study Eli's and Elizabeth's solutions in Figure 2.17. What is the evidence that they understood or misunderstood the problem situation?

Figure 2.17 Eli's and Elizabeth's responses to the Thrill Amusement Park

Eli's response

$10 + 1 = 11$

Yes

$$\frac{11\text{ dollars}}{1\text{ ride}} \xrightarrow{\times 3} \frac{33\text{ dollars}}{3\text{ rides}}$$

They both Are mulipled by 3

Elizabeth's response

No Maria's reasoning is not correct. I Know this because the admissions fee for getting into the park is $10. Not $10 per ride. It is $1 per ride after the admissions fee

The cost for going on 3 rides is

$$\underset{\text{admission fee}}{10} + \underset{\text{\# of rides}}{\overset{\$\text{ per ride}}{1}x} = \underset{\text{the price}}{y} \qquad 10 + 1(3) = \$13$$

The evidence in Eli's solution indicated that he does not understand the problem situation. In contrast, Elizabeth's solution provides evidence of recognizing which aspect of the problem involved the rate ($1 per ride) and how to apply it in a non-proportional situation like this one in which there was an initial fee.

Assuring that students encounter problems that involve both proportional and non-proportional situations while engaging in instruction on proportions has the potential to help students think first about the situation—asking first, "Is this a proportional situation or is there some other relationship(s) in this problem?" instead of assuming that because it is during a proportion unit, the situation must be proportional.

The focus in the next section involves qualitative reasoning problems. In qualitative reasoning problems no numerical values are assigned to the quantities in the problems.

Interpreting the Effects of Changes on Variables Without Numerical Values

Cramer (2017) indicates that Lamon (2007) "suggests that relating quantities that are not quantitated is an important kind of reasoning" (p. 27) referred to as qualitative reasoning. In these problems, students are asked to make sense of the impact of changing quantities on a resulting rate. For example, when considering problems involving distance, rate and time, if two runners compete in the same race and one runner takes less time, is that runner's average speed (rate) faster, slower or the same as that of the other runner? Engaging students in problem situations that involve sense making in this way puts the focus on the interactions of the quantities not on numerical solutions. To begin to understand the structure of these problems, solve the problems in Figure 2.18. How are these problems alike, and how are they different? In what ways do these problems help focus students on the relationships between the quantities in the problems?

Figure 2.18 Sample non-numerical problems (qualitative)

Problem 1

Nick ran twice as many laps around the school track today as he ran yesterday. It took Nick twice as much time to run today's laps. Which statement below best describes Nick's running speed? Explain your choice.

a. Nick ran faster today.

b. Nick ran slower today.

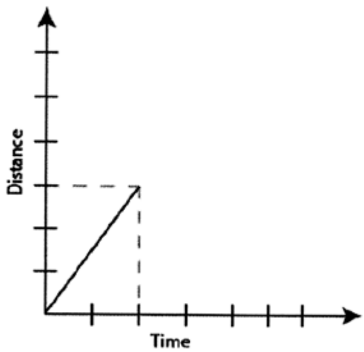
c. Nick ran the same speed today and yesterday.

d. There is not enough information to compare today's speed with yesterday's speed.

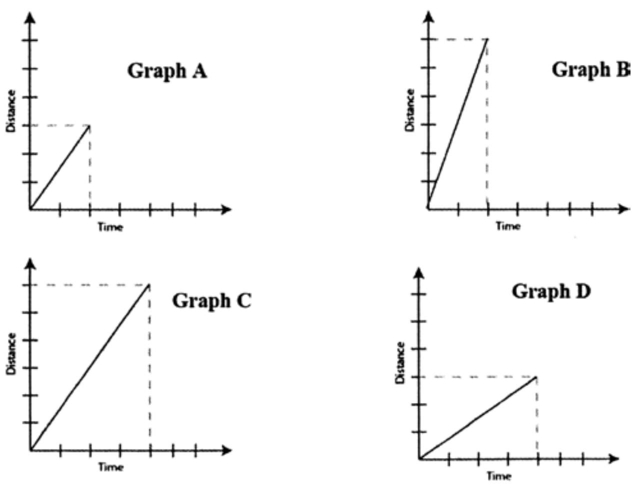
Figure 2.18 Continued.

Problem 2

Joanne and Maria traveled to school today on their bicycles. This graph describes Joanne's speed.



Maria traveled twice as far as Joanne and traveled at twice the speed of Joanne. Which graph describes Maria's speed? All the graphs below have the same scale as Joanne's graph above. Explain your choice.



Problem 3

Tucker Forest and Sherwood Forest have the same number of trees.

Tucker Forest has a smaller area than Sherwood Forest.

Which forest has more trees per acre? Explain your answer.

The one similarity is that each of these problems lacks any numbers, focusing rather on the relationships between the quantities in the problems. While Problems

1 and 2 are situated in average speed context, Problem 1 is described in text and Problem 2 is represented on coordinate planes. Problem 3 involves the context of density, which is often unfamiliar to students. Notably, each of these problems demands that students interpret and demonstrate understanding of the multiplicative relationships between the quantities in the problem situation without numbers influencing their thinking.

Now study Liam's, Logan's and Isabelle's solutions in Figure 2.19. What understanding or misunderstanding is evidenced in each solution? What questions might you ask each of these students to deepen their understanding of the situation?

Figure 2.19 Liam's, Logan's and Isabelle's responses

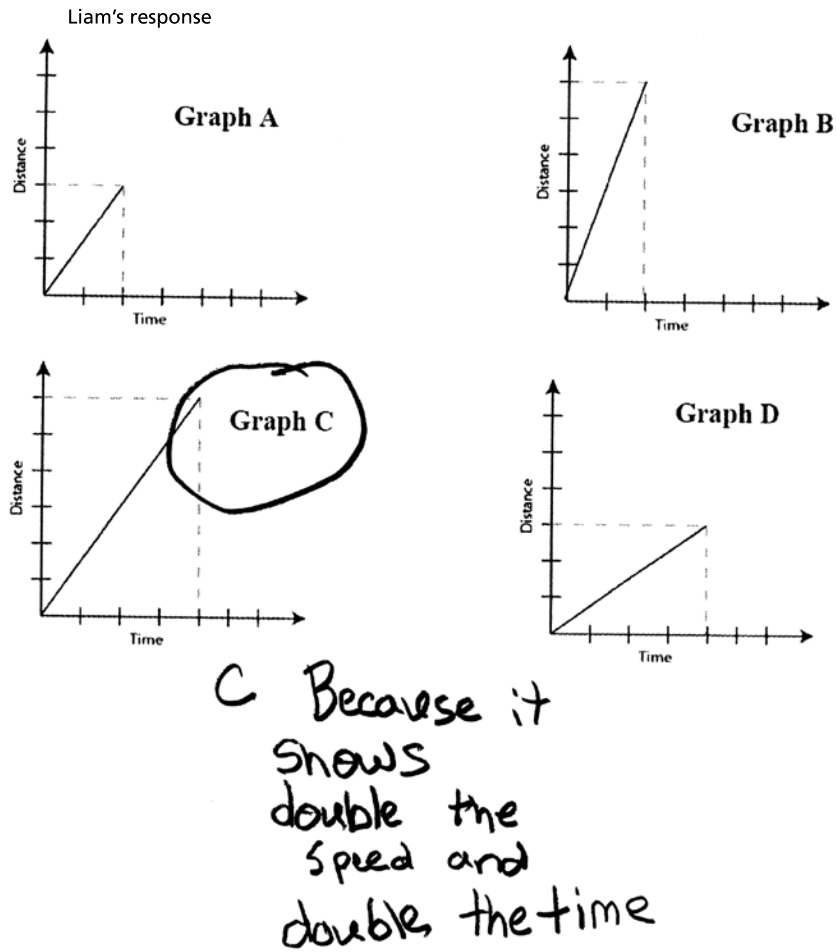
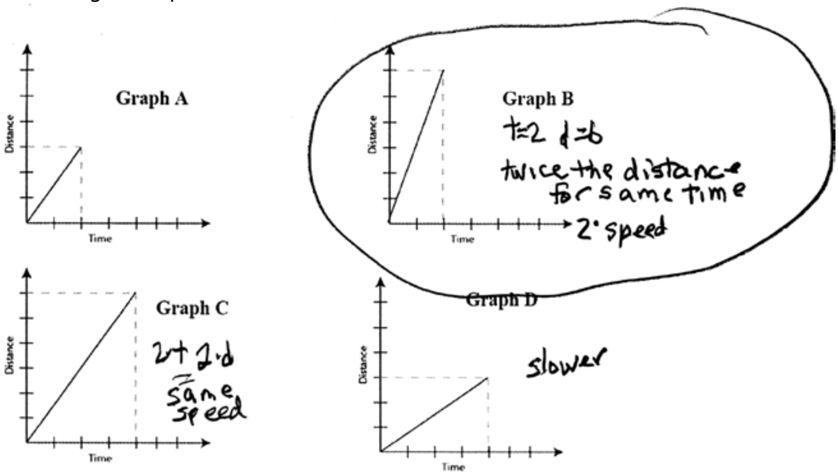
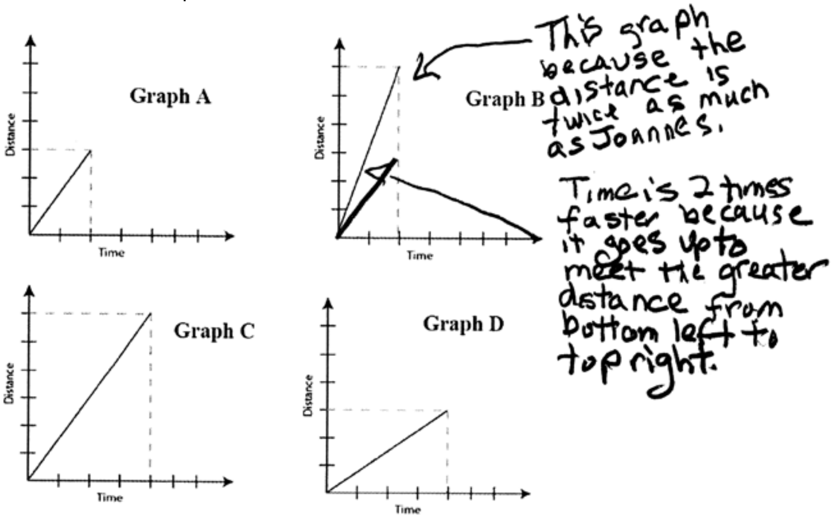


Figure 2.19 Continued.

Logan's response



Isabelle's response



In qualitative problems, the important evidence comes from student explanations that support their choice. Liam incorrectly selected Graph C, interpreting twice as far and twice as fast as meaning both time and distance are doubled. This is important evidence, as it shows that in terms of representing the situation graphically, Liam needs support in understanding the relationship between distance and time when considering speed.

Logan correctly selected Graph B, indicating that twice the distance and twice the speed means Maria will cover twice the distance in the same amount of time as Joanne. Note, however, that Logan inserted numbers on the graph to solve the problem. Also note that Logan provides additional evidence of understanding by providing an interpretation of Graph C as twice the distance and time, so Maria was traveling at the same speed as Joanna; and Graph D as Maria biking slower than Joanne. Isabelle also correctly chose Graph B. Of the three solutions, Isabelle's solution was the only solution to consider the slope of the line as also inferring a faster speed, although she never directly indicated she was considering slope, but her solution is certainly an opportunity to discuss the meaning of the steepness of the line on a graph with all the students.

In non-numerical problems, some students directly interpret the relationships in the problem (e.g., Isabelle in Figure 2.19 and Matt in Figure 2.21), while others may impose numerical values, as was evidenced in Logan's solution in Figure 2.19 and in Kaitlyn's response in Figure 2.20. Other times, students may create a visual model to help interpret the situation as found in Noah's solution in Figure 2.21.

Figure 2.20 Kaitlyn's response

Nick ran twice as many laps around the school track today as he ran yesterday. It took Nick twice as much time to run today's laps. Which statement below best describes Nick's running speed? Explain your choice.

a. Nick ran faster today.

b. Nick ran slower today.

c. Nick ran the same speed today and yesterday.

d. There is not enough information to compare today's speed with yesterday's speed.

Kaitlyn's response

I think C because she has
to run as much as he did yesterday.
So let's say he ran 15 laps yesterday,
It would be double the laps in
double the time

15 X 2 = 30
10 X 2 = 20

30 laps in 20 min

Figure 2.21 Matt's and Noah's responses to the Tucker Forest problem

Tucker Forest and Sherwood Forest have the same number of trees.

Tucker Forest has a smaller area than Sherwood Forest.

Which forest has more trees per acre? Explain your answer.

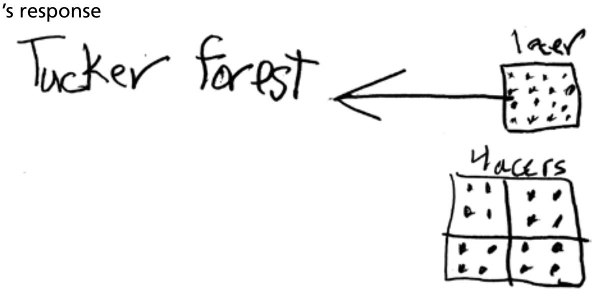
Figure 2.21 Continued.

Matt's response

tucker foresd because...
Sherwood forest has more area, so the trees will be spread out, leaveing less trees close together for acres.

Tucker foresd has less arae So all the Trees are close to geather. more per acre

Noah's response



One way to understand the complexity and importance of qualitative reasoning is to consider the range of outcomes given changes in variables. On the surface, these problems may seem easy, but when we consider the relationship between distance, rate (speed) and time, there are nine potential outcomes. Study Table 2.6.

Table 2.6 Relationships among distance, rate and time

		Distance		
		No change in distance	*Increase in distance*	*Decrease in distance*
Time	No change in time	Speed is the same	Speed Increases	Speed decreases
	Increased Time	Speed decreases	Can't tell unless you know distance and time	Speed decreases
	Decreased Time	Speed increases	Speed Increases	Can't tell unless you know distance and time

While the data in the table specifically focuses on the relationships between distance, speed and time, the concept can be used in a range of rate situations such as consumption (e.g., miles per gallon), scaling (e.g., miles per inch), package size (e.g., ounces per box), mixtures (e.g., paint), conversions (U.S. dollars per Canadian dollar) and density (e.g., population per square mile).

Go to Looking Back question 6. This question provides you an opportunity to write some qualitative problems using these different contexts. With practice, you could create and use these on a regular basis and as you introduce a new rate.

Inversely Proportional Relationships

As we have seen, proportional situations are characterized by two quantities changing together multiplicatively. For example, multiplying each quantity in a ratio by 4, or dividing each quantity by 2, results in new ratios that are proportional to the original. In this section, we briefly examine *inversely proportional relationships*. As you will see, these relationships involve multiplication and division but are not proportional.

An *inversely proportional relationship* is one in which one quantity in a ratio is multiplied by a positive number and the other quantity is divided by the same positive number (or multiplied by the reciprocal of the number). This results in one quantity increasing and the other decreasing. Table 2.7 illustrates an inversely proportional situation. In this case, we examine the number of painters it takes to paint a house.

Table 2.7 Number of painters it takes to paint a house

Hours to Paint a House	96	48	24	16	12
Number of Painters	1	2	4	6	8

It is logical that as the number of painters increase, the number of hours it takes to paint a house decrease. More specifically, these changes in the number of hours and the number of painters are related multiplicatively. This is shown in Table 2.8.

Table 2.8 The inversely proportional relationship between the number of hours and the number of painters

$\div 4$ or $\times \frac{1}{4}$

Hours to Paint a House	96	48	24	16	12
Number of Painters	1	2	4	6	8

$\times 4$

Notice when the number of painters increase by a factor of 4, the number of hours decrease by a factor of 4. Comparing any two related quantities (number of hours and number of painters) involves both multiplication and division by the same factor. This is an important feature of all inversely proportional relationships.

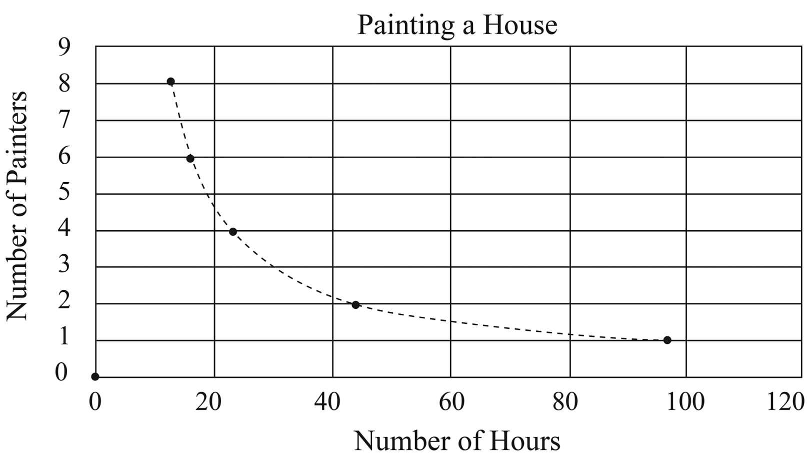
Another feature of inversely proportional relationships is that the product of the number of hours and the related number of painters is the same for all the related quantities. In this case:

$$
\begin{aligned}
&96\,\text{hours} \times 1\,\text{painter} = 96\\
&48\,\text{hours} \times 2\,\text{painters} = 96\\
&24\,\text{hours} \times 4\,\text{painters} = 96\\
&16\,\text{hours} \times 6\,\text{painters} = 96\\
&12\,\text{hours} \times 8\,\text{painters} = 96
\end{aligned}
$$

The product, 96, is the number of *painter hours* it takes to paint a house. Thus, no matter the number of painters, in this situation, it takes 96 *painter hours* to paint a house.

It is also important to note that the graph of an inversely proportional relationship is not linear, and the line does not pass through the origin. Figure 2.22 is a graph of the house painting situation we examined earlier.

Figure 2.22 Graph representing the inverse proportional relationship between number of painters and hours to paint a house



Student Solutions to Inversely Proportional Problems

As you have seen, inversely proportional relationships differ from proportional relationships in significant ways. Regardless, it is common for students to attempt to solve inversely proportional situations by using proportional strategies. Study the problem and Richard's and Emma's solutions to the problem in Figure 2.23. What do you notice about the relationships among the quantities in the problem? How did Richard interpret the relationships among the quantities? How did Emma interpret the relationships between the quantities?

Figure 2.23 Richard's solution

Tom travels 40 miles per hour for the first four hours of a trip. How long will it take Tom to travel the same distance if he is driving at 80 miles per hour?

Richard's response

$$\div 10 \; \frac{40\text{mph}}{4\text{hrs}} = \frac{80\text{mph}}{8\text{hrs}} \; \div 10$$

Emma's response

I know that Tom is traveling twice as fast so ~~it~~ must take him half the time to travel the same distance – 2hrs

Richard incorrectly solved this problem using a proportional strategy. The evidence in Emma's solution suggests that she understood that a faster speed results in less time. She observed that Tom was traveling twice as fast, so it should take him half the time to travel the same distance.

Other common inversely proportional situations are found in Table 2.9.

Table 2.9 Examples of situations in which quantities are inversely proportional

Situation	*Relationship*
Speed and time	As speed increases, the time to travel the same distance decreases, and if speed decreases, the time to travel the same distance increases.
Work problems in which number of people completing a job increases	As the number of people to complete a job increases, the time it takes to complete the job decreases.
Intensity of light and distance from light	The closer you are to the light, the brighter the light. As your distance from the light increases, the brightness of the light decreases.

Instructional Considerations

An understanding of inversely proportional relationships is not specifically stated in the K–8 CCSSM standards. However, students in grade 8 are expected to quantitatively describe the functional relationship (linear and non-linear situations) between two quantities by analyzing a graph. They should also be able to sketch a graph of a linear and non-linear function (CCSSM, 2012). As stated earlier in the chapter, students should be provided regular and ongoing opportunities to analyze, reason about and represent a variety of both proportional and non-proportional situations. This variety of situations provides students valuable opportunities to differentiate between different situations and contexts and to apply and transfer their developing understanding of proportional and non-proportional relationships,

CCSSM Expectations for Ratio and Proportion Representations in Grades 6 and 7

Student understanding and facility with representations are central to many of the CCSSM expectations for ratio and proportions at grades 6 and 7. Students in grade 6 use rate and ratio reasoning to solve problems involving ratio tables, tape diagrams, double number lines and equations. In addition, sixth-grade students are expected to use ratio tables to compare ratios. The use of representations continues in grade 7 with particular focus on ratio tables, equations and the coordinate plane, representations that are foundational for algebra. Students in seventh grade are expected to use ratio tables and the coordinate plane to decide whether two quantities are in a proportional relationship. Students also identify the constant of proportionality in a variety of situations including equations, ratio tables and graphs. Finally, seventh-grade students are expected to use equations to represent proportional equations and to explain the meaning of a point on a graph of a proportional relationship (CCSSM/NGA, 2010).

Representations in grades 6 and 7 are an integral part of student understanding of proportionality and their abilities to solve problems flexibly and accurately. Thus, effective ratio and proportion instruction should include intentional and systematic use of representations.

Upcoming Chapters Return to the Concepts From Chapters 1 and 2

Chapters 1 and 2 provided a close investigation of concepts and ideas that underpin understanding of ratios and proportions. The focus in Chapters 3 and 4 turns to student learning and understanding the OGAP Ratio and Proportion Framework and Progression. In these chapters, you will see multiple examples of student solutions, many of which contain elements we discussed in these first two chapters, such as iterating a composed unit and use of ratio tables.

Chapters 5, 6 and 7 focus on the impact of three aspects of ratio and proportion problem situations and their influence on student solutions to ratio and proportion problems: problem structures (e.g., magnitude of numbers, multiplicative relationship between quantities), problem types (e.g., missing value, rate comparisons) and contexts (e.g., similarity, constant speed). Developing student understanding and fluency with the ratio and proportion concepts demands an understanding of the mathematics discussed in Chapters 1 and 2 as well as the mathematics education research on how students develop understanding of ratio and proportions, common errors that students may make and preconceptions or misconceptions that may interfere with learning new concepts or solving related problems discussed in other chapters throughout the book.

Chapter Summary

- Engaging students in the use of a variety of representations focusing on how the quantities and the multiplicative relationship between the quantities are represented (in ratio tables, tape diagrams, double number lines, coordinate planes or equations) can deepen understanding of the multiplicative nature of proportional situations.

- Engaging students in qualitative reasoning problems and in problems that require discriminating between non-proportional and proportional situations has the potential to deepen student understanding of proportional relationships as well as to have students focus on the problem situation and not assume that in a proportion unit, all the problems will be proportional.

Looking Back

1. Consider the following situation and answer the questions that follow.
   4 packages of bagels cost $13.00

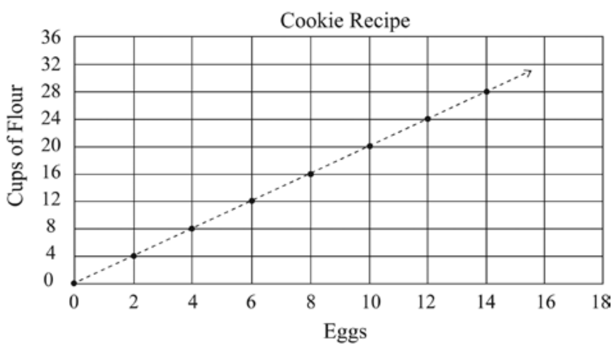
Table 2.10 Cost of bagels

Cost of Bagels	
Number of Packs of Bagels	*Cost*
0	
2	
	$13.00
	$16.25
10	

   a. Complete the ratio table (Table 2.7) for this situation by determining the missing quantities in the table.
   b. What are the two quantities in this situation?
   c. What is the associated rate connected to each ratio? How did you determine this rate?
   d. Rewrite this table to make the associated rate obvious.

2. Study the coordinate grid in Figure 2.24.

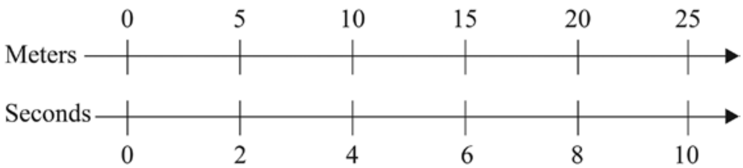
Figure 2.24 Graph of flour and eggs in cookie recipe



a. Construct either a tape diagram or a double number line to represent this information. Which visual model makes the most sense for this situation?
b. Use the visual model you constructed to answer the questions that follow.
   - How many eggs are needed for 4 cups of flour?
   - How many cups of flour are needed for 9 eggs?

3. Study the double number line in Figure 2.25.

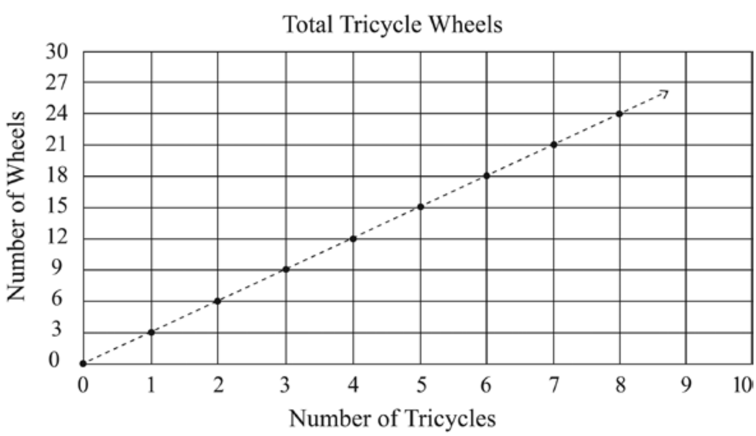
Figure 2.25 Double number line representing meters and seconds



a. The quantities 5 and 2 are located the same distance from 0. What does this mean?
b. How far from 0 is the number pair 15 and 6 in relation to the number pair 5 and 2?
c. How far from 0 is the number pair 15 and 6 in relation to the number pair 10 and 4?
d. What is the associated rate in this situation? Make this visible on the double number line.

4. Study the coordinate grid in Figure 2.26 and answer the question that follow.

Figure 2.26 Coordinate grid showing total tricycle wheels



a. What are the two quantities in this situation and how are they represented in this visual model?
b. What is the associated rate connected to each ratio in this situation? How is it represented in this visual model?

5. Compare and contrast the differences in how a ratio table, a tape diagram, a double number line and a coordinate grid represent the two quantities in a ratio and the associated rate. What does this suggest about instructional use of these visual models?

6. Write or find in your instructional materials three qualitative rate problems for each of the following contexts. For each problem, indicate how the variables interact as described in Table 2.6.
   a. Speed
   b. Buying or selling
   c. Scale

7. Write or find in your instructional materials two problems that involve discriminating between proportional and non-proportional situations. One of the problems should require students to interact with a visual model.

Instructional Link

Use the questions that follow to analyze ways your math instruction and program provides students opportunities to build fluency and understanding of important ratio and proportion concepts and skills.

1. To what degree do your math instruction and program use the following to deepen understanding of the concepts underpinning ratios and proportions?
   a. Representations (which ones)
   b. Discriminating between proportional and non-proportional situations
   c. Engaging in non-numerical qualitative problems
2. Based on your analysis, in what ways would you fill the gaps in your instruction/math problem for each of 1 a–c?

3

Understanding the OGAP Proportionality Framework and Progression

Big Ideas

- The OGAP Proportionality Framework is based on math education research on how students develop ratio and proportion understanding and fluency. It is designed as a tool for teachers to gather evidence of student thinking to inform instruction and monitor student learning.
- Accumulating evidence by researchers indicates that knowledge and use of learning progressions positively impacts teachers' knowledge and instruction as well as students' motivation and achievement.

Chapters 1 and 2 introduced concepts that underpin understanding of ratio and proportions as well as how different representations can be used to develop understanding. Chapters 3 and 4 turn to student learning. Chapter 3 focuses on understanding the OGAP Framework and Progression, and Chapter 4 focuses on using the OGAP Framework and Progression to inform instruction and support student learning. It is suggested that you read Chapters 3 and 4 in order.

About the OGAP Proportionality Framework

The OGAP Proportionality Framework is based on mathematics education research on how students learn ratio and proportion concepts and is a valuable tool to help teachers select or design tasks, understand evidence in student work, make instructional decisions and provide actionable feedback to students.

It is suggested that you download the OGAP Proportionality Framework and refer to it as you read this chapter and any references made to the Framework in other chapters of the book. An electronic copy can be found at https://www.routledge.com/9780367374051.

There are two major elements in the OGAP Ratio and Proportion Framework:

1. Problem Contexts and Structures (front page) and Sample Problems (back page)
2. The OGAP Ratio and Proportion Progression (centerfold) shows sample student strategies along a continuum of student understanding for ratios and proportions.

These two parts of the OGAP Ratio and Proportion Framework are interrelated. That is, students' development of ratio and proportion understanding and fluency is often influenced by the structures of the problems they engage with. Student work samples are used throughout this chapter to describe and exemplify the different levels on the progression. Strategies to help move student understanding along the progression are embedded in this chapter, Chapter 4—and other chapters throughout the book.

The OGAP Ratio and Proportion Progression

> *The OGAP Ratio and Proportion Progression is designed to help teachers gather descriptive evidence of developing understanding of concepts and skills as well as to identify the underlying issues and errors that may interfere with students learning new concepts or solving ratio and proportion problems.*

The OGAP Ratio and Proportion Progression also provides some instructional guidance on how to help transition student understanding and strategies from one level to the next with the goal of developing procedural fluency with understanding. Importantly, the progression provides guidance on helping students develop deeper understanding of ratio and proportion concepts and more mature, efficient and generalizable solution strategies.

Researchers indicate that students may struggle with the use and understanding of formal algorithms if their knowledge is dependent on memory rather than anchored with a deep understanding of the foundational concepts (e.g., Battista, 2012; Carpenter, Franke, & Levi, 2003; Empson & Levi, 2011; Fosnot & Dolk, 2001; Kaput, 1989). The importance of developing fluency based on understanding cannot be overstated.

Additionally, there is accumulating evidence that knowledge and instructional use of learning progressions together with the mathematics education research that underpins progressions positively affect instructional decision-making and student motivation and achievement in mathematics (Supovitz, Ebby, Remillard, & Nathenson, 2018; Clements, Sarama, Spitler, Lange, & Wolfe, 2011; Clarke, 2004; and others). This research supports the use of a learning progression as an effective strategy to gather and act on evidence of student thinking as students develop understanding and fluency with ratios and proportions.

When you review the OGAP Ratio and Proportion Progression, you will notice that the levels reflect different kinds of evidence that might be found in student work as students learn new concepts and solve problems. Each level on the progression is briefly described in this section. In addition, subsequent chapters provide numerous opportunities to deepen your understanding of the progression.

OGAP Ratio and Progression Levels

The OGAP Ratio and Proportion Progression levels represent the continuum of evidence from Non-Proportional to Proportional Strategies that is visible in student work as students develop their understanding and fluency when working with ratio and proportion concepts. The levels are at a grain size that is usable by teachers to gather actionable evidence across the development of ratio and proportion concepts and skills.

Open to the centerfold of the OGAP Proportionality Framework. Notice the five levels along the left side of the progression: Non-Proportional Strategies, Early Ratio Strategies, Early Transitional Strategies, Transitional Strategies and Proportional Strategies.

The progression is designed to provide an expected pathway based on mathematics education research that supports the development of procedural fluency founded on conceptual understanding. The Transitional Proportional Strategy level (including

Early Transitional) of the progression is the bridge between Early Ratio Strategies and Proportional Strategies.

The remainder of the chapter is organized by the progression levels—Early Ratio, Transitional Strategies, Proportional Strategies, and Non-Proportional Strategies. Each section provides examples of student solutions that exemplify evidence at the respective level. Naturally, all the possible examples of student solutions are not included, but each section should provide enough examples to help you become comfortable with the characteristics of the evidence at each level that can inform effective instructional actions.

Early Ratio Strategies

Evidence at the Early Ratio level involves:

- Using the pre-ratio strategy of iterating composed units with and without models; and/or
- Using an additive rather than multiplicative strategy throughout the problem.

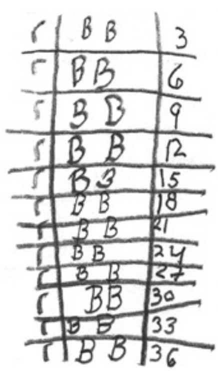
Early Ratio Strategy—Iterating Composed Units by Repeatedly Adding

Some students begin determining equivalent ratios by iterating the composed unit using repeated addition. Researchers refer to this strategy as a pre-ratio strategy, which was discussed in depth in Chapter 1. That is, they iterate the ratio but do not use the multiplicative relationship within the quantities in the ratio or between the ratios. This pre-ratio strategy is classified as an Early Ratio Strategy on the OGAP Ratio and Proportion Progression.

Go To Chapter 1 for a discussion on the pre-ratio strategy of iterating composed units using addition.

Study Alex's solution to the marble problem in Figure 3.1. What is the evidence that Alex used a pre-ratio strategy?

Figure 3.1 Alex's response to marble problem



The ratio of red marbles to black marbles in a bag is 1 : 2. Sue opened the bag and found 12 red marbles.

Alex's solution is a clear example of iterating the composed unit represented by the 1 red marble (R) to 2 black marbles (BB). Alex appeared to use an additive strategy to determine that there were 36 total marbles in the bag.

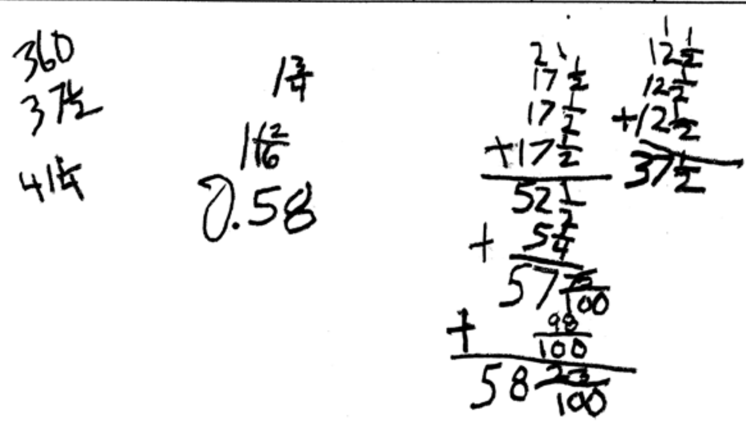
Early Ratio Strategy—Builds Up or Down Using Addition

Solutions that contain evidence of using addition to solve a ratio or proportion problem or use addition as a building-up strategy are also classified as Early Ratio Strategies. Study the Pancake problem in Figure 3.2. Then study Sofia's solution to the Pancake problem. What is the evidence Sofia used addition to solve the problem?

Figure 3.2 Sofia's responses

Using the table that follows, determine the number of cups of pancake mix needed to make 400 pancakes.

	Pancake Recipe				
Number of Pancakes	12	24	36	120	400
Cups of Pancakes	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	$17\frac{1}{2}$	$58\frac{23}{100}$
Cups of Milk	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	$12\frac{1}{2}$	



The fractions and the non-integral multiplicative relationships between quantities in this problem (e.g., the scale factor between 12 pancakes and 400 pancakes is $33\frac{1}{3}$) make it challenging for students as they are developing proportional strategies. This solution contains evidence of first building up to 360 pancakes by adding $17\frac{1}{2}$ cups of pancake mix 3 times and then building to 396 pancakes by adding $5\frac{1}{4}$ cups of pancake mix.

The correct answer to this problem is $58\frac{1}{3}$ cups of pancake mix to make 400 pancakes.

While Sofia used a very interesting approach to the problem, the strategy evidenced in her solution involved addition and not the multiplicative relationships between the quantities in the problem. Therefore, Sophia's solution is classified as an Early Ratio Strategy.

Transitional Strategies

Transitional strategies are important because they provide a pathway that links Early Ratio with Proportional strategies.

The Transitional Proportional Strategy level is characterized by the use of visual models and other representations such as ratio tables that can help students apply multiplicative rather than additive strategies to ratio and proportion problems. For example, at the Transitional Proportional Strategy level, one may see students using building-up strategies that incorporate multiplication. In this section, the examples of Transitional Proportional Strategies on the OGAP Ratio and Proportion Progression will be discussed in depth.

Open the OGAP Ratio and Proportion Progression in the centerfold of the Framework. Review the solutions at the Transitional Proportional Strategy Level. Discussion of each of these solutions follows.

In general, the solutions at the Transitional Strategy Level show evidence of:

- understanding ratio relationships
- scaling up or down
- some evidence of using multiplicative relationships between quantities

In addition, solutions at the Early Transitional Level may include both additive and multiplicative strategies.

Transitional Strategies build toward efficient proportional strategies based on multiplicative relationships between the quantities in the problem situations. You'll notice this level is divided into Early Transitional and Transitional. Those solutions in the Early Transitional Level are bridges from Early Ratio strategies to Transitional Proportional Strategies. The strategies at the Transitional Proportional Level are bridges to Proportional Strategies.

Understanding the Role of the Transitional Strategy Level as a Bridge From Early Ratio Strategies to Proportional Strategies

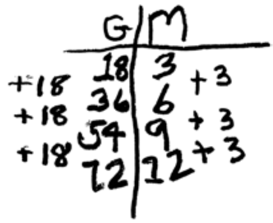
To begin to understand how Transitional Strategies bridge Early Ratio Strategies to Proportional Strategies, study Mohammad's, Charlie's, Sophia's and Isabelle's solutions in Figure 3.3. The evidence in these four pieces of student work show the transition from an additive strategy in Mohammad's Early Ratio solution to a Proportional Strategy of finding and applying a unit rate evidenced in Isabelle's solution.

Figure 3.3 Mohammad's, Charlie's, Sophia's and Isabelle's solutions

Mohammad's response: *Early ratio—build up using addition*

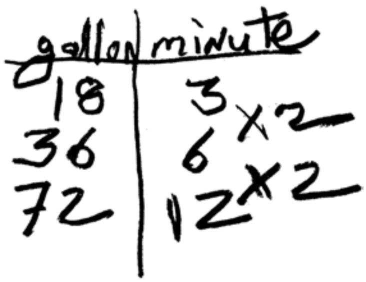
Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 12-minute shower?

Figure 3.3 Continued.



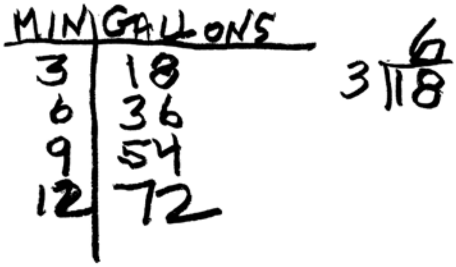
Charlie's response: *Early Transitional Strategy—Uses scaling by a factor of 2 (doubling) to build up both gallons and minutes*.

Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 12-minute shower?



Sophia's response: *Transitional Strategy—Finds and applies the multiplicative relationship between minutes and gallons—organized in ratio table*.

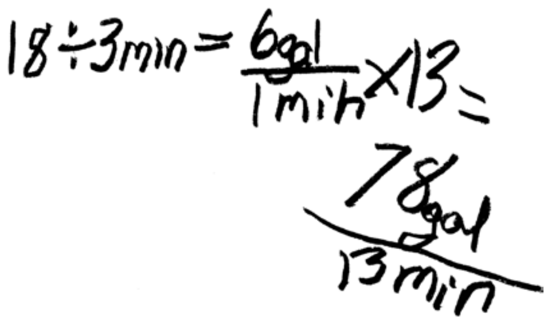
Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 12-minute shower?



Isabelle's response: *Proportional Strategy—finds the unit rate, 6 gallons per minute, then applies it to a 13-minute shower*. (Note that this is a 13-minute shower, not 12 minutes like the other examples.)

Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?

Figure 3.3 Continued.



In reviewing these solutions, it is important to note that all four of the solutions have evidence of correctly identifying and using the quantities gallons and minutes in the problem situation. This means that instruction can focus on transitioning students to more efficient/sophisticated strategies.

From Early Ratio to Early Transitional

The evidence in Mohammad's and Charlie's responses shows both students using a building-up strategy. The important difference between the strategies is that Mohammed used addition to build up, while Charlie used multiplication to scale up the number of gallons and the number of minutes. This move from applying addition to using multiplication is an example of a transition from the Early Ratio Level to the Early Transition Level on the OGAP Ratio and Proportion Progression. "While the use of building up strategies is an important milestone on the path toward proportional reasoning, developing more sophisticated reasoning is crucial for solving more complex problems and understanding multiplicative relationships" (Steinthorsdottir & Sriraman, 2009, p. 5). Charlie's and Mohammad's work illustrates the limiting nature of these building-up strategies. That is, if the shower was 13 minutes long instead of 12 minutes, their strategies might be problematic.

From Early Transitional to Transitional

Of course, while Charlie's multiplicative scaling shows an important progression from Mohammad's Early Ratio Strategy, it is not until one studies Sophia's response that there is evidence of using the multiplicative relationship between the quantities minutes and gallons in the problem situation. This move to finding and applying the multiplicative relationship between the quantities in the situation in a ratio table is an important example of transition from Early Transitional Strategies to Transitional Strategies.

From Transitional to Proportional Strategies

Moving from Transitional Strategies to Proportional Strategies is about flexible and efficient application of the multiplicative relationships in ratio and proportion situations without interference by the number type or number relationships. While Sophia did apply the multiplicative relationship between gallons and minutes, she also applied that ratio to four different quantities. This is in contrast to the evidence in Isabelle's solution of finding the unit rate and applying it directly to a 13-minute shower.

The use of unit rates is not the only strategy at the Proportional Strategy Level. However, all the strategies at the Proportional Strategy Level involve finding and applying the multiplicative relationships between quantities directly rather than building up, using ratio tables or using visual models. Examples at the Proportional Strategy Level are discussed in detail later in this chapter.

The student solutions in Figure 3.3 will be revisited from an instructional perspective in Chapter 4. Specifically, the solutions will be used to illustrate how to use student solutions in classroom discussions to help all students move toward Transitional and Proportional Strategies.

Transitional-Level Strategies

Early Transitional Strategy—Building Up or Down Using Multiplication

Charlie's solution in Figure 3.3 involving the building-up strategy using a multiplicative scale factor is one example of an Early Transitional Strategy. It is classified as Early Transitional rather than Transitional because there is evidence of using doubling rather than the scale factor between 3 minutes and 18 minutes (× 6).

Early Transitional Strategy—Using the Multiplicative Relationship for Only Part of the Problem

Another type of Early Transitional Strategy involves using the multiplicative relationship in a problem but for only part of the problem, as exemplified by Noah's and Richard's solutions found in Figure 3.4. Study the two solutions. What is the evidence that they used the multiplicative relationship for part of the problem? What other strategies did they use?

Figure 3.4 Noah's and Richard's early transitional strategies

Noah's response

Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?

$$\frac{18\text{gal}}{3\text{min}} + \frac{18\text{gal}}{3\text{min}} + \frac{18\text{gal}}{3\text{in}} + \frac{18\text{gal}}{3\text{min}} = \frac{72\text{gal}}{12\text{min}} + \frac{6\text{g}}{1\text{m}}$$

$$\frac{78\text{gal}}{13\text{min}}$$

$18 \div 3 = 6$gal per min.

$$\begin{array}{r} 13 \\ \times\ 6 \\ \hline 78 \end{array}$$

Figure 3.4 Continued.

Richard's response

Paul's dog eats 20 pounds of food in 30 days. How long will it take Paul's dog to eat a 45-pound bag of dog food?

$$\frac{20\text{ lb.}}{30\text{ days}} = \frac{45\text{ lb.}}{65\text{ days}}$$

$$65 = 30 \times 2 + 5$$
$$45 = 20 \times 2 + 5$$

Notice that Noah added the ratios $\left(\frac{18\text{ gallons}}{3\text{ minutes}}\right)$ four times to determine that 72 gallons are used in 12 minutes. Notably, when confronted with the 13th minute of the shower, Noah found and applied the unit rate of 6 gallons per minute. This is in stark contrast to Richard's response in Figure 3.4, in which he initially applied a multiplicative relationship (doubling) but dealt with the non-integral multiplicative relationship in the problem by doubling and adding 5 to each quantity.

Early Transitional Strategy—Applies Multiplicative Relationships but Uses the Incorrect Ratio Referent

The term *ratio referent* in the title of this section is used to describe evidence of student work in this section and in other chapters. This topic is more fully explained in Chapter 6 in the section Ratio Problems and Problem Types. However, while the term may be new to you, the concept is not. That is, a ratio referent is a term used to describe the relationship between the quantities in a ratio (e.g., part-to-part; part-to-whole). Sometimes the quantities in the relationship used to solve the problem are explicitly stated. In problem situations involving ratios, it is common for students to use the wrong ratio referent when solving a problem, as you will see in examples in this section.

Go To: Go to Chapter 6 for an in-depth discussion on ratio referents.

Solve the problem and then study Annabelle's solution in Figure 3.5. What quantities were compared in her solution? What quantities should be compared?

Figure 3.5 Annabelle's response

Dana and Jamie ran for student council president at Midvale Middle School. The data below represents the voting results.

	Seventh-Grade Votes	
	Jamie	*Dana*
Boys	24	40
Girls	49	20

Figure 3.5 Continued.

John said, "About half the students who voted for Jamie were boys." Mary said, "About one-third the students who voted for Jamie were boys." Who is correct?

Johns correct because $24 \times 2 = 48$ which is only 1 away from 49 so that's almost one half.

Annabelle's solution, like the other responses, at the Early Transitional Level showed some evidence of finding and applying the multiplicative relationship, but in this case, it used the incorrect ratio referent. That is, she used the part-to-part ratio of the seventh-grade boys who votes for Jamie to seventh-grade girls who voted for Jamie instead of the part-to-whole ratio of seventh-grade boys who voted for Jamie to all the seventh-grade students who voted for Jamie.

One final thought about the solutions at the Early Transitional Level. The solutions at the Early Transitional Level and at all levels are not organized on the Framework in a hierarchy. That is, for example, Richard's doubling (Figure 3.4) is not more sophisticated or efficient than Charlie's doubling in a table (Figure 3.3). Rather, all the solutions at the Early Transitional Level show some evidence of finding and applying a multiplicative relationship between quantities, but they also contain evidence of additive reasoning or, in the case of Annabelle's response in Figure 3.5, applying the multiplicative relationship to the wrong quantities.

Transitional Strategy—Uses Multiplicative Relationship in a Ratio Table

The use of a ratio table based on the multiplicative relationship between the quantities is an example of a Transitional Proportional Strategy. Study Kim's solution in Figure 3.6. How is her solution the same as or different from Noah's or Richard's solution in Figure 3.4?

Figure 3.6 Kim's response—uses multiplicative relationship in a ratio table

Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?

MIN	GAL
3	18
6	36
9	54
12	72
13	78

$$3\overline{)18}^{\,06}$$

Bob will use 78 gallons of water for a 13 min. shower

Kim's solution shows evidence of finding and using the multiplicative relationship between minutes and gallons, that is, there are 6 times more gallons than minutes. She used this multiplicative relationship to determine the number of gallons used in 13 minutes. Kim's solution, in contrast to Noah's solution in Figure 3.4, illustrates the direct application of the scale factor (× 6) between minutes and gallons within the ratio table.

The sample Transitional Proportional Strategies provided here share clear evidence that students understood and worked with the ratios given in the problem and used a multiplicative relationship found in the situation in their solution.

Transitional Strategy—Use of Visual Models Multiplicatively

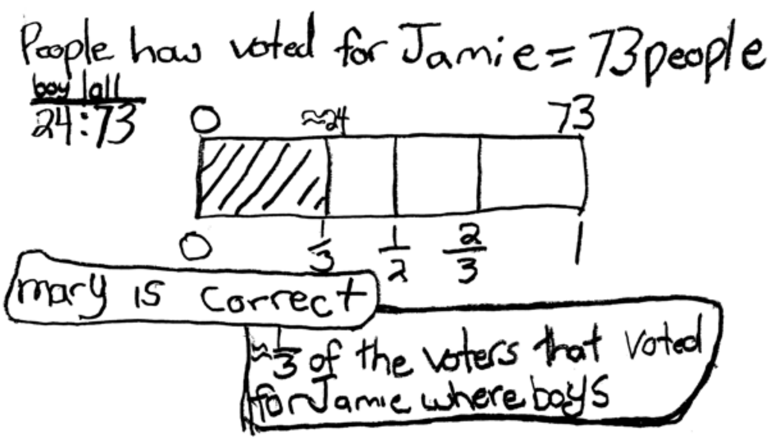
Chapter 2 provides an in-depth discussion about the role that visual models play in developing ratio and proportion understanding and fluency. Solutions classified at the Transitional Proportional Strategy level must illustrate an understanding of the meaning of the ratio or proportion in the problem situation. Study Hanna's response in Figure 3.7. In what ways do the visual models in her solution show an understanding of the ratio concepts in the problem? Of the multiplicative relationship in the problem?

Figure 3.7 Hanna's response—uses of visual models and the correct ratio referent

Dana and Jamie ran for student council president at Midvale Middle School. The data below represents the voting results.

	Seventh-Grade Votes	
	Jamie	*Dana*
Boys	24	40
Girls	49	20

John said, "About half the students who voted for Jamie were boys." Mary said, "About one-third the students who voted for Jamie were boys." Who is correct?



Hanna's visual models illustrate understanding of the part-to-whole relationship implied in the problem situation (24 votes for Jamie to all 73 students in the seventh grade who voted for Jamie). Hanna correctly sketched a visual model to show that 24 votes are about ⅓ of 73 votes. Importantly, Hanna used division to represent the fractional part of the votes, illustrating her understanding of the multiplicative relationships between the quantities.

Go to Chapter 2 for an in-depth discussion on how visual models are used to build understanding of ratio and proportion concepts.

Proportional Strategies

Proportional Strategies on the OGAP Ratio and Proportional Progression all have the following characteristics in common. There is evidence that:

- The ratios in the student solutions are consistent with the problem situation.
- The multiplicative relationships applied are consistent with the problem situation.
- The strategies were efficient.

As mentioned earlier in the chapter, the solution strategies at this level are NOT organized to reflect a hierarchy in the sophistication of the strategies; rather, they are organized to show different examples of Proportional Strategies.

These strategies include:

1. Using $y = kx$ and relationships between coordinate graphs, tables and equations
2. Using cross products
3. Comparing fractions, rates or ratios
4. Applying multiplicative relationships within and between ratios
5. Finding and applying unit rates
6. Using the correct ratio referent and applying a multiplicative relationship

Importantly, a student with flexible use of these strategies will choose the best strategy given the problem situation. This does not mean that there is a correct strategy to use for a given situation or a single efficient strategy for a situation. Rather, students should be allowed to choose a strategy that best fits their understanding of the problem in order to develop flexibility. In fact, it is more concerning if students only apply one proportional strategy to all proportional situations even if the strategy is less efficient given the problem situation. We will revisit this idea while discussing problem and solution strategies at the Proportional Level.

Study the solutions in the Proportional Strategy section of the OGAP Ratio and Proportion Progression and in Figure 3.7. What is the evidence in each solution of a Proportional Strategy?

Proportional Strategy—Finds and Applies the Unit Rate

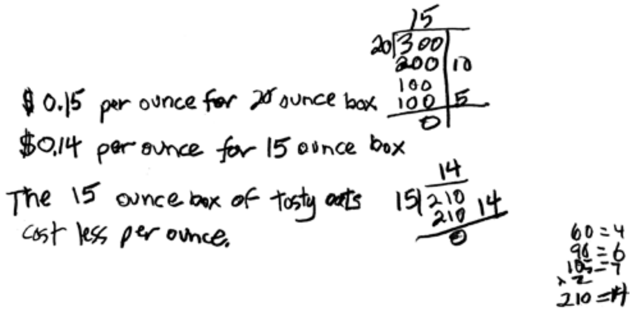
At the Transitional Proportional Level, there were two instances of solutions that found and applied the unit rate. Study the problems and solutions in Figure 3.8. How are the problems alike, and how are they different? In what ways did Karl and Patty find and apply a unit rate?

Figure 3.8 Opportunity to find and apply unit rates

Karl's response

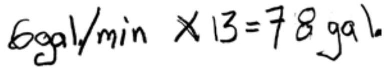
A 20-ounce box of Toasty Oats cereal costs \$3.00. A 15-ounce box of Toasty Oats cereal costs \$2.10. Which box costs less per ounce?

Figure 3.8 Continued.



Patty's response

Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?



Toasty Oats is a rate comparison problem. One way to solve this problem is by finding and comparing the unit rate (dollars/ounces) for each box of Toasty Oats. Bob's Shower, as you know from earlier examples, is different. It is a missing value problem. In this case, one could determine the unit rate (gallons/minute) and apply that rate to determine the number of gallons used in a 13-minute shower.

Go to Chapter 6 for an in-depth discussion on different problem types and how they might influence difficulties students encounter as well as strategies they might use.

Proportional Strategy—Compares Fractions, Rates or Ratios

The context of a problem plays a large part in providing opportunities for students to compare ratios or rates in problems involving proportionality. For example, the Toasty Oat problem in Figure 3.8 requires some comparison of the cost per ounce. Solve the Raccoon problem in Figure 3.9 to understand the structure of other problems that provide the opportunity to compare rates or ratios.

Now study Hunter's, Emma's and Elizabeth's solutions in Figure 3.9. How does the problem provide the opportunity to compare the rates or ratios? What is the evidence in each solution of making rate or ratio comparisons? How are the strategies for comparison alike? How are they different?

Figure 3.9 Hunter's, Emma's and Elizabeth's responses to the Raccoon problem

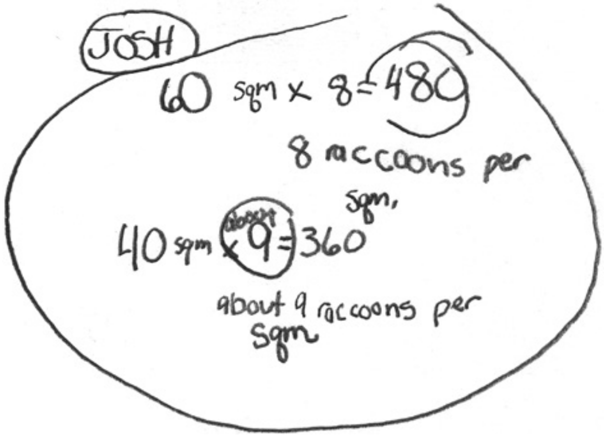
The chart below shows the population of raccoons in two towns and the areas of the two towns.

Town A	*Town B*
60 square miles	40 square miles
480 raccoons	380 raccoons

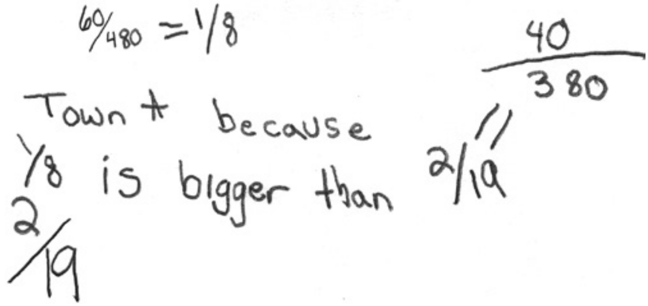
Figure 3.9 Continued.

Karl says that Town A has more raccoons per square mile than Town B. Josh says Town B has more raccoons per square mile than Town A. Who is right?

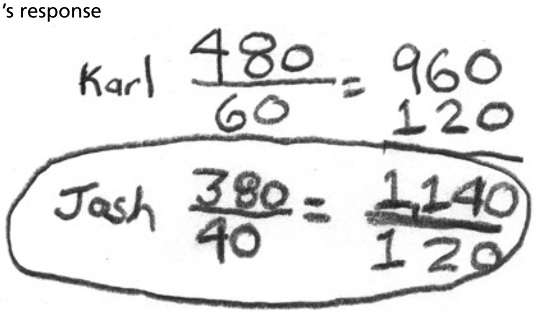
Hunter's response



Emma's response



Elizabeth's response

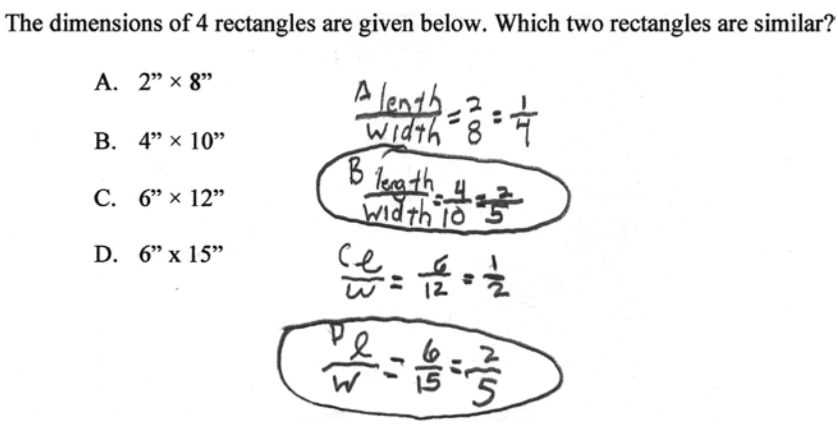


The problem is asking for a comparison without specifically indicating how the population density should be compared. All three responses contain evidence of making comparisons, but in very different ways. Hunter approximated the unit rates and then compared them, concluding correctly that about 9 raccoons per square mile is greater than 8 raccoons per square mile. Emma compared the simplified ratios $\frac{60 \text{ miles}}{480 \text{ racoons}} = \frac{1 \text{ mile}}{8 \text{ racccoons}}$ to $\frac{40 \text{ miles}}{380 \text{ raccoons}} = \frac{2 \text{ miles}}{19 \text{ raccoons}}$ making the erroneous conclusion Town A has a greater raccoon population density than Town B by comparing miles to raccoons, not raccoons per square mile. Elizabeth essentially found equivalent ratios with a common denominator of 120 (acres), concluding correctly, since $\frac{1140}{120} > \frac{960}{120}$.

[Note: There is a discussion later in this chapter about collecting evidence of underlying issues or errors that are evidenced in these and other solutions (e.g., missing units in Elizabeth's solution and fraction comparison of square miles to raccoons error in Emma's solution). In Chapter 4 there is a strategy for gathering this type of evidence and using it to inform instructional decisions.]

Sometimes the problems in which this strategy can be applied are not as obvious as in the Toasty Oats or the Raccoon problems. Case in point is the similarity problem in Figure 3.10. Study the problem and solution in Figure 3.10. In what ways did Oliver's solution in Figure 3.10 show evidence of finding and comparing rates or ratios?

Figure 3.10 Oliver's response



Notice that the solution to the problem in Figure 3.10 involves simplifying and comparing the ratio of the length to the width, or recognizing that rectangles are similar is tantamount to understanding that the dimensions of the rectangles have the same ratios.

Go To Go to Chapter 7 for a discussion of similarity and scale factor contexts.

Proportional Strategy—Applies Multiplicative Relationship and Scale Factor

Study Jackson's and Mia's responses in Figure 3.11. What is the evidence that Jackson and Mia found and applied the multiplicative relationship between the quantities in the problem? Is this a unit rate? Why or why not?

Figure 3.11 Jackson's and Mia's responses

Jackson's response

Paul's dog eats 20 pounds of food in 30 days. How long will it take Paul's dog to eat 45 pounds of dog food?

$45 \div 20 = 2.25$

$30 \times 2.25 = 67.5$ days

Mia's response

Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?

$$\frac{18 \text{ gallons}}{3 \text{ min}} \begin{matrix} \times 4\frac{1}{3} = \\ \times 4\frac{1}{3} = \end{matrix} \frac{78 \text{ gal}}{13 \text{ min.}} \qquad 3\overline{)13}\ \ 4\frac{1}{3}$$

If Bob takes a 13min shower he will use 78gal of water.

Both Mia and Jackson used the scale factor in the proportions between like units. In Jackson's case, he found and applied the scale factor between 20 pounds and 45 pounds (× 2.25) of food and applied the scale factor to 30 days (30 days × 2.25 = 67.5 days). Mia found the scale factor between 3 minutes and 13 minutes and applied the scale factor to the number of 18 gallons (18 gallons × 4 ⅓ = 78 gallons). Notice that there are no units associated with the scale factor in each of these solutions. This distinguishes this strategy from a unit rate strategy in which there are associated units (e.g., 6 gallons/minute).

The scale factor can be used to determine if rectangles are similar. Study Ava's and Isabelle's solutions in Figure 3.12. How was the scale factor used to determine if the rectangles were similar in each solution? What understanding does it appear Ava and Isabelle have about similar rectangles?

Figure 3.12 Ava's and Isabelle's responses

Ava's response

The dimensions of 4 rectangles are given below. Which two rectangles are similar?

A. 2" × 8"

B. 4" × 10"

C. 6" × 12"

D. 6" x 15"

B+D because both of the fiest numbens go into the second Number 2.5 times.

Isabelle's response

The dimensions of 4 rectangles are given below. Which two rectangles are similar?

E. 2" × 8"

F. 4" × 10"

G. 6" × 12"

H. 6" x 15"

The Scale factor is 1.5

Ava and Isabelle's solutions illustrate understanding of similarity in two different ways. Ava used the multiplicative relationship between the length and the width of each rectangle (4 inches × 2.5 = 10 inches and 6 inches × 2.5 = 15 inches). Ava found B and D had the same multiplicative relationships between their dimensions and therefore are similar. Isabelle identified the scale factors between corresponding sides of rectangles B and D (4 inches × 1.5 = 6 inches and 10 inches × 1.5 = 15 inches), and therefore B and D are similar.

Go To

Go to Chapter 7 for a discussion of similarity and scale factor contexts.

Proportional Strategy—Uses y = kx *(Either Symbolic or Graphical Representation Moving Between Graphs, Tables and Equations)*

Chapter 2 provides an in-depth discussion on the relationships between equations, graphs and ratio tables focused particularly on how these relationships can build and strengthen ratio and proportion understanding and flexibility. To help gather evidence about this developing understanding, students need to engage in problems in which they interpret data in tables and on coordinate planes and represent those situations with equations. The problems and student solutions that follow illustrate responses at the Proportional Strategy Level that illustrate understanding and application of the relationships between tables, graphs and equations.

Study Finley's solution in Figure 3.13. What is the evidence that Finley used his understanding of the relationships between tables and equations to solve the problem?

Figure 3.13 Finley's response

The table below shows the towns at which the train is scheduled to stop.

If the train travels at the average rate of 45 miles per hour between Central Station and Green Valley, how many hours did the train travel when it reached Green Valley?

Towns	*Miles From Central Station*	*Number of Hours From Central Station*
Central Station	O	0
Elmwood	45	?
Green Valley	180	?
Watertown	225	?
Woodland	270	?

$d \div 45 = t$

speed

time

$$\begin{array}{r} 4 \\ 45 \overline{)180} \\ -180 \\ \hline 0 \end{array}$$

distance

$45 \times 4 = 180$

$\boxed{\text{A: 4 hours}}$

The equation ($d \div 45$ mph $= t$) in Finley's solution correctly matches his calculations representing the relationships between distance, time and speed in the problem situation.

Study the problem and Aurora's solution in Figure 3.14. What is the evidence in her solution of interpreting the graph in this problem situation? What other Proportional Strategies are evidenced in Aurora's solution?

Figure 3.14 Aurora's response

Sam is working to earn $200 to buy a bicycle. He always earns the same amount for each hour that he works. Sam started a graph to show how much money he has earned.

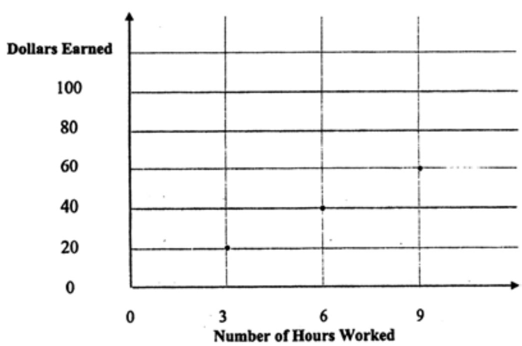


Figure 3.14 Continued.

a. How many hours must he work to earn $200?

in 3 hours Sam makes $20
20×10=200
3×10=30

$\boxed{30 \text{ hours}}$

b. Based on his earnings in the graph, how much does Sam earn per hour?

$$
\begin{array}{r}
06.66 \\
3\overline{)20.00} \\
-18 \\
\hline
20 \\
-18 \\
\hline
20 \\
-18 \\
\hline
\end{array}
$$

$\boxed{\$6.66 \text{ an hour}}$

Aurora's solution contains evidence that she interpreted the multiplicative relationship between dollars earned and hours worked in a number of ways. First, Aurora found that it would take Sam 30 hours to earn $200 using a scale factor of × 10 based on the interpretation of the graph at 3 hours earning $20. Next, Aurora used this same relationship to obtain the hourly rate of $6.66 ($20 ÷ 3 hours ≈ $6.67 an hour).

Solve the problem and study Jamie's solution in Figure 3.15. What is the evidence that Jamie used his understanding of proportional relationships in his solution? What is the evidence that this solution is classified at the Proportional Strategy Level?

Figure 3.15 Jamie's response

Thrill Amusement Park charges a $10 admission fee plus $1 per ride. Maria determined that the cost for going on three rides at the Thrill Amusement Park is $33. Her reasoning is shown below. Is Maria's reasoning correct? Why or why not?

$$\frac{\$11}{1\text{ride}} = \frac{\$33}{3\text{rides}}$$

No, Marias resoning is not correct.
She is saying that 1 ride is 11$
but 1 ride cost one dollare
+admission

The equation is: y=x+10 NOT y=$11x
y=1·3+10 y=$11·3
y=$13 y=$33

The Amusement Park problem in Figure 3.15 involves recognizing that the situation is not proportional. That is, the initial \$10 admission fee is only applied once and then added to the cost of \$1 per ride. To support Jamie's answer that Maria is wrong, Jamie showed the equation that represents the situation ($y = x + \$10$) and made a point of identifying the relationship ($y = \$11x$) that Maria incorrectly used to solve the problem. Both the accurate representation of the proportional and non-proportional situation with equations puts this solution at the Proportional Strategy Level.

Go To

Go to Chapter 2 for a discussion of the importance of providing students the opportunity to distinguish between proportional and non-proportional relationships.

Proportional Strategy—Using Cross Products

Using cross products can be an important procedure for solving proportion problems. An example of the application of cross products is found on the OGAP Proportion and in Figure 3.16.

Figure 3.16 Harper's response—uses cross products

Bob's shower uses 14 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a shower for 8 minutes?

$$\frac{3 \text{ minutes}}{14 \text{ gallons}} = \frac{8 \text{ minutes}}{x}$$

14 gallons × 8 min = 112   112 ÷ 3 minutes = $37\frac{1}{3}$ gallons

As with all procedural solutions, one does not know whether Harper applied the cross-product procedure with conceptual understanding or simply followed a procedure they were shown.

An in-depth discussion of this strategy, an understanding of the mathematics underlying the algorithm and when researchers suggest is the appropriate time to introduce this procedure is found in Chapter 8.

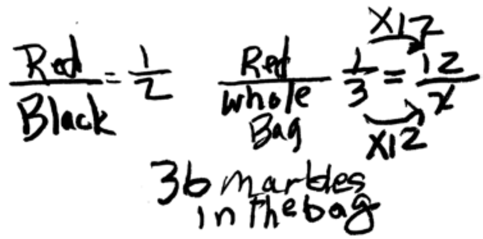
Proportional Strategy—Applies the Correct Ratio Referent in a Ratio Problem and Applies Multiplicative Relationships

As described earlier in the chapter, identifying and using the correct ratio relationships/referents in the problem and applying the multiplicative relationships between the quantities in a ratio is foundational for solving ratio problems. For a ratio problem to be classified at the Proportional Strategy Level, both these conditions need to be met.

Study Arlo's solution in Figure 3.17. What is the evidence that this solution is at the Proportional Strategy Level?

Figure 3.17 Arlo's solution

There are red and blue marbles in a bag. The ratio of red to blue marbles in the bag is 1 : 2. Sue opened the bag and found 12 red marbles. How many marbles are in the bag altogether?



Arlo used the part-to-whole relationship (1 red marble to 3 marbles) implied in the ratio of 1 red to 2 blue marbles. He then used the scale factor between 1 red marble and 12 red marbles to determine that there are 36 marbles in the bag (3 marbles × 12 = 36 marbles in the bag).

Non-Proportional Strategies

Not discussed yet but important for gathering evidence for instructional planning is the Non-Proportional Level on the OGAP Ratio and Proportion Progression. Evidence of non-proportional strategies includes:

1. Using additive differences between quantities in a ratio
2. Using the incorrect ratio referent and the additive difference between quantities in a ratio
3. Comparing numbers not ratios
4. Guessing or using random operations
5. Misinterpreting vocabulary or a related concept
6. Solving a non-proportional problem using a proportion

As students first engage in ratio and proportion topics, or when they encounter a new context or structure, non-proportional strategies are often evidenced in their work.

Non-Proportional—Using Additive Differences Between Quantities in a Ratio; or Compares Numbers, Not Ratios

Figure 3.18 contains the examples for *using additive differences* and *comparing numbers not ratios*. What is the evidence that these examples are non-proportional strategies?

Figure 3.18 Amelia's and Ken's responses

Amelia's response

Bob's shower uses 14 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a shower in 8 minutes?

Figure 3.18 Continued.

$$3\text{min} \xrightarrow{+5} 8\text{min}$$
$$14\text{gal} \xrightarrow{+5} 19\text{gal}$$

Ken's response

Karl says that Town A has more raccoons per square mile. Josh says Town B has more raccoons per square mile. Who is right? Justify your answer.

Town A	*Town B*
60 square miles	40 square miles
480 raccoons	380 raccoons

$$\begin{array}{r} 480 \\ -\ 60 \\ \hline 420 \end{array} \qquad \begin{array}{r} 380 \\ -\ 40 \\ \hline 340 \end{array}$$

420 is bigger than 340 so Town A has more racoons

Amelia's response shows evidence of finding the additive difference between 3 minutes and 8 minutes (+ 5 minutes) not the multiplicative relationship between 3 minutes and 8 minutes $(2\frac{2}{3} \times 3 \text{ minutes} = 8 \text{ minutes})$. Notable is that the comparison is between minutes, and the evidence points to the idea that the number of gallons should be increased the same way as the minutes.

Ken's response shows evidence of using the additive difference between square miles and raccoons (480 raccoons – 60 square miles = 420?). Of course, the answer 420 has no meaning. That is, it is not possible to subtract square miles from the number of raccoons.

Non-Proportional—Uses the Incorrect Ratio Referent and Makes an Additive Comparison Between the Quantities

Like Amelia's and Ken's responses in Figure 3.18, Liam's work in Figure 3.19 contains evidence of using the additive difference between quantities. Additionally, what is the evidence that this solution should be classified as non-proportional?

Figure 3.19 Liam's response

Dana and Jamie ran for student council president at Midvale Middle School. The data below represents the voting results.

	Seventh-Grade Votes	
	Jamie	*Dana*
Boys	24	40
Girls	49	20

Figure 3.19 Continued.

John said, "About half the students who voted for Jamie were boys." Mary said, "About one-third the students who voted for Jamie were boys." Who is correct?

49 40
-24 -20 Niether
25 20

In addition to using the additive differences between quantities, there is evidence in Liam's work of not using the correct ratio referents. He calculated the differences between girls and boys who voted for either Jamie or Dana, NOT the ratio of seventh-grade boys who voted for Jamie to all the seventh-grade students who voted for Jamie.

Non-Proportional—Solves a Non-Proportional Problem Using a Proportional Strategy

Chapter 2 has an in-depth discussion on the importance of students being provided opportunities to discriminate between proportional situations and non-proportional situations by analyzing problem contexts as well as graphs and equations of both proportional and non-proportional situations.

Go To Go to Chapter 2 for the discussion on importance of students discriminating between proportional and non-proportional situations.

Solve the problem in Figure 3.20. Is this a proportional situation? What is the evidence?

Figure 3.20 Kim and Bob running on the track

Kim and Bob were running equally fast around a track. Kim started first. When she had run 9 laps, Bob had run 3 laps. When Bob had run 15 laps, how many laps had Kim run?

This situation is not proportional. It is actually an additive situation. That is, there will always be a 6-lap difference between Kim and Bob. So when Bob has run 15 laps, Kim will have run 21 laps.

Study Mason's and Scarlett's solution to the track problem in Figure 3.21. Scarlett's solution has evidence of correctly using the additive relationship in the problem. What did Mason do? Why do you think Mason thought that the situation was proportional?

Figure 3.21 Mason's and Scarlet's responses

Kim and Bob were running equally fast around a track. Kim started first. When she had run 9 laps, Bob had run 3 laps. When Bob had run 15 laps, how many laps had Kim run?

Figure 3.21 Continued.

Scarlet's response

21

Because Kim had ran 6 laps more than BOB. So if BoB ran 15 laps than Kim ran 21.

Mason's response

$$\frac{\text{Kim's laps}}{\text{Bob's laps}} = \frac{9}{3} = \frac{n}{15} \quad n = 45$$

Mason treated the problem as a proportional problem by setting up a proportion and scaling up using the scale factor of × 5 instead of using the additive difference. In an OGAP study involving 82 sixth-grade students, 39 (48%) of the students solved this problem with a proportional strategy. One can hypothesize the possible reasons for these responses. One reason could be that during instruction on proportions, students were not asked to discriminate between proportional and non-proportional situations. Couple this with the fact that the problem involved three knowns and asked for a related fourth quantity. This structure is common to proportion problems, although in this case the situation was additive.

As you learned in Figure 3.15, the Amusement Park problem is also a non-proportional problem because it involves an initial cost of $10 that is added to the per-ride cost of $1. Study Eric's solution in Figure 3.22. What is the evidence that this solution should be classified at the Non-Proportional Strategy Level?

Figure 3.22 Eric's response

Thrill Amusement Park charges a $10 admission fee plus $1 per ride. Maria determined that the cost for going on three rides at the Thrill Amusement Park is $33. Her reasoning is shown below. Is Maria's reasoning correct? Why or why not?

$$\frac{\$11}{1\text{ride}} = \frac{\$33}{3\text{rides}}$$

10+1=11 ×3 yes

$$\frac{11 \text{ dollars}}{1 \text{ ride}} = \frac{33 \text{ dollars}}{3 \text{ rides}}$$

×3

At first glance, one notices that Eric found the multiplicative relationship in the proportion that Maria used to solve the problem. However, the problem is not proportional, and there is no evidence in his solution that he understood the problem situation. Because the goal of the problem was to recognize that the situation is not proportional, this solution is classified at the Non-Proportional Level even though Eric recognized the multiplicative relationship in the proportion that the problem posed.

Non-Proportional—Misinterprets Vocabulary or a Related Concept

As students encounter new topics like similarity or density, the vocabulary in the problem might interfere with successfully solving problems, or the solution may be less sophisticated or efficient. We saw students using Proportional Strategies in Figure 3.12 to solve the similarity problem shown in Figure 3.23. What concept and associated vocabulary may interfere with students successfully solving this problem?

Figure 3.23 Similar rectangles problem

The dimensions of four rectangles are given below. Which two rectangles are similar? Explain your choice.

A. 2 inches × 8 inches

B. 4 inches × 10 inches

C. 6 inches × 12 inches

D. 6 inches × 15 inches

In order to solve this problem successfully, a student needs to understand that similar rectangles are proportional to each other. That is, the ratio of length to width for one rectangle is equivalent to the ratio of length to width for the other rectangle. Study Leo's solution in Figure 3.24. What is the evidence that he may not understand the concept of similarity?

Go To: Go to Chapter 7 for an in-depth discussion of similarity and scaling.

Figure 3.24 Leo's solution

The dimensions of four rectangles are given below. Which two rectangles are similar? Explain your choice,

A. 2 inches × 8 inches

B. 4 inches × 10 inches

C. 6 inches × 12 inches

D. 6 inches × 15 inches

A and B because there dimentions like
2"x8" are 6 apart from eachother
and 4"x10" are 6 apart from each other.

Leo focused on the additive differences between the given dimensions. As already discussed, solutions to proportion problems that focus on additive differences are classified as Non-Proportional on the OGAP Ratio and Proportion Progression. This evidence, coupled with other evidence from Leo's work, might point to the concept of similarity being the problem, particularly if Leo successfully uses a Transitional or Proportional Strategy on other proportion problems.

Table 3.1 provides a summary of the major distinguishing evidence between OGAP Ratio and Proportional Progression levels.

Table 3.1 Summary table—distinguishing evidence between levels on OGAP ratio and proportion progression

	Strategy Level	*Distinguishing Evidence*
Proportional		✓ The ratios worked with are consistent with the problem situation. ✓ The multiplicative relationships applied are consistent with the problem situation. ✓ The strategies are efficient. (Some examples: finds and applies unit rate; uses $y = kx$; compares fractions, rates or ratios; uses cross products; finds and applies scale factor)
TRANSITIONAL	Transitional	✓ The ratios worked with are consistent with the problem situation. ✓ Provides evidence of finding and applying the multiplicative relationship represented in the problem ✓ Strategies may not be efficient (Some examples: uses visual model multiplicatively; applies multiplicative relationship in a ratio table)
TRANSITIONAL	Early Transitional	✓ Includes some use of multiplicative relationships between quantities ✓ May include evidence of both additive and multiplicative reasoning (Some examples: uses multiplicative relationship for part of the problem; builds up or down multiplicatively)
Early Ratio		✓ Uses the pre-ratio strategy of iterating composed units with and without models ✓ Involves an additive rather than multiplicative strategy (Some examples: iterates composed unit in model or table; build up using skip counting or repeated addition)
Non-Proportional		✓ Uses additive differences between quantities in a ratio ✓ Uses the incorrect ratio referent and the additive difference between quantities in a ratio ✓ Compares numbers not quantities in a ratio ✓ Guesses or uses random operations ✓ Misinterprets vocabulary or a related concept ✓ Solves a non-proportional problem using a proportion

Using the OGAP Ratio and Proportion Progression involves understanding other concepts involved with using a learning progression beyond the classification of student solutions. The next section focuses on these other important ideas.

Important Ideas About the OGAP Ratio and Proportion Progression

This section focuses on the following important points to keep in mind when using the OGAP Ratio and Proportion Progression:

1. Movement along the progression is not linear.
2. Students' strategies will be at different levels on the progression at different times.
3. The progression provides instructional guidance.
4. The progression is not evaluative.
5. Collection of underlying issues and errors is important.

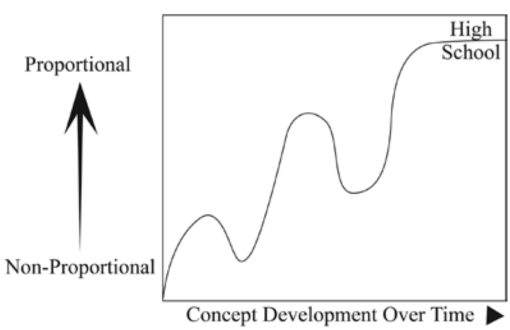
Movement along the OGAP Ratio and Proportion Progression is not linear. While the progression looks linear, student development of understanding and fluency is a more complex pathway. As students are introduced to new concepts, different problem structures for the same concept and more complicated numbers or are asked to apply their proportional reasoning knowledge to new mathematical topics their solution strategies may move back and forth between Proportional, Transitional, Early Ratio and Non-Proportional strategies.

Students' strategies will be at different levels on the progression depending on the concepts being taught and learned or the problems they are solving. That is, a student may use a Proportional Strategy when solving a consumer problem and a Non-Multiplicative Strategy when solving a similarity or density problem.

The graphic in Figure 3.25 illustrates these important points. That is, as ratio and proportion understanding and fluency develop and as students are introduced to new concepts or interact with different problem structures for the same concept, students' solutions may move up and down the progression levels (Kouba & Franklin, 1993; OGAP, 2006). On the left side of the OGAP Ratio and Proportion Progression, there is the two-way arrow representing this important idea.

Ideally, by high school, ratio and proportion fluency and understanding should be stabilized at the Proportional Strategy Level when solving ratio and proportion problems, understanding that there will be instances when a high school student will use a visual model to make sense of a complex problem or a new mathematical topic.

Figure 3.25 Hypothesized movement on the progression as concepts are introduced and developed across grades



Source: (adapted from Petit, Laird, Marsden, & Ebby, 2016).

The OGAP Ratio and Proportion Progression provides instructional guidance. Along the right side of the progression, there is an arrow that contains key concepts (focus on visual models, multiplicative relationships, equivalence, unit rates, contexts, coordinate plane, ratio tables, equations) that are important for the development of ratio and proportion fluency with understanding. These concepts serve as instructional guidance on how to help move student understanding and strategies from one level to the next. For example, if a student is still solving problems using a building-up strategy, the instructional focus might turn to unit rates. Chapter 4 focuses on strategies to collect evidence in student work using the progression as well as ways in which the evidence can be used to inform instruction and student learning.

Using the OGAP Ratio and Proportion Progression to guide instruction is not about direct instruction on specific strategies or concepts. Rather, it involves the interaction of foundational concepts with targeted instructional strategies (e.g., connecting mathematical ideas, classroom discourse, sharing solutions and purposeful questioning) to engage students in thinking and reasoning about these concepts and relationships.

Go To

Go to Chapter 4 for guidance on using the progression to inform instruction and student learning.

The OGAP Ratio and Proportion Progression is not evaluative. You'll notice there are no numbers associated with the levels on the progression. A learning progression is designed to help teachers gather descriptive evidence about student learning to inform instruction and student learning, not to assign a number or grade. The descriptive evidence includes the level on the OGAP Ratio and Proportion Progression, the substrategy, underlying issues or errors and evidence of solution accuracy (e.g., Transitional, building up, calculation error, incorrect). In addition to the progression not being evaluative, it is important to remember that when classifying responses into the different levels on the progression, it is the solution that has evidence at that level, not the student.

Example:

Appropriate way to discuss a solution: "The evidence in Oliver's solution is at the Transitional level.

Inappropriate way to discuss a solution: "Oliver is at the Transitional level."

Collection of Underlying Issues and Errors is important. At the bottom of the progressions, there is a list of potential underlying issues or errors that may interfere with students learning new concepts and solving related problems. Sometimes evidence of errors or underlying issues does not influence where the evidence is classified along the progression. For example, consider a student who uses a unit rate at the Proportional Strategy Level but makes a calculation error. It is important to record this calculation error because it affects accuracy, but it does not change the level of the evidence on the progression. Other times, an error can influence the placement on the progression—errors like using the wrong operation or guessing. This information, coupled with the location of the strategy used along the progression, provides teachers with actionable evidence to inform instruction and learning.

OGAP Ratio and Proportion Progression

Some chapters of *A Focus on Ratios and Proportions: Bringing Research to the Classroom* include a section on the OGAP Ratio and Proportion Progressions describing what the evidence in the student work for the topic under discussion would look like at different levels of the progression. We also suggest that you analyze the student work examples throughout the book through the lens of the OGAP Ratio and Proportion Progressions.

The icon to the left is used throughout the book to indicate where the OGAP Proportionality Framework or the Ratios and Proportions Progressions are discussed.

Chapter Summary

This chapter focused on learning progressions and specifically the OGAP Ratio and Proportion Progression.

- The OGAP Proportionality Framework consists of two sections: Problem Contexts and Structures and the OGAP Ratio and Proportion Progression.
- The OGAP Ratio and Proportion Progression provides examples of a learning progression founded on mathematics education research, written at a grain size that is usable across a range of ratio and proportion teaching and learning by teachers and students in a classroom.
- The OGAP Ratio and Proportion Progression was specifically designed to inform instruction and monitor student learning from a formative assessment perspective.
- The OGAP Ratio and Proportion Progression illustrates how student strategies progress from Early Ratio to Transitional to Proportional Strategies as they develop understanding of foundational concepts as well as engage in a range of problem types, contexts and other problem structures.
- Analyzing evidence in student work using the OGAP Ratio and Proportion Progression provides important information about where students are in their understanding of concepts and use of Ratio and Proportion Strategies.

Looking Back

1. **Become familiar with the OGAP Proportionality Framework:** The OGAP Ratio and Proportion Framework is comprised of two sections: Problem Contexts and Problem Structures on page 1 with sample problems on page 4 and the Ratio and Proportion Progression on the centerfold.
   Examine these two sections of the OGAP Proportionality Framework and answer the two questions that follow.
   a. What is the main purpose of each section of the OGAP Proportionality Framework?
   b. In what ways is the information in the OGAP Ratio and Proportion Framework important to teachers who teach ratio and proportion?

2. Briefly explain the instructional importance of each of these features of a learning progression:
   a. Movement along the progressions is not linear.
   b. Students' strategies will be at different levels on the progression at different times.

c. The progressions provide instructional guidance.
d. The progressions are not evaluative.
e. Collection of underlying issues and errors is important.

3. Study each of the solutions in Figure 3.26.
   a. How are they alike and how are they different?
   b. Decide—how is each solution classified on the progression?
   c. What is the evidence?

Figure 3.26 Solutions A, B and C

Donna runs at an average rate of 12 minutes per 1 mile. At this rate, how many miles does Donna run in 28 minutes?

Solution A

$12 \times 2 = 24$ $\quad 24 + 4 = 28$

$12 \times 1/3 = 4$

$2 + 1/3 = 2\,1/3$ $\quad$ $2\,1/3$ miles

Solution B

$\overset{\div 3}{12} : \overset{\div 3}{1}$

$=$

$4 : \frac{1}{3}$ mile

min

Donna runs $2\frac{1}{3}$ miles in 28 minutes

Minutes: 4, 8, 12, 16, 20, 24, 28

Miles: $\frac{1}{3}$, $\frac{2}{3}$, 1, $1\frac{1}{3}$, $1\frac{2}{3}$, 2, $2\frac{1}{3}$

Solution C

Donna runs 2.33 miles in 28 minutes

$$\begin{array}{r} 2.33 \\ 12\overline{)28} \\ 24 \\ \hline 40 \\ 36 \\ \hline 40 \\ 36 \\ \hline 4 \end{array}$$

Go To Read Chapter 4: Using the OGAP Progression to Inform Instruction and Student Learning. In the Looking Back section of that chapter, there are opportunities to use the OGAP Ratio and Proportion Progression and make instructional decisions.

Instructional Link

Use the questions that follow to analyze ways your math instruction and program provide students opportunities to build fluency and understanding of important ratio and proportion concepts and skills.

1. To what degree do your math instruction and program focus on regularly gathering descriptive information about student learning to inform your instruction?
2. What are the ways in which your instruction focuses on the multiplicative relationships within and between ratios in a proportion?
3. How do you and your math program use visual models, representations, equivalence to help students transition from Transitional to Proportional Strategies?
4. Based on this analysis, can you identify some specific ways that you can enhance your math instruction by utilizing ideas from this chapter?

4

Using the OGAP Ratio and Proportion Progression to Inform Instruction and Student Learning

Big Ideas

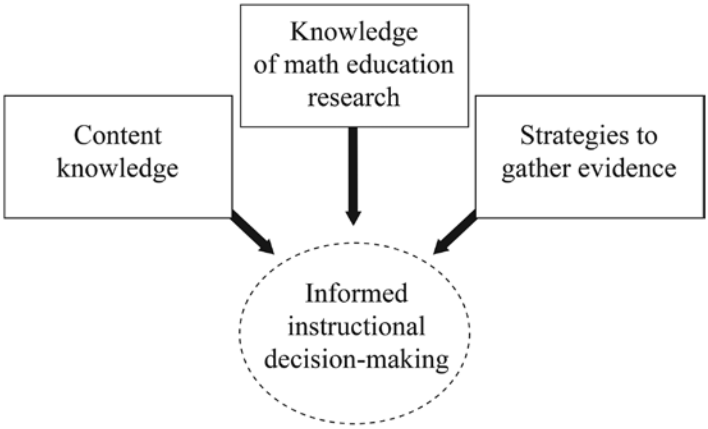
- OGAP is an intentional and systematic approach to using a formative assessment system based on understanding the OGAP Proportionality Framework and Progression.
- The OGAP Sort is a strategy to gather evidence of student understanding in relationship to the progression, including errors students make and accuracy of their solutions.
- There are many instructional responses to evidence in student work.

Chapter 4 focuses on using learning progressions to inform instruction and support student learning. Specifically, this chapter provides a strategy for analyzing student work, recording the evidence from that analysis and making instructional decisions based on the evidence from the whole class and individual students. The ideas in this chapter are dependent on understanding the concepts in Chapter 3; thus it is recommended that you read Chapter 3 prior to reading this chapter.

The formative assessment process described in this chapter is cyclical. The process involves eliciting evidence of student thinking and adjusting instruction on an ongoing basis as students are learning during classroom discussions, while observing students working and collecting evidence from each student at the end of lessons using an exit card. Popham (2012) proposes that, "The essence of formative assessment is the relentless attention to evidence of student thinking" (p. ix). Wiliam (2011, p. 43) indicates that to be considered formative assessment, the evidence must be "elicited, interpreted, and used by teachers and learners."

These ideas are at the heart of the OGAP formative assessment system. Strategies to systematically collect evidence to inform instruction and student learning through the lens of the OGAP learning progression are essential. The suggested strategies and routines have been developed and refined since 2003 through interactions with thousands of teachers. Importantly, the instructional decision-making that is at the core of this process is based on knowledge of mathematics, the related mathematics education research reflected in the OGAP Ratio and Proportion Progression and instructional strategies that elicit actionable evidence. See Figure 4.1.

Figure 4.1 Informed instructional decision-making



Gathering Evidence Using the OGAP Sort

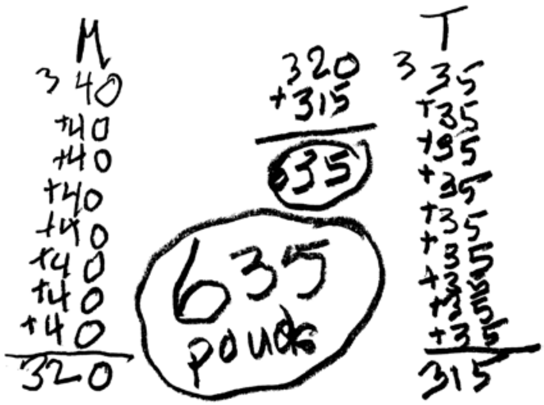
Teachers who effectively use the OGAP Ratio and Proportion Progression are able to adjust instruction during classroom discussions and as students are working. Another formative assessment strategy teachers have adopted is the regular use of exit questions at the end of a lesson (Fennell, Kobett, & Wray, 2017). An exit question at the end of a lesson provides evidence from all students that can be analyzed to inform instruction for the next lesson.

Exit questions can provide teachers with a view of a student's understanding, particularly when the question asked has been engineered to elicit specific understanding related to the learning goals of the lesson. One strategy for analyzing exit tickets from students is the OGAP Sort. The OGAP Sort, when used with the OGAP Ratio and Proportion Progression, provides evidence to inform instruction with respect to three aspects of student solutions: (1) strategies students used to solve a problem and the corresponding level of each solution on the progression; (2) underlying issues or errors present in each solution; and (3) accuracy of each solution.

Many teachers are initially surprised by the priority of the order of the OGAP Sort analysis. Traditionally, when looking at student work, the accuracy of the solution is considered first. However, OGAP studies have shown that accuracy alone may produce a *false positive*. That is, just considering the accuracy of a solution may disguise the inefficiency of the solution strategy with unexpected consequences. For example, a fifth-grade student who consistently uses repeated addition to solve multiplication problems, even if they are all correct, will be at a significant disadvantage when engaging in middle school topics dependent upon strong multiplicative reasoning and fluency (e.g., proportions). A correct answer may hide the fact that the student may not have the needed multiplicative understanding or strategies to engage in middle school math topics. Study Anna's solution in Figure 4.2. Anna is a beginning sixth-grade student. What strategy did Anna use to solve this multiplication problem? If this is Anna's primary strategy when solving multiplication problems, how might this interfere with her solving ratio and proportion problems?

Figure 4.2 Anna's response

Max and Thomas each delivered vegetables to a store. Max delivered 8 bags of vegetables with 40 pounds in each bag. Thomas delivered 9 bags of vegetables with 35 pounds in each bag. How many pounds of vegetables were delivered by Max and Thomas altogether?



Notice first that Anna found the correct answer to this problem. If a teacher only looked at the answer, they might overlook the fact that Anna did not use multiplication to solve the problem. Instead, Anna used repeated addition to determine the total number of vegetables delivered to the store. If this is Anna's only strategy to solve multiplication problems, it is likely that she will be at a disadvantage when solving ratio and proportion problems.

To focus the analysis of evidence on a progression perspective, the OGAP Sort begins by organizing the student work into piles of strategies used, which correspond to each of the progression levels as shown in Figure 4.3.

Figure 4.3 Sorting student work into Proportional, Transitional, Early Transitional, Early Ratio and Non-Proportional Strategies

Once sorted, the information is then recorded on an OGAP evidence collection sheet such as the one pictured in Figure 4.4. Next, one returns to the student work and makes notes about the strategies used within a level and any underlying issues or errors. In the example in Figure 4.4 the teacher also circled the solutions that were incorrect. This collection sheet parallels the OGAP Ratio and Proportion Progression Levels and provides teachers with a picture of where the evidence in the student solutions is along the progression, the errors that may interfere with learning and the accuracy of the answers. Together, these pieces of information can help inform instruction for the whole class, for small groups and/or for individual students. Note on the evidence collection sheet, student names are listed under the strategy evidenced (e.g., Transitional), the substrategies (e.g., used visual model) are noted, student names are listed under underlying issues or errors if evidenced and incorrect solutions are circled.

Figure 4.4 Evidence collection sheet

Item #	Problem Structures	Proportional	Transitional Proportional: Transitional	Transitional Proportional: Early Transitional	Early Ratio	Non-proportional
1	Raccoons Density	Harper, Amelia, Mason } unit rate Michael, Amy } Mult. Relationship	William, Mia } Visual Model James, Elijah, Evelyn } ratio table	Ava, Charlotte, Oliver } Builds Multipli. Sophia, Isabelle, Lucas } used mult. for part of the problem	Noah, Olivia } Iterates a composed unit	Liam, Ben, Emma } Added factors

Uses additive strategy to find multiplicative relationship:	Remainder error: Sophia Isabelle Lucas Ava	Error in equation:	Computational error: Mia Evelyn Ava
Misinterprets the meaning of quantities: Noah Olivia Liam Ben	Missing or inconsistent use of units: Harper Mason Amelia James William Elijah	Rounding error:	Other:

Once the evidence is recorded, the analysis turns to looking across the whole class, using the following questions to help make instructional decisions:

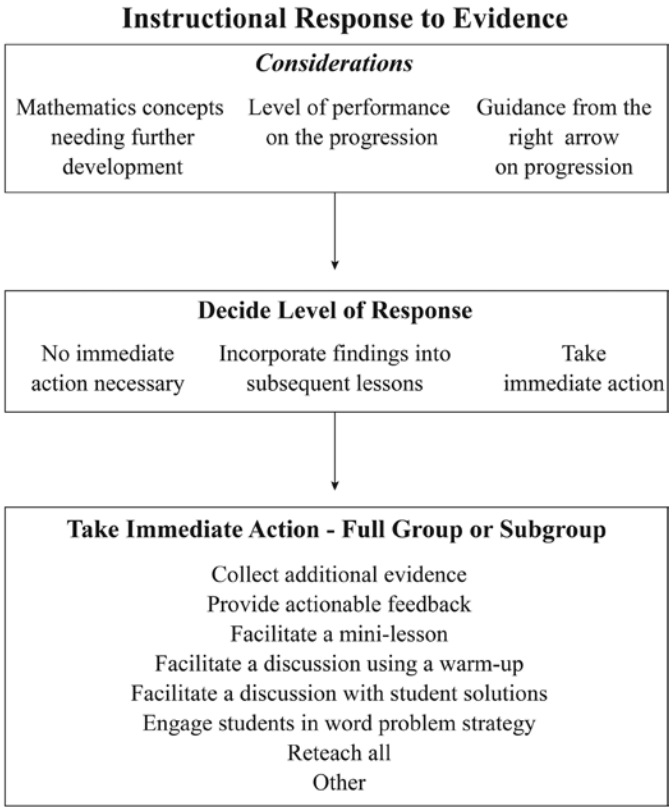
1. What are the developing understandings that can be built upon?
2. What issues or concerns are evidenced in student work?
3. What are potential next instructional steps for the whole class, for small groups and/or for individuals?

This next section provides a framework for instructional decision-making based on the evidence in student work. It is followed by a case study that illustrates how a teacher might address each of these questions and provides examples of a range of instructional responses to the evidence.

Instructional Responses to Evidence

In working with teachers, the authors have found that many teachers struggle with how to respond to evidence in student work beyond putting students into small groups. To help teachers think about a range of potential instructional responses, OGAP professional-development facilitators have found it helpful for teachers to think about three levels of decision-making. Figure 4.5 provides a graphic illustrating the relationships between these decision-making processes. The process involves starting with the evidence to be considered (e.g., the mathematics; the level of performance on the progression), then deciding on the level of response (e.g., no immediate action necessary) and finally, if action is necessary, deciding on an instructional strategy (e.g., collect additional evidence; implement a mini-lesson).

Figure 4.5 Instructional responses to evidence



The case study that follows provides an example of a classroom implementing an OGAP formative assessment cycle over a four-day period. The case study will provide an example of several of the instructional strategies listed on Figure 4.5.

This icon will accompany any examples of instructional responses found throughout the book.

Case Study—Four Days in Ms. Collins's Seventh-Grade Classroom

Day 1

Ms. Collins, a seventh-grade math teacher, had just finished a lesson that focused on engaging students in a range of missing value and unit rate problems focusing on the multiplicative relationships in each problem situation. During the lesson, the students solved both missing value and unit rate problems and studied the relationship between the problems and the solutions. At the end of the lesson, Ms. Collins administered the following exit question to the students.

> Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 12-minute shower?

Ms. Collins realized that this problem involved small numbers and that the multiplicative relationships between the quantities involved whole numbers. That is, 18 gallons ÷ 3 minutes = 6 gallons per minute and 12 minutes ÷ 3 minutes = 4 (scale factor). She understood that students could solve this problem using either the unit rate of 6 gallons per minute or the scale factor of 4. She also understood that because both the unit rate and the scale factor involved in the problem were integral (whole number), students could use either a multiplicative or repeated addition strategy. She was interested in gathering evidence about the strategies that students used to solve the problem.

Her record of the evidence is found in the evidence collection sheet in Figure 4.6.

Figure 4.6 OGAP evidence collection sheet from Ms. Collins's Class Day 1

Bob's shower uses 18 gallons of water in 3 minutes. How many gallons of water does Bob use if he takes a 12-minute shower?

Item #	Problem Structures	Proportional	Transitional Proportional		Early Ratio	Non-proportional
			Transitional	Early Transitional		
1	Bob's Shower Integral Mult. relationships	Paul Karen Isabelle unit rate Victoria Liam use Scale factor	Sophia Iris ratio table Keith visual model	Charlie Max Margo Builds up Mult.	Alex Mohamed Build up adding Mia Madison Logan Iterated Composed unit	Harper Elijah Oliver Added difference

Uses additive strategy to find multiplicative relationship:	Remainder error:	Error in equation:	Computational error:
		Isabelle	Iris Chris Paul Liam
Misinterprets the meaning of quantities:	**Missing or inconsistent use of units:**	**Rounding error:**	**Other:**
	Paul Karen Victoria Max Margo		

After completing the OGAP Sort and recording the evidence, Ms. Collins addressed the three questions explained on page 4 in this chapter. Note that the process starts with evidence that can be built upon.

1. What are developing understandings that can be built upon?
   Almost all of the students showed evidence of understanding the quantities in the problem (three students did not) and used a viable strategy given the problem situation. Students used strategies across the levels of the

progression. The solutions at the proportional and transitional level had evidence of either the unit rate or the scale factor strategy. Those solutions at the Early Transitional Level scaled multiplicatively but did not use the single scale factor between the quantities. Those at the Early Ratio Level used repeated addition.

2. What issues or concerns are evidenced in student work?
   Three students incorrectly used additive differences between quantities. Eight students used either additive or building-up strategies, which were not related to the multiplicative relationships between the quantities. Additionally, four students made calculation errors, and five students did not include units to define quantities in their solutions.
3. What are potential next instructional steps for the whole class, for small groups and/or for individuals?
   Ms. Collins used guidance from the arrow on the right side of the progression and focused the instructional response on the multiplicative relationships in the problem. She decided to do this by *facilitating a discussion using the students' solutions* at the beginning of the next lesson.

Day 2

Facilitate a discussion using student solutions

Ms. Collins's goal for Day 2 was to help students to move toward using Proportional Strategies when solving problems with whole-number multiplicative relationships between quantities. She decided to take immediate action by facilitating a discussion using student solutions.

Ms. Collins chose these solutions for two reasons. First, the solutions all included a ratio table representing the relationship between gallons and minutes. Second, the ratios in each table were arrived at differently. Mohammad used addition. Charlie scaled

Figure 4.7 Mohammad Bob's shower solutions (earlier discussed in Chapter 3)

Mohammad's response

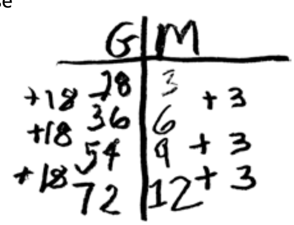
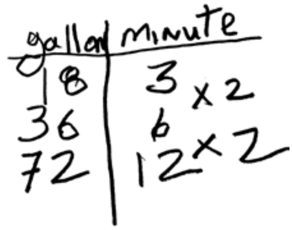
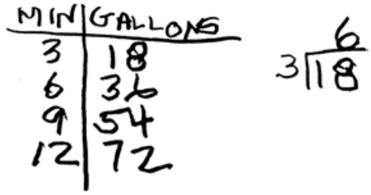


Figure 4.7 Continued.

Charlie's response



Sophia's response



up the ratios using doubling. Sophia used the multiplicative relationship between minutes and gallons. There are different ways these three solutions could be used as a tool for facilitating a discussion. Ms. Collins decided to focus students' attention on how the quantities in the problem are shown in Mohammad's, Sophia's and Charlie's ratio tables. She displayed the three solutions and asked the following questions of all the students. She used a think, pair, share strategy to provide all students, including those students who used a Non-Proportional Strategy, the opportunity to be involved in the conversation.

1. Did each of the students arrive at the same correct answer to the problem? What strategy or relationships did each student use?
2. How are the ratios in these tables alike or different?
3. Does the order in which the ratios appear in the tables matter (gallons to minutes vs. minutes to gallons)?
4. In our discussions we have identified that ratio and proportion problems involve the multiplicative relationships between the quantities in the problem situation.
   a. Which solution(s) involve a multiplicative relationship between the quantities in the problem situation?
   b. In what ways did Sophia and Charlie use a multiplicative relationship between the quantities?
   c. Study each of the ratio tables—what is the multiplicative relationship between gallons and minutes in each table? What is the multiplicative relationship between 3 minutes and 12 minutes in the problem?
   d. How can these multiplicative relationships be used to solve the problem?

Ms. Collins then asked her students to solve a number of problems with a similar structure to this problem during the lesson. At the end of the lesson, Ms. Collins administered the two questions in Figure 4.8 as exit questions. Why do you think she gave these questions to her students?

Figure 4.8 Ms. Collins's exit questions

Question 1:

It takes Joe 12 minutes to type 600 words.

At this rate, how long will it take Joe to type 150 words? Show your work.

Question 2:

Donna runs at an average rate of 12 minutes a mile. At this rate, how many miles does Donna run in 28 minutes?

Ms. Collins had two goals when she administered these questions. First, she wanted to gather evidence of the strategies that students used when the numbers were greater than the numbers in the Bob's Shower problem and which would require students to build down instead of up. The second goal was to gather evidence from a unit rate problem in which the scale factor between quantities was fractional (28 minutes ÷ 12 minutes = $2\frac{1}{3}$) in preparation for the next day's planned lesson.

After administering the questions, Ms. Collins analyzed the work using the OGAP Sort and then recorded evidence in an evidence collection sheet. The analysis for both these problems is found in Figure 4.9. Study the results. Based on this evidence, how are the student responses different between these two items? What could be the source of so many errors in the Donna Running problem?

Figure 4.9 Evidence collection sheet (Joe Typing and Donna Running). Circled names indicate an incorrect answer.

Item #	Problem Structures	Proportional	Transitional Proportional		Early Ratio	Non-proportional
			Transitional	Early Transitional		
1	Joe's Typing Integral mult. relationship	Paul, Karen, Isabelle, Victoria – unit rate; Sophia, Charlie, Iris, Keith – mult. relat.	May, Marge, Alex, Mohamed, Oliver – ratio table unit rate	Mia, Madison – mult. for part of problem; Harper, Eliza – Build up mult.		
2	Donna's Running Fractional Mult. relat.	Paul, Karen		Sophia, Charlie, Keith, Max, Marge, Alex, Liam, Victoria, Oliver, Isabelle, Eliza All circled students made a remainder error	Mohamed, Logan, Mia	Harper

Uses additive strategy to find multiplicative relationship	Remainder error: All circled students	Error in equation:	Computational error:
Misinterprets the meaning of quantities:	Missing or inconsistent use of units:	Rounding error:	Other:

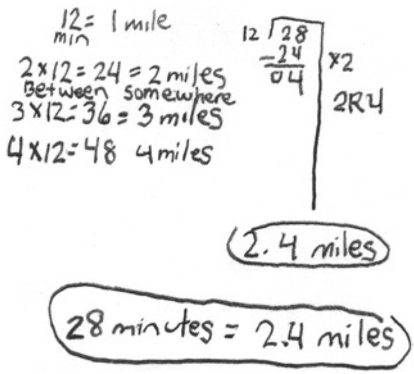
Ms. Collins noticed that the student solutions to the Joe's Typing problem were more sophisticated than the solutions to the Bob's Shower problem, and almost every student used a multiplicative relationship to solve the problem.

This was not the case when students solved the Donna Running problem. Ms. Collins noticed that the students showed evidence of determining that Donna can run 2 miles in 24 minutes. Fifteen students out of 17 students arrived at the incorrect answer; specifically, they struggled with the remainder, 4, which occurs in this problem because 28 is not a multiple of 12. Figure 4.10 contains samples of the typical errors students made on this problem. Study the student solutions in Figure 4.10. Where on the progression are these solutions? What type of errors are evidenced in these student solutions?

Figure 4.10 Student responses to Donna's Running

Donna runs at an average rate of 12 minutes a mile. At this rate how many miles does Donna run in 28 minutes?

Sophia's response



Max's response

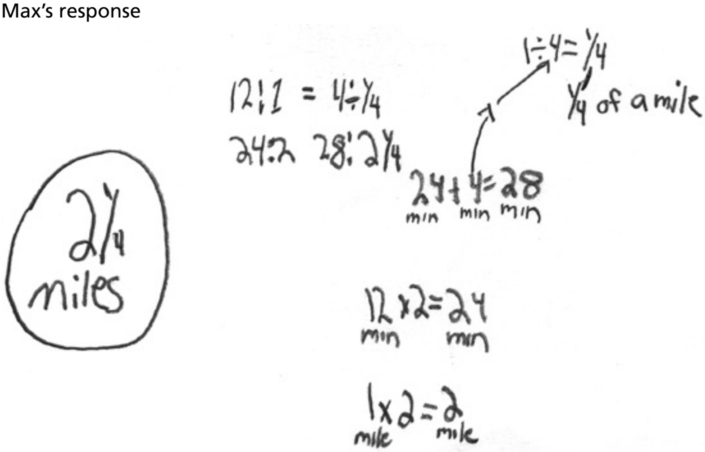
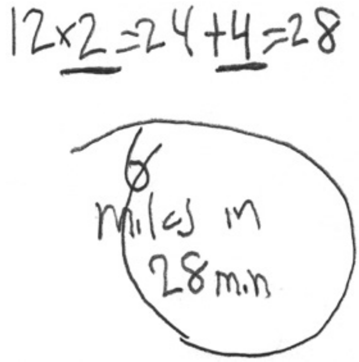
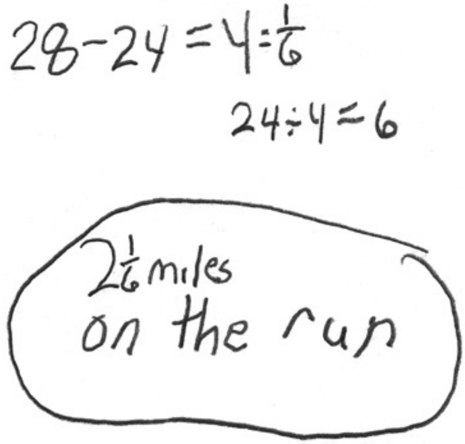


Figure 4.10 Continued.

Alex's response



Oliver's response



As noted earlier, these solutions show evidence that students understood Donna ran 2 miles in 24 minutes at a rate of 12 minutes per mile. Ms. Collins was concerned about the number of incorrect solutions. While she knew her instruction was about to focus on problems that involved fractional (non-integral) multiplicative relationships, she was surprised by how this one structure (a fractional scale factor between quantities) of the problem resulted in such a change in performance. Then she remembered the important point about progressions discussed in Chapter 3: "As students are introduced to new concepts, different problem structures . . . their solution strategies may move back and forth between proportional, transitional, early transitional, early ratio, and non-proportional." This was a clear example of this situation.

Looking closely at the student work in Figure 4.10, the four students struggled with how the extra 4 minutes related to the quantities in the problem. Alex added the 4 minutes to the 2 miles to get 6 miles. Sophia annexed the remainder 4 onto 2 as a decimal remainder to get 2.4 miles. Max created a new ratio relationship, 1 mile to 4, minutes incorrectly represented as $\frac{1}{4}$ of a mile. Oliver divided 24 minutes by 4 minutes and then changed the quotient (6) into $\frac{1}{6}$ of a mile.

In contrast, study Paul's and Karen's solutions in Figure 4.11. What is the evidence that Karen and Paul understood the quantities in the problem? Considering the contrast between Paul's and Karen's responses and the rest of the class, what might be a logical next instructional move. That is, what would be the mathematical focus, and what instructional strategy(s) might you use to address this evidence?

Figure 4.11 Paul's and Karen's responses to the Donna Running problem

Donna runs at an average rate of 12 minutes a mile. At this rate how many miles does Donna run in 28 minutes?

Paul's response

Donna runs $2\frac{1}{3}$ miles

$$\times 2\tfrac{1}{3} \begin{pmatrix} 12 : 1 \\ 28 : 2\tfrac{1}{3} \end{pmatrix} \times 2\tfrac{1}{3}$$

$$12 \times 2 = 24$$
$$12 \times \tfrac{1}{3} = 4$$
$$12 \times 2\tfrac{1}{3} = 28$$

Karen's response

min mile

$$\div 12 \begin{pmatrix} 12 : 1 \\ 1 : \tfrac{1}{12} \end{pmatrix} \div 12$$

MIN mile

$1 \times 28 = 28$

$\frac{1}{12} \times 28 \quad \frac{28}{12} = 2\frac{4}{12} = 2\frac{2}{6}$

$28 : 2\frac{2}{6}$

min mile

Donna runs $2\frac{2}{6}$ miles 28 minutes

Ms. Collins observed that Paul found the scale factor between 12 minutes and 28 minutes ($\times 2\frac{1}{3}$) and applied it to scaling 1 mile to $2\frac{1}{3}$ miles. She noticed that Karen found and applied the unit rate of $\frac{1}{12}$ mile per minute. Ms. Collins recognized that all but two students were having difficulty with the non-integral multiplicative relationship in the problem. She also wondered if the context of the problem, i.e., minutes per mile, was causing problems for some of her students.

Day 3

Incorporate findings into subsequent lessons and use a warm-up problem at the start of a lesson.

Ms. Collins decided that she would use two instructional strategies to address the evidence in Day 2's exit question and engage students in problems that involved fractional multiplicative relationships between quantities:

1. Incorporate her understanding of the errors that students made into subsequent lessons.
2. Start the next day's lesson using a warm-up problem to gather additional information.

She realized that in addition to the multiplicative relationship between quantities being fractional (non-integral), the rate in the Donna Running problem—minutes per mile—was not a familiar rate for the students. Study the warm-up problem in Figure 4.12. Why do you think she chose this warm-up problem?

Figure 4.12 Ms. Collins's warm-up problem

> Chris bikes at an average rate of 10 miles per hour. At this rate, how many hours will it take for Chris to bike 35 miles? Show your work.

The problem, as you probably noticed, involves a more familiar rate—miles per hour—in contrast to minutes per mile in the Donna Running problem. As her students were solving the problem, Ms. Collins walked around the room making notes in regard to strategies the students were using and how they were dealing with the remainder in this situation. In this case, she did not complete a formal OGAP Sort and did not record the evidence on an evidence collection sheet but used her understanding of the progression and the mathematics in the problem to consider her next instructional step based on her observations.

She observed the following:

1. All but four students correctly dealt with the quantities in the problem, finding that it will take $3\frac{1}{2}$ hours to ride 35 miles at an average rate of 10 miles per hour.
2. Many students made sketches of double number lines to solve the problem. Some used tape diagrams (Transitional Level).

3. Some of the students found and applied the scale factor between 10 miles and 35 miles ($3\frac{1}{2}$) to solve the problem (Proportional).

The fact that so many students were more successful with this problem in comparison to the way they solved the Donna Running problem reminded Ms. Collins how important it is to pay attention to the structures in problems during instruction and when selecting problems for formative as well as summative purposes. She was also reminded how important it is for students to have experience with the many structures in proportion problems.

Go To

Go to Chapters 5, 6 and 7 to learn more about the impact of different problem structures on student solutions.

During lessons over the next couple of days, Ms. Collins intentionally included problems with fractional multiplicative relationships between quantities in familiar and unfamiliar contexts to start building flexibility.

Day 4

The goal of the lesson on Day 4 was to extend student understanding of rate problems when the unit rate is not explicitly stated and the multiplicative relationship between the quantities is non-integral (not a whole number). Ms. Collins anticipated this change of structure might make these problems difficult for some students. In order to help the students focus on the meaning of the quantities and the relationships between the quantities in this new situation, Ms. Collins decided to use a strategy she learned recently that was adapted from a research-based reading strategy: read, retell, and anticipate what will happen next (Gambrell, Koskinen, & Kapinus, 1991; Morrow, 1985).

Making sense of word problems

Ms. Collins used the four-part strategy for making sense of word problems described here (Hulbert et al., 2017).

1. Remove the question from the problem and have students read and retell the situation (read and retell).
2. Have students generate questions that can be asked and answered given the problem situation (anticipate next).
3. Have students solve the problems that are generated.
4. Read the original question and have students solve it.

Ms. Collins selected the problem in Figure 4.13. Ms. Collins carefully selected the numbers and the multiplicative relationships in the task. Specifically, she chose whole-number quantities, a unit rate that was non-integral ($16\frac{2}{3}$ miles to the gallon) and a scale factor between 250 miles and 200 miles as non-integral ($\times \frac{4}{5}$). Study and solve the problem in Figure 4.13. What difficulties might students encounter? What strategies might students use?

Figure 4.13 Original word problem

> A car travels 250 miles on 15 gallons of gas. About how many gallons of gas will the same car use when traveling 200 miles?

Step 1: Remove the question from the word problem.
Ms. Collins took the original word problem and projected it without the question.

A car travels 250 miles on 15 gallons of gas. Another day, the same car travels 200 miles.

Step 2—Read and retell: Provide time for students to read and retell the situation to partners and to discuss as a full class.

Ms. Collins then had the students retell the story in their own words. As they generated information, she wrote the information on the board. As students retold the story, she anticipated the following facts would be identified.

- A car traveled 250 miles one day.
- The same car traveled 200 miles another day.
- When the car traveled 250 miles, it used 15 gallons of gasoline.

Step 3—Generate questions: Some possible questions that might be generated.

Ms. Collins asked her students what mathematical questions they could generate from this information. After working for a few minutes independently, they came up with the following list:

1. How much further did the car travel on one day than the other?
2. How many times further did the car travel on one day than the other day?
3. How far can the car travel using 5 gallons of gas?
4. How far can the car travel using 10 gallons of gas?
5. How far can the car travel using 25 gallons of gas?
6. How much gas is needed to travel 300 miles?
7. How much gas is needed to travel 125 miles?
8. How many gallons per mile were used on the day the car traveled 250 miles?

Step 4—Solve the problems generated by the class.

Because the goal was to have students focus on the quantities and the multiplicative relationships [the unit rate (in miles per gallon) and the scale factor between the original miles and the new mileage ($\times\frac{4}{5}$), Ms. Collins decided to have the students solve only those problems that moved them to that understanding. For example, scaling up and down mileage (questions 6–7) or quantity of gasoline used (questions 3–5) to see the impact on each other (e.g., traveling further means using more gasoline; traveling shorter distances means using less gas; using more gas means traveling a greater distance).

Show the original problem:

> A car travels 250 miles on 15 gallons of gas. About how many gallons of gas will the same car use when traveling 200 miles?

Step 5—Solve the original problem.

Ask, "Will the car use more or less gas to go 200 miles than 250 miles?" During the rest of the lesson, students engaged in other problems in which the unit rate is not explicitly stated. At the end of the lesson, Ms. Collins gave the following exit question. How is the problem similar to the problem at the beginning of the lesson? How is the problem different from the problem at the beginning of the lesson?

Figure 4.14 Exit question

> A car travels 280 miles on 12 gallons of gas on Monday. The same car uses 9 gallons of gasoline on Tuesday. About how far will the car travel on Tuesday?

Both problems involve gas mileage. Both problems involve non-integral multiplicative relationships. The main difference between the question used in the Day 4 lesson and the exit question is the nature of the quantities given in the problems and the missing quantities. The question used in the lesson requires students to determine the number of gallons, and the Exit question requires students to determine the number of miles. Ms. Collins made the decision to use this problem to see if the students could extend their understanding to this slightly modified problem situation.

Ms. Collins's case study provides insight into how a formative assessment cycle can be used in a classroom—including how different instructional strategies might be used based on evidence in student work. In summary, during the four days, Ms. Collins used the evidence from exit questions (analyzed evidence using the OGAP Sort), made observations during class and provided a warm-up problem to gather additional evidence to inform instruction. Based on the evidence she gathered during these four lessons Ms. Collins used four different instructional strategies to engage students in the targeted mathematics.

1. Incorporated findings into subsequent lessons
2. Collected additional evidence using a warm-up problem
3. Facilitated a discussion using student solutions
4. Used a reading strategy adapted to help students understand the math word problems

Of course, Ms. Collins might have engaged the students in the targeted mathematics in different ways. She could have provided actionable feedback to individuals or all students. She could have used small-group instruction.

> *The important point is that she used evidence of student thinking to make instructional decisions in an intentional and systematic way, both to maximize the benefits of using formative assessment and to engage students in the process.*

Ms. Collins was a seasoned user of formative assessment. With her knowledge of the OGAP Progression, she constantly had her eye on the evidence of student thinking and strategies that students were using during instruction. In addition, she collected formative information at the end of lessons to inform the next day's instruction and learning. If you are starting to use formative assessment, you might start by administering a couple of exit questions a week and trying out some of the instructional strategies listed in Table 4.1, building over time toward Popham and William's vision.

Table 4.1 provides brief rationales for decision making in regard to the levels of response and sample instructional strategies that may be helpful as you analyze evidence of student thinking and then make instructional decisions.

Table 4.1 Instructional response strategies based on analysis of evidence

Level of Response	*Potential Rationale*
No immediate action necessary	The class as a whole is ready for the next mathematics concept.
Incorporate findings into subsequent lessons	The instructional materials support further development of the concepts that students are struggling with or there is a common error students are making that can be incorporated into subsequent lessons (e.g., students are not labeling answers).
Take immediate action	Additional instruction is necessary before proceeding with new concepts.

Take Immediate Action
Plan Instruction and Implement for Full Class or Subset of the Students
*Strategies that can take up to 10–15 minutes of instructional time

Instructional Strategy	*Rationale*
Collect additional evidence	For a variety of reasons, you need to collect additional evidence of student understanding to understand students' thinking or strategies.
Provide actionable feedback	Regardless of which instructional strategy is used, students should be provided with feedback. It can be whole-class feedback or individual feedback. Students should be provided time to address the feedback.
Implement a mini-lesson*	There is an instructional issue that merits full-group discussion that focuses on a specific concept.
Facilitate a discussion using student solutions*	Select and sequence student solutions to focus on a specific aspect of a problem, the mathematics or errors that students are making or to extend thinking to a new concept or a higher level on the OGAP Progression.
Facilitate a discussion using a warm-up problem*	Can be used in a variety of situations (e.g., to gather more evidence at the beginning of a lesson; to bridge one concept to a new concept)

Table 4.1 Continued.

Instructional Strategy	*Rationale*
Engage students in making sense of word problems*	There are a variety of ways to engage students in understanding word problems. During the case study, this chapter illustrated one such reading-based strategy.
Reteach to all	None of the students have progressed based on the evidence; the instructional materials did not further student understanding or help them develop strategies. This does not mean repeating the same lesson. Rather, it means focusing on the same goal using a different instructional approach and different instructional materials.

Final Thoughts

This chapter provided an example of how a teacher can implement a *formative assessment cycle* that involves gathering evidence as students learn, analyzing the evidence and making instructional decisions based on the levels on the OGAP Ratio and Proportion Progression. *This is a continuous process that occurs during a lesson as students are working and engaging in discussions along with gathering evidence from each student at the end of a lesson (e.g., exit questions) for the sole purpose of informing instruction and student learning*. Table 4.1 contains examples of some ways teachers may respond to formative assessment evidence. Teachers may use instructional strategies identified in Table 4.1 or others from research or experience. Importantly, these responses take place *as students are learning* and *are informed by knowledge of the mathematics content and knowledge of mathematics education research*.

Chapters 3 and 4 focused on the OGAP Ratio and Proportion Progression and how the levels can be used to analyze evidence of student understanding using the OGAP Sort and inform instructional decisions. Chapters 5, 6 and 7 introduce readers to additional aspects of the mathematics education research on the impact different contexts and problem structures may have on students learning new concepts or solving related ratio and proportion problems. Chapter 9 attends to research on how students develop understanding, fluency, and flexibility with percents as well as common errors or misconceptions that may interfere with student developing understanding and fluency. In each of these chapters, you will find case studies that model potential instructional responses to different types of evidence discussed in these chapters.

Chapter Summary

- OGAP formative assessment is an intentional and systematic approach to using formative assessment based on understanding the OGAP Ratio and Proportion Framework and Progression.
- The OGAP Sort is a strategy to gather evidence of student understanding in relationship to the progression, including errors students make and accuracy of the solutions.

- While there are many instructional responses to evidence in student work, some teachers believe that the only response is small-group instruction. This chapter provides examples of potential instructional strategies to use based on the evidence of student thinking.

Looking Back

1. **Practice engaging students in making sense of word problems:** In this question, you will use the word problem strategy highlighted in this chapter. Read the train problem that follows and then respond to a–d.

   Train A travels from Newton to Derby, a distance of 310 miles. Train A's average speed is 60 miles per hour. Train B travels from Newton to Sudbury, a distance of 225 miles. Train B's average speed is 50 miles per hour. Which train reaches its destination in less time?

   a. (Read and retell) You will remove the question—Which train reaches its destination in less time?—and have your students read only the first four factual sentences. Generate a list of the facts that you hope students will identify in the problem when they retell the story.
   b. (Generate questions—anticipate next) Ask your students, "What questions could we answer given this story?" Make a list of five questions that you anticipate your students might ask.
   c. (Solving the problems generated by the class) Describe different ways you can use the questions that you anticipate students will generate.
      i. Are there some questions you would not assign to your students? Why?
      ii. Are there some questions you might assign to some students but not all students?
      iii. Given the list that you generated, would you anticipate letting students select the questions to answer? Why or why not?
   d. (Solving the original problem) Anticipate strategies that students might use after you unveil the original question.

2. **Try the word problem strategy discussed in this chapter with your class** using a problem from your mathematics program/textbook materials.
   a. Choose a problem. Why did you choose this problem?
   b. Try the process with your class using the problem that you selected.
   c. Write a brief summary about how the strategy worked, including things you might modify the next time you use the strategy.

1. **Practice using the OGAP Ratio and Proportion Progression to inform instruction:** The OGAP Sort refers to the process of understanding the evidence in student solutions and determining the level of the progression that the evidence in the solution matches best. Sorting student work is the first step in using evidence to inform instruction and is described in this chapter.

Analyze the student work in Figures 4.15 through 4.18 using the OGAP Ratio and Proportion Progression. Record your analysis on a copy of the evidence collection sheet in Figure 4.19 as was modeled in this chapter.

a. For each solution, identify:
   i. The level on the progression at which the evidence is found. What is the evidence?
   ii. Any underlying issues or concerns.
   iii. Highlight/circle any solutions that are incorrect.
b. Assuming that these pieces of work are representative of the student work across the classroom, answer the following.
   i. What understandings are evidenced in the solutions that can be built upon? What is the evidence?
   ii. What mathematical concepts need to be further developed? What is the evidence?
c. How might you use this work to engage the class in a discussion that will strengthen the underlying mathematics?
   i. If you were to select and sequence student solutions from this group of solutions, which pieces of work might you use, and how would you sequence it? Explain your decisions.
   ii. Based on the pieces of work you selected, make a list of five questions that you could ask your students about these solutions. Explain why you chose these questions.

Problem: Paul's dog eats 20 pounds of food in 30 days. How long will it take Paul's dog to eat a 45-pound bag of dog food? Show your work.

Figure 4.15 Alex's response

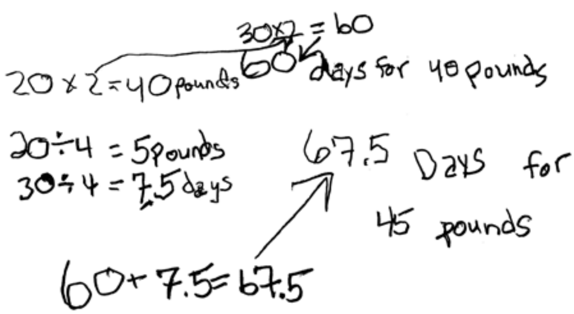


Figure 4.16 Margo's response

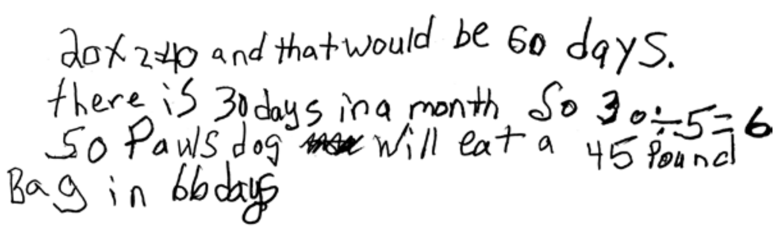


Figure 4.17 Antonia's response

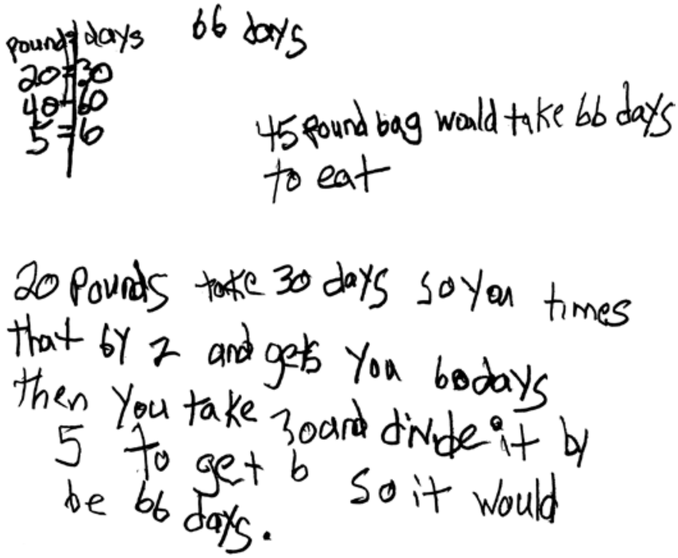
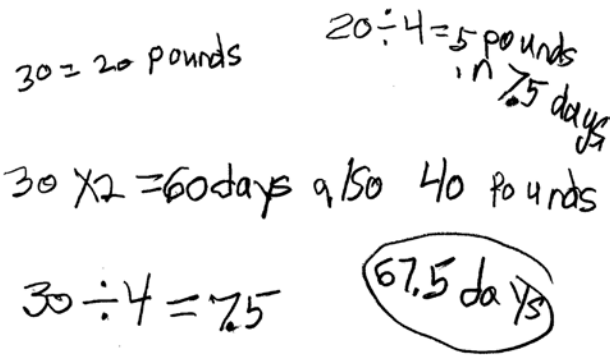


Figure 4.18 Mia's response



4. **Try this in your class:** In Question 3, you practiced analyzing student solutions using the OGAP Ratio and Proportion Progression and the OGAP evidence collection sheet. Most importantly, you considered instructional implications in light of the evidence you collected.
   a. Try this process with your students. Follow these three steps.
      i. Design or select a ratio or proportion question based on the mathematical goal of the upcoming lesson.
      ii. Administer the question as an exit question at the end of the lesson.

Item #	Problem Structures	Proportional	Transitional Proportional		Early Ratio	Non-proportional
			Transitional	Early Transitional		
1						

Uses additive strategy to find multiplicative relationship:	Remainder error:	Error in equation:	Computational error:
Misinterprets the meaning of quantities:	Missing or inconsistent use of units:	Rounding error:	Other:

Figure 4.19 OGAP student work evidence collection sheet

iii. Analyze your students' responses and record the information on a copy of the OGAP evidence collection sheet shown in Figure 4.19.

b. Use the evidence you collected in 4a to answer these questions:
   i. What are some developing understandings you noticed in the solutions that can be built upon in future lessons?
   ii. What are some underlying issues or concerns across your class that future lessons should address?
   iii. What are some implications for instruction?
   iv. Make a decision about an instructional intervention:
      a. Describe the targeted mathematics and the level on the progression. Use guidance from the arrow on the right side of the progression to help you make a decision about next steps.
      b. What level of response does the evidence suggest? Provide a rationale for that level of response.
      c. Select an instructional strategy to use if the evidence suggests that you take immediate action. Use Table 4.1 as a guide.

Instructional Link

Use the questions that follow to analyze the ways that your math instruction and program provides students with opportunities to build fluency and understanding of important ratio and proportion concepts and skills.

1. To what degree do your math instruction and program focus on regularly gathering descriptive information about student learning to inform your instruction?
2. In what ways does your math instruction use strategies found in Table 4.1 to respond to evidence in student work?
3. Based on this analysis, identify specific ways you can enhance your math instruction by utilizing ideas from this chapter.

5
Structures in Proportion Problems

Big Ideas

- The structure of a ratio and proportion problem can impact its difficulty and thus influence a student's choice of solution strategy.
- Flexibility when solving ratio and proportion problems can be increased by varying the structures in problems, which facilitates students' conscious analysis of the relationships and quantities in the problem situations.
- Your knowledge of these structures will impact your ability to intentionally and systematically select and/or design questions for your students that vary the structures in the problems in a purposeful way.

Introduction to Problem Structures in Ratio and Proportion Problems

This chapter is the first of three chapters that focus specifically on problem structures that influence the difficulty of ratio and proportion problems. Problem structures refer to how the problems are built—that is, how the features of problems are organized and interact with each other (Hulbert et al., 2017). Figure 5.1 provides a list of the structures that researchers have found that can impact the difficulty of ratio and proportion problems and the strategies that students use to solve them. Take a minute and study Figure 5.1. Which terms are familiar? Which are unfamiliar?

Figure 5.1 Problem structures in ratio, rate and proportion problems

- Number
- Multiplicative relationships within and between ratios
- Internal structure of proportion situations
- Ratio relationships
- Ratio referents
- Language and notation
- Representations
- Problem types
- Problem contexts

Some of these structures were introduced in earlier chapters, like the important idea of using different representations to build an understanding of the multiplicative relationships in ratio and proportion problems (Chapter 2), how the language in ratio and rates problems can be confusing for students (Chapter 1), and an introduction to ratio relationships and ratio referents (Chapter 3). There will be additional discussion of ratio referents in Chapter 6. Problem types and contexts will be discussed in Chapters 6 and 7.

Before reading this chapter any further, look at the front page of the OGAP Proportionality Framework. There you will find more detail about each of the structures listed in Figure 5.1. The quote at the top of the framework, referred to in previous chapters and shown here describes the impact that modifying structures in ratio and proportion problems can have on student learning.

> Depending upon the strength of proportional reasoning students may move back and forth between using proportional, transitional, and non-proportional strategies as they interact with different problem structures and problem contexts.
>
> (Cramer et al., 1993; Karplus, Pulos, & Stage, 1983; OGAP, 2006, 2007)

Ultimately, a proportional reasoner should not be influenced by context, problem types, the quantities in the problems and their associated units, numerical complexity, or other problem structures (Cramer et al., 1993; Silver, 2006). While that is the goal, students may struggle with these structures as they are learning ratio and proportion concepts and skills.

This chapter focuses specifically on the impact that the following structures have on both problem difficulty and the strategies students use to solve problems.

- Multiplicative relationships within and between ratios in ratio and proportion situations
- Number types
- Internal structure of proportion problems

Multiplicative Relationships Within and Between Ratios in Ratio and Proportion Situations

To begin to understand these ideas, solve the two different versions of the Bob's Shower problem in Figure 5.2. How are these two problems similar and different? Based on these differences, which is easiest? Which is most difficult? What strategies do you expect students to use for each problem and what difficulties might they encounter?

Figure 5.2 Bob's Shower Problems 1 & 2

> Problem 1: Bob's shower uses 18 gallons of water every 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?
>
> Problem 2: Bob's shower uses 18 gallons of water every 3 minutes. How many gallons of water does Bob use if he takes a 12-minute shower?

You probably noticed a number of important similarities and differences in these problems. Both problems are based on the context of using water when showering. Both problems involve the ratio of 18 gallons per 3 minutes or the unit rate of 6 gallons per minute. Here is where the similarities end. Problem 1 involves a 13-minute shower compared to 12 minutes in Problem 2.

Researchers find that the type of number (e.g., whole number, fraction, decimal) in a problem and the type of multiplicative relationship (integral vs. non-integral) between the quantities in the problem impacts the difficulty of problems and the strategies that students use to solve the problems (Artut & Pelen, 2015; Steinthorsdottir, 2006; Karplus et al., 1983)

This research helps us better understand why Problem 1 may be more difficult for students than Problem 2. That is, the multiplicative relationship between a 3-minute shower and a 13-minute shower is non-integral (i.e., not a whole number), while the multiplicative relationship between a 3-minute shower and a 12-minute shower is integral (i.e., a whole number). This idea was discussed briefly in Chapter 4. This intuitive sense of the difficulty is supported by research that shows that when multiplicative relationships in a proportional situation are integral, it is easier for students to solve than when they are non-integral (Cramer, Post & Currier, 1993; Karplus et al., 1983; Ongoing Assessment Project (OGAP), 2005, 2007).

To illustrate this point, compare the multiplicative relationships in Bob's Shower Problems 1 and 2. Let's begin by seeing the two ratios in Problem 2 as two different but related situations. The first is the ratio of 18 gallons to 3 minutes. The second within the context of this problem is the related but different situation, the ratio of *x* gallons to 12 minutes. Notice that the multiplicative relationship of minutes to gallons *within* the ratio $\frac{18 \text{ gallons}}{3 \text{ minutes}}$ is integral (× 6). Notice also that the multiplicative relationship of 3 minutes to 12 minutes *between* the ratios is also integral (× 4). Thus, the number of gallons it takes to complete a 12-minute shower can be determined using either of the relationships. That is, 6 gallons/minute × 12 minutes = 72 gallons, or 4 × 18 gallons = 72 gallons. Figure 5.3 illustrates the two integral multiplicative relationships in Problem 2.

Figure 5.3 Within and between multiplicative relationships in Bob's Shower Problem 2

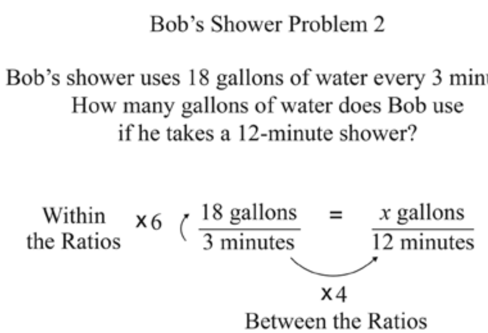


Figure 5.4 shows the multiplicative relationships in Problem 1. Notice that the multiplicative relationship *within* the ratio, $\frac{18 \text{ gallons}}{3 \text{ minutes}}$, is integral (× 6). However, the multiplicative relationship *between* the ratios, 3 minutes to 13 minutes, is not integral $(\times 4\frac{1}{3})$.

This one difference in the structure of the problem (i.e., integral versus non-integral relationship) is significant because it impacts the difficulty level of each problem and the strategies students use to solve them. Study the student solutions to Bob's Shower Problem 1 in Figure 5.5. In what ways does the non-integral relationship impact these solutions?

Figure 5.4 Within and between multiplicative relationships in Bob's Shower Problem 1

Bob's Shower Problem 1

Bob's shower uses 18 gallons of water every 3 minutes.
How many gallons of water does Bob use
if he takes a 13-minute shower?

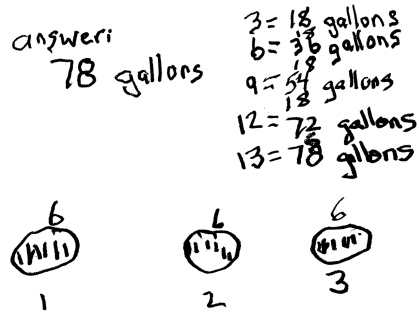
Within the Ratios $\times 6$

$$\frac{18 \text{ gallons}}{3 \text{ minutes}} = \frac{x \text{ gallons}}{12 \text{ minutes}}$$

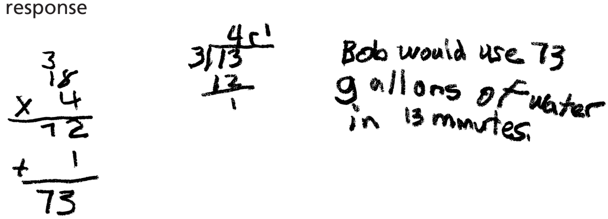
$\times 4\,{}^{1}/_{3}$
Between the Ratios

Figure 5.5 Bob's shower uses 18 gallons of water every 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?

Sam's response



Ted's response

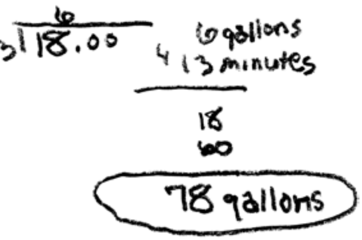


As you probably noticed, Sam solved the problem by building up the ratio 18 gallons in 3 minutes using addition and then *dealing out* 18 gallons to find how many gallons are used in 1 minute. On the other hand, there is evidence in Ted's solution that he recognized the multiplicative relationship between gallons and minutes but did not show evidence of understanding the meaning of the remainder in the situation. Instead of identifying the multiplicative relationship between gallons and minutes as $\times 4\frac{1}{3}$, he saw it as $\times 4+1$.

While we do not know if Sam would have seen this problem as multiplicative if the multiplicative relationships within and between the quantities in the ratios were integral, research has consistently shown that when the multiplicative relationships in ratio and proportion situations are non-integral, students often either retreat to additive strategies or make calculation errors when using fractions, as was evidenced in Ted's solution (Artut & Pelen, 2015; Steinthorsdottir, 2006; Karplus et al., 1983; Ongoing Assessment Project, 2006, 2007). Solutions like these and others seen throughout this book are clear evidence of the important role procedural fluency plays in a student's success when working with ratio and proportional relationships.

In Figure 5.6, Catherine used the multiplicative relationship between the quantities in the ratio $\frac{18 \text{ gallons}}{3 \text{ minutes}}$ to find and apply the unit rate of 6 gallons/minute. Study Catherine's response.

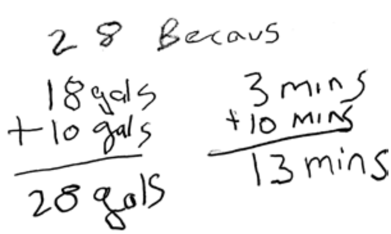
Figure 5.6 Catherine's response to Bob's Shower Problem 1, the 13-minute shower



In work with teachers, OGAP facilitators found that some teachers taught students to set up a proportion in a missing value problem and use the multiplicative relationship in one prescribed way rather than first making sense of the contextual situation by a *conscious analysis of the relationships* between the quantities in the problem (Lamon, 2007). While we do not know if Catherine made a conscious decision to use the integral rather than the non-integral relationship in the problem, one can see how a careful analysis of the relationships between the quantities in the problem could have helped Ted choose a more effective strategy.

In Chapter 1, there was a discussion of the difficulty some students have when moving from additive situations in elementary school to multiplicative situations in their study of ratios and proportions. Kaitlyn's solution in Figure 5.7 exemplifies this retreat to an additive solution. In this case Kaitlyn determined that 3 minutes + 10 minutes = 13 minutes and then added 10 gallons + 18 gallons = 28 gallons. In the Looking Back section at the end of this chapter, you will return to Kaitlyn's solution to think about ways you can help her better understand the quantities and relationships in the problem.

Figure 5.7 Kaitlyn's solution to Bob's Shower Problem 1, the 13-minute shower, using additive reasoning and not multiplicative reasoning



Case Study: Collecting additional evidence by changing problem structures and using student solutions

The solutions to the Bob's Shower problem provide a wealth of information to inform instruction, particularly because one of the multiplicative relationships was integral and the other was non-integral. An effective approach to inform instructional decisions involves both carefully analyzing the evidence in student solutions and recognizing the structure of the problem that may have influenced the students' strategic choices.

Mr. King, a seventh-grade math teacher, chose a couple of instructional responses to address the evidence in the student work. First, he wondered if changing some structures in the problem would allow him to gather clearer evidence of his students' understanding.

He decided first to study Sam's solution and other solutions like Sam's, considering both the structure of the problem and the strategies that the students used.

Some things Mr. King observed in solutions like Sam's that can be built upon to inform instruction:

- Some students used the integral multiplicative relationship between gallons and minutes $(\times 6)$.
- Most students effectively built up the ratio 18 gallons in 3 minutes to find the number of gallons used in 12 minutes.
- Some students effectively dealt out the 18 gallons into three 1-minute groups to determine the number of gallons used in 1 minute (the unit rate) while other students ignored the extra minute or dealt with it incorrectly (e.g., Ted's solution, Figure 5.5).
- Some solutions were correct and others were not.

Modifying the problem structure

Mr. King created two new problems by slightly modifying the structure of the original problem. In one problem, Mr. King increased the length of the shower and kept all other aspects of the original problem the same. In the second problem, Mr. King increased the length of the shower and included the unit rate in the text of the problem. To justify these new problems, Mr. King told his students that Bob's mother wanted him to understand the impact of his long showers on water usage.

Problem 1 rationale: Increase the magnitude of one of the numbers to make building up a less attractive approach to see if students use a more efficient strategy.

Modified problem: Bob's shower uses 18 gallons of water every 3 minutes. How many gallons of water does Bob use if he takes a 43-minute shower?

Problem 2 rationale: Change the ratio (18 gallons to 3 minutes) to a unit rate to see if students apply the unit rate directly or still use a building-up strategy.

Modified problem: Bob's shower uses 6 gallons of water for every minute. How many gallons of water does Bob use if he takes a 43-minute shower?

He decided to use one or both of these problems as an exit card. He wasn't sure if he would have some students solve the first problem and others the second problem or have everyone solve one of these problems. He would make the decision as the lesson progressed.

Facilitating a discussion using student solutions to advance understanding

At the beginning of the next class, Mr. King decided to use Sam's and Catherine's solutions in Figures 5.5 and 5.6 to focus on the relationship between Catherine's unit rate solution and Sam's building-up and dealing-out strategy.

Mr. King used Sam's and Catherine's solutions and the following questions to facilitate a full-group discussion.

a. How are the solutions alike?
b. How are they different?
c. What do the sketches mean in Sam's solution?
d. How do you think Sam used the information in the sketches?
e. How did Catherine come to her conclusion?
f. What assumptions can you make about Catherine's thinking?

You will have an opportunity in the Looking Back section to apply this analysis and instructional decision-making to other examples to practice implementing this type of strategic analysis in your classroom.

An OGAP Study—Integral Versus Non-Integral Multiplicative Relationships

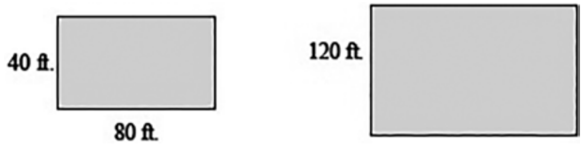
To better understand the impact of the number type and integral versus non-integral relationships on student solutions, OGAP conducted a study in which 153 middle school students solved the three problems in Figure 5.8 on different days during the same week: Problem 1 on Monday, Problem 2 on Wednesday and Problem 3 on Friday. No additional instruction on the topic was given between Monday and Friday. Solve all three problems in Figure 5.8. How are these problems similar, and how are they different? How might the differences in these problems impact student success?

Figure 5.8 Three pilot problems

Pilot 1

A school is enlarging its playground. The dimensions of the new playground are proportional to the dimensions of the old playground. What is the length of the new playground?

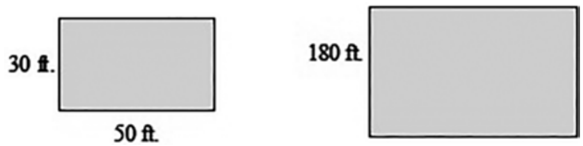
Old Playground New Playground (not drawn to scale)



Pilot 2

A school is enlarging its playground. The dimensions of the new playground are proportional to the dimensions of the old playground. What is the length of the new playground?

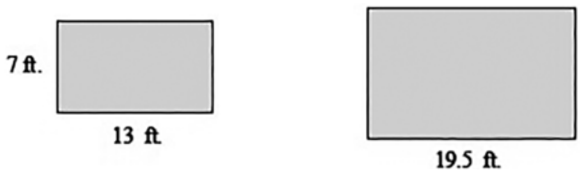
Old Playground New Playground (not drawn to scale)



Pilot 3

A school is enlarging its garden. The dimensions of the new garden are proportional to the dimensions of the old garden. What is the length of the new garden?

Old Garden New Garden (not drawn to scale)



All three problems utilized the identical situation of scaling up a rectangular structure (garden or playground). The problems differ in the nature of the multiplicative relationships between the dimensions of the old and new figures. Note also that Pilot 3 had different dimensions missing than Pilots 1 and 2. In addition, one of the dimensions in Pilot 3 included a decimal. Study Table 5.1.

Table 5.1 Multiplicative relationships in Pilots 1–3

	Multiplicative Relationships Between Ratios and Within Ratios		
	Relationships	*Within*	*Between*
Pilot 1	Both integral	$40\text{ feet} \times 2 = 80\text{ feet}$	$40\text{ feet} \times 3 = 120\text{ feet}$
Pilot 2	One non-integral	$30\text{ feet} \times 1\frac{2}{3} = 50\text{ feet}$	$30\text{ feet} \times 6 = 180\text{ feet}$
Pilot 3	Both non-integral	$7\text{ feet} \times 1\frac{6}{7} = 13\text{ feet}$	$13\text{ feet} \times 1\frac{1}{2} = 19.5\text{ feet}$

The data in Table 5.2 show the results of the analysis of the student responses from this OGAP Study. One can see that students were overall more successful when the multiplicative relationship between and within the ratios was integral. And not surprisingly, the students had the least success when both multiplicative relationships were non-integral.

Table 5.2 OGAP 2006 Pilot (n = 153)

	Multiplicative Relationships Between Ratios and Within Ratios	*Percent Correct Responses*
Pilot 1	Both integral	80%
Pilot 2	One non-integral and one integral	65%
Pilot 3	Both non-integral	35.5%

Study Kaitlyn's responses in Figure 5.9 to the three pilot questions. While there is evidence that Kaitlyn used and applied the multiplicative relationship between the lengths of the old and new playground for Pilots 1 and 2, her solution in Pilot 3 shows evidence of reverting to an additive strategy. These findings are consistent with findings of several other researchers: that students tend to revert to an additive strategy when faced with non-integral relationships (Artut & Pelen, 2015).

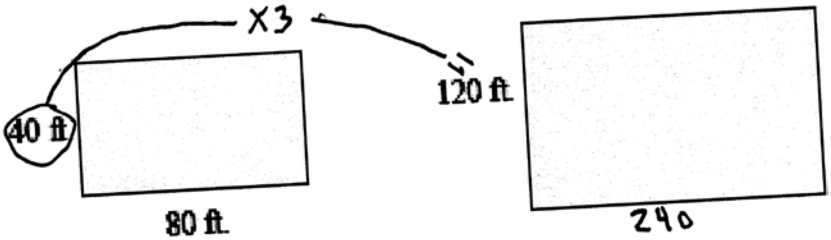
Figure 5.9 Kaitlyn's responses in Pilots 1, 2 and 3 with evidence of identifying and applying the multiplicative relationships in Pilots 1 and 2 while reverting to an additive relationship in Pilot 3

Pilot 1

A school is enlarging its playground. The dimensions of the new garden are proportional to the dimensions of the old garden. What is the length of the new playground?

Old Playground New Playground (not drawn to scale)

Figure 5.9 Continued.

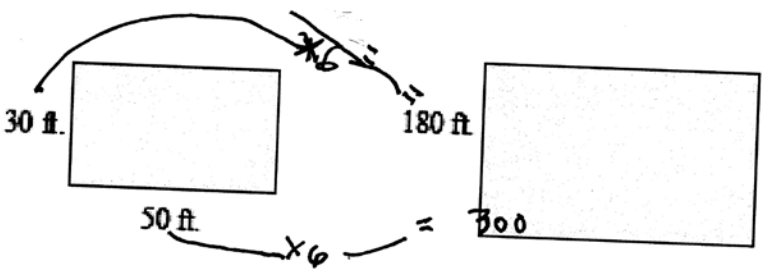


If 40x3= 120 the length of the playground would be 240

Pilot 2

A school is enlarging its playground. The dimensions of the new playground are proportional to the dimensions of the old playground. What is the length of the new playground?

Old Playground New Playground (not drawn to scale)



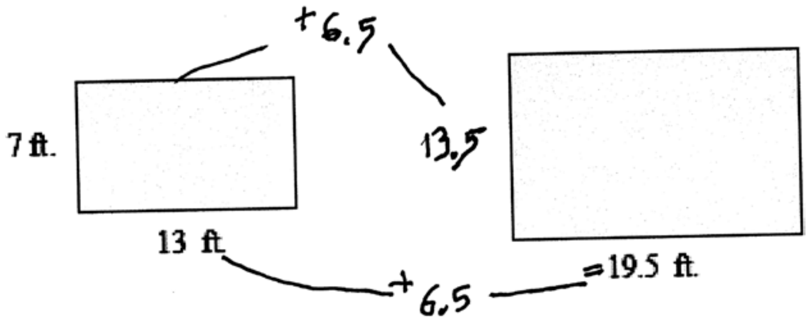
30x6= 180 So you need to X 50x6 that Would= 300

Pilot 3

A school is enlarging its garden. The dimensions of the new garden are proportional to the dimensions of the old garden. What is the length of the new garden?

Old Garden New Garden (not drawn to scale)

Figure 5.9 Continued.



If you add 6.5 to each one you should come up with an answer.

Note also that none of Kaitlyn's solutions included units to define the quantities in the problem. This issue can be a focus of instruction over time to reinforce the idea that units are an important aspect of ratio and proportional situations.

The Pancake Problem—*Impact of Number Type*

Up to this point you have examined the impact that the multiplicative relationships within and between ratios can have on problem difficulty and student solutions. The Pancake problem in Figure 5.10 integrates a number of other structures that can also impact problem difficulty and student solutions. Solve the Pancake problem in Figure 5.10. Besides the non-integral relationships within and between the quantities in the problem, what other structures might influence the difficulty of the problem? What strategies might students use to solve the problem?

Figure 5.10 The Pancake problem

Use the data in the table below to determine the number of cups of pancake mix to make 400 pancakes.

	Pancake Recipe				
Number of Pancakes	**12**	**24**	**36**	**120**	**400**
Cups of Pancake Mix	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	$17\frac{1}{2}$	
Cups of Milk	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	$12\frac{1}{2}$	

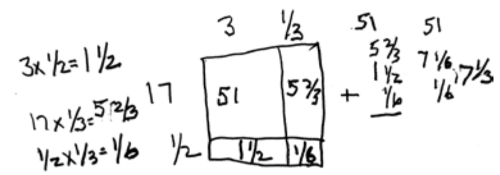
A number of factors make this situation difficult for some students. For one, the multiplicative relationships within the ratios (e.g., number of pancakes to cups of mix) and between the ratios are all non-integral. In addition, you probably noticed that the ratios are organized in a table, and the quantities are both whole numbers and fractions. While the recipe context is familiar to most students, the addition of the table and fractional values along with the non-integral multiplicative relationships within and between the quantities in the ratios make this problem challenging as students solidify their proportional reasoning. To be successful with this problem, students have to make sense of the contextual situation and the quantities by a *conscious analysis of the relationships* (Lamon, 2007) in the problem and be fluent when using multiplicative relationships involving fractions.

Now study the two solutions in Figure 5.11. What do you notice about the impact of the structures in this problem on the strategies students used? What is the evidence that these students made conscious decisions about the relationships in the problem?

Figure 5.11 Kelyn's and Asher's responses to the Pancake problem

Kelyn's response

	Pancake Recipe				X 3⅓
Number of Pancakes	12	24	36	120	400
Cups of Pancake Mix	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	$17\frac{1}{2}$	58⅓
Cups of Milk	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	$12\frac{1}{2}$	



Asher's response

	Pancake Recipe				
Number of Pancakes	12	24	36	120	400
Cups of Pancake Mix	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	$17\frac{1}{2}$	58.33
Cups of Milk	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	$12\frac{1}{2}$	

$1\frac{3}{4} \div 12 =$

$12 \div 12 = 1$

$1\frac{3}{4} \div 12 = 0.1458333$

$1 \times 400 = 400$

$0.145833 \times 400 = 58.33$ cups of mix

Both solutions are correct, and there is evidence that Asher and Kelyn made conscious decisions on how to deal with the fractions as well as with the non-integral multiplicative relationships in the problem. Kelyn found and applied the multiplicative relationship between 120 pancakes and 400 pancakes ($\times 3\frac{1}{3}$) to determine the number of cups of mix needed to make 400 pancakes ($17\frac{1}{2} \times 3\frac{1}{3} = 58\frac{1}{3}$ cups of mix). Asher used a similar approach but found and applied the multiplicative relationship between 12 pancakes and 1¾ cups of mix ($400 \times 0.1458333 = 58.33$ cups of mix). The big difference between Asher's and Kelyn's solution is Kelyn used fractions, and Asher probably relied on a calculator for his decimal calculations. Importantly, however, both students found a strategy to effectively deal with the non-integral relationships in the problem by finding and applying a multiplicative relationship.

The next section includes a discussion of the importance of providing students with opportunities that facilitate a mindset of analyzing problem situations intentionally by altering yet another structure in proportion problems.

Internal Structures of Missing Value Proportion Problems

Bob's Shower, the Playground, and the Pancake problem are all missing value proportion problems. That is, three values in a proportion are given and the fourth is to be determined (e.g., we knew the gallons used for a 3-minute shower but not the gallons used for the 13-minute shower). As we have seen, the nature of the multiplicative relationships within and between the ratios can influence the difficulty of the problems as well as the strategies students use to solve the problems. Study Problems 2a and 2b that follow. How are they alike, and how are they different? Which of the problems do you think is more challenging for students? What is the evidence?

Problem 2a: Bob's shower uses 18 gallons of water every 3 minutes. How many gallons of water does Bob use if he takes a 12-minute shower?

Problem 2b: Bob's shower uses 18 gallons of water every 3 minutes. If Bob takes a 12-minute shower, how many gallons will he use?

Problems 2a and 2b above are similar in that the multiplicative relationships between the ratios (× 4) and within the ratios (× 6) are the same and both integral. According to the research already discussed, this makes these problems equally challenging. However, the difference between these two problems is significant. Researchers indicate that the location of the missing value in the problem may affect performance (Harel & Behr, 1989). Figure 5.12 provides an example of the difference between parallel structures where the quantities are in the same order in the prompt as they are in the question, and a non-parallel structure where the quantities are not stated in the same order.

Solve the problem in Figure 5.13. Is the structure of the problem parallel or non-parallel? In what ways might the location of the missing value in the problem influence the student solutions?

What is notable in Liam's response is his work showed evidence of effectively using fractions as well as using the multiplicative, not additive, difference between quantities albeit the wrong quantities. That is, Liam may have disregarded the meaning of quantities, using just the order in which the quantities were presented in the prompt as cues to set up a proportion. His answer was in miles not gallons. In addition, it appears that Liam did not consider the reasonableness of his answer (4200 miles) given that for 240 miles, 12 gallons of gas were used. He also used 210 miles not 200 miles indicated in the problem.

Figure 5.12 Internal structure of Bob's Shower Problems 2a and 2b

Parallel Structure

Bob's shower uses *18 gallons of water* every *3 minutes*

How many *gallons of water* does Bob use if he takes a *12-minute shower*?

Non-parallel Internal Structure

Bob's shower uses *18 gallons of water* every *3 minutes*

If Bob takes a *12-minute shower*, how many *gallons of water* will Bob use?

Figure 5.13 Liam's response to a gallons-of-gas problem

> A car travels 240 miles on 12 gallons of gas. About how many gallons of gas will the same car use when traveling 200 miles?

$$\frac{240}{12} = \frac{n}{210}$$

$$\times 17\tfrac{1}{2}$$

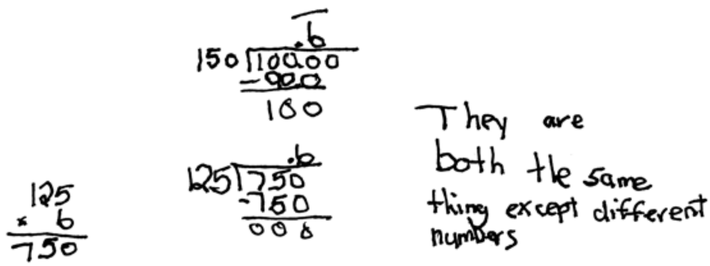
$$240 \times 17\tfrac{1}{2} = 4{,}200 \text{ mi.}$$

The non-parallel structure can also impact ratio comparison problems. Study the Big Horn Ranch problem in Figure 5.14 and Alec's response to the problem. In what way did the non-parallel structure influence Alec's response? What is the evidence?

Figure 5.14 Alec's solution to the Big Horn Ranch problem

> Big Horn Ranch raises 100 horses on 150 acres of pasture. Jefferson Ranch raises 75 horses on 125 acres of pasture.
>
> Which ranch has more acres of pasture per horse?

Figure 5.14 Continued.



There is evidence in Alec's solution that he compared horses per acres instead of acres per horse and interpreted his results incorrectly, indicating that 0.6 is equal to $0.\overline{6}$. These are common errors, particularly when the quantities in the problems students regularly encounter are always presented in the same order. Take a minute and look through the sets of problems in the instructional materials that your students experience to see if the structures having to do with multiplicative relationships or internal structure differ, forcing students to make a conscious analysis of the problem situation rather than just practicing responses to problems in similar structure.

Chapter 6 focuses on structures within ratio and rate problems. While the problem examples in this chapter were proportion problems, the structures discussed in this chapter (number type, multiplicative relationship between the quantities in ratio situations) also apply to ratio and rate problems.

Engineering Problems

Earlier in this chapter, you read two examples of how one might use the evidence in students' solutions while simultaneously considering the influence of the structure of the problem on the strategy that students used or the errors that they made. Teachers have told OGAP facilitators that understanding the structures in problems has helped them to:

1. Anticipate the kinds of solutions that students might generate and the challenges they might experience.
2. Consider and plan responses to students' solutions.
3. Select problems based on the goals of a lesson.
4. Balance the types of problems and the structures in problems that students encounter.
5. Analyze instructional materials to assure that students interact with a variety of structures.

"Engineering problems by itself will not do all the work of strengthening student understanding of ratios and proportions, but carefully selected problems in combination with classroom discussion can help develop students' conceptual understanding and multiplicative reasoning" (Hulbert et al., 2017, p. 123).

Chapter Summary

- The structure of a problem may influence the strategy that students use to solve the problem.

- Structures in a problem influence its difficulty.
- Flexibility when solving ratio and proportion problems can be increased by varying the structures in problems.
- Your knowledge of these structures will impact your ability to intentionally and meaningfully select and/or design problems.

Looking Back

1. Study each of the five problems found in Figure 5.15.
   a. Describe the different structures (types of numbers, multiplicative relationships, format of the problems, and internal structures).
   b. Based on the structures:
      i. Describe a couple of difficulties that students might encounter.
      ii. Provide a couple of examples of strategies students might use to solve the problem.
      iii. Identify errors students might make or difficulties they may have based on the structure of the problem.

Figure 5.15 Five examples

Example 1: Donna runs at an average rate of 12 minutes per mile. At this rate, how many miles will Donna run in 28 minutes?

Example 2: Carrie is packing apples. It takes 3 boxes to pack 2 bushels of apples. How many boxes does Carrie use to pack 7 bushels?

Example 3: Mia is cooking $2\frac{1}{2}$ cups of dry rice. Use the information in the table to determine how much water she should add to $2\frac{1}{2}$ cups of dry rice.

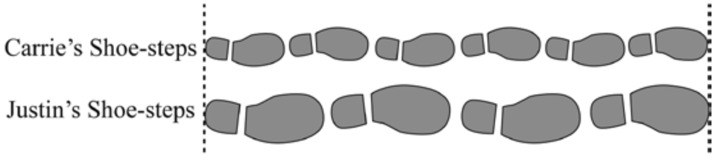
Cups of Dry rice	Cups of Water
1	1¾
2	3½
3	5¼

Example 4: Holly has a photograph that has a width of 8 inches and a length of 10 inches. If she enlarges the photograph so that the width is 10 inches, what is the length of the enlarged photo?

Example 5: Carrie measured the distance from the front door to the flagpole by placing one shoe in front of the other. Carrie counted 27 shoe-steps from the door to the flagpole. Use the diagram below that shows the ratio of Carrie's shoe-steps

Figure 5.15 Continued.

to Justin's shoe-steps to determine the number of shoe-steps Justin will make to walk the same distance.



2. In the next series of questions, you will be asked to analyze the evidence in student work like you did earlier in the chapter with Sam's responses to the Bob's Shower problem. When providing examples of next intentional steps, be specific (use strategies from Table 4.1 or others of your own). For example, if "collecting additional evidence" by modifying the problem is one strategy, provide an example of how you could modify the problem. If "asking additional questions" is a part of your strategy given the evidence and the structures in the problem, provide examples of questions with rationales as to why they are important questions to ask.
   A. Review the evidence in Catherine's response in Figure 5.6. Even though her solution was correct and a Proportional Strategy (finding and applying a unit rate), it is still important to consider the next instructional step.
      i. What solution strategy is evidenced in Catherine's response?
      ii. Where along the OGAP Progression is the evidence on Catherine's response?
      iii. Identify additional evidence you might collect to help you challenge Catherine.
      iv. What additional information about Catherine's understanding of proportions could be used to challenge Catherine?
      v. Provide three potential instructional moves/responses based on this analysis. Be specific.
   B. Review the evidence in Kaitlyn's response in Figure 5.7.
      i. What solution strategy is evidenced in Kaitlyn's response?
      ii. Where along the OGAP Progression is the evidence on Kaitlyn's response?
      iii. Identify additional evidence you might collect to help her teacher provide appropriate next instructional steps.
      iv. Provide three potential instructional moves/responses based on this analysis. Be specific.

3. Study the problem that follows. What structures may have influenced Margo's solution to the problem? Provide three potential next instructional steps that might help Margo understand the quantities in the problems.

Figure 5.16 Margo's response

Paul's dog eats 20 pounds of dog food in 30 days. How long will it take Paul's dog to eat 45 pounds of dog food? Show your work.

Figure 5.16 Continued.

$$\frac{20\text{ lb.}}{30\text{ days}} = \frac{45\text{ lb.}}{65\text{ days}}$$

$$65 = 30 \times 2 + 5$$
$$45 = 20 \times 2 + 5$$

Instructional Link

1. To what degree do you or your instructional materials provide a range of opportunities for students to engage in proportion problems with different number types, integral and non-integral multiplicative relationships and parallel and non-parallel structures?
2. Based on this analysis, what adjustments do you need to make to your instructional materials to assure students are solving problems that vary in structure so that students are encouraged to make conscious decisions about the relationships of the quantities in problems?

6
Structures in Ratio and Rate Problems

Big Ideas

- Two basic structural elements, *problem contexts* and *problem types*, may appear similar but differ in important ways.
- Ratio and proportion problems can have the *same problem context* while structured as *different problem types* with differing mathematical demands.
- Ratio relationships can be *part-to-part* or *part-to-whole*. These relationships can be *explicitly* stated or *implied* from the information provided in a problem.
- Unit rates are ratios that apply to more than just one situation and can be extended to a whole range of situations.
- Buying and consumerism problems are generally easier for students than problems involving constant speed and density.

Chapter 5 introduced the idea that problem structures impact the difficulty of ratio and proportion problems and the strategies students use to solve problems. Two structures briefly discussed in Chapter 5 are problem contexts and problem types. Problem contexts and problem types may sound similar, but they differ in important ways. To begin to understand how they differ, solve the problems in Figure 6.1. How are these problems alike? How are they different?

Figure 6.1 Problem contexts and problem types

Problem 1: Sally walks 2 miles every day. It takes her 30 minutes. Ashley walks 5 miles every day, and it takes her 2 hours. Who walks faster?

Problem 2: Donna runs at an average rate of 6 miles per hour.
At this rate, how many miles can she run in 2 hours?

Problem 3: Tanya walks 3 miles in 30 minutes. If Tanya maintains this rate, how many miles can she walk in 60 minutes?

Even though all three problems involve the *same context*, constant speed, each represents a *different problem type* with differing mathematical demands that provide students opportunities to apply different strategies. How these problems differ is explained in Figure 6.2.

Figure 6.2 Three problem types and features that distinguish them from each other

> Problem 1 is a *rate comparison problem*. In this example, the distance and time are given for both walkers. To solve this problem, one has to compare the ratios and choose the one that indicates a faster speed.
>
> Problem 2 is a *unit rate problem*. That is, the unit rate of 6 miles per hour is one of the quantities given. To solve this problem, one has to use the unit rate as well as the amount of time stated in the problem to determine how far one might run in a different amount of time.
>
> Problem 3 is a *missing value problem*. That is, three quantities are given (in this case two amounts of time and one distance), and the student is asked to find the missing quantity (distance for walking 60 minutes).

While these problems all involve the same context, constant speed, each of these three problem types (rate comparison, unit rate and missing value) could be situated in different contexts such as density, buying and consuming, measurement conversions or concentrations.

On the front of the OGAP Ratio and Proportion Framework and in Table 6.1 are the different ratio and proportion problem contexts and topics and problem types students encounter in the middle grades.

Table 6.1 OGAP Ratio and Proportion Framework—Problem Contexts & Topics and Problem Types

Contexts & Topics	*Problem Types*
Ratios	Ratio
Density	Unit Rate
Constant Speed ($d = rt$)	Ratio and Rate Comparisons
Buy/Consume	Missing Value
Concentrations	Scale Factor
Measurement Conversions	Qualitative
Similarity	Non-proportional
Scale	
Probability	
Percents	
Slope	
Sampling	

It is important to restate that various problem types can be situated within different contexts and topics. For example, one could utilize a scale factor problem type positioned in a concentration context or a unit rate problem type in a slope context.

CCSSM and Problem Contexts

Table 6.2 shows the problem topics and contexts identified in the CCSSM/NGA (2010) for grades 6, 7 and 8. Note that grade 6 involves working with familiar contexts like buying, selling and speed. Unfamiliar topics like density, slope and similarity are introduced and mastered in grades 7 and 8.

This seems to be consistent with research that suggests students are initially more successful when solving problems that are set in familiar contexts like buying and selling than unfamiliar contexts such as density or similarity (Cramer, 2017).

Table 6.2 The CCSSM—Math Topics & Contexts by grade

Grade 6	*Grade 7*	*Grade 8*
Ratio Rate • Buying and consuming • Constant speed ($d = rt$) Measurement conversions Percent of a quantity	Ratio and proportion situations in real world problems • Density • Constant speed • Buying and consuming • Measurement conversions • Scale • Other	Slope Similarity

This chapter focuses on problem types specific to ratios, unit rates and ratio and rate comparison problems providing, where applicable, examples across the different contexts and topics. Missing value problems, scale factor, non-proportional problems and qualitative change problems are discussed in other chapters: missing value—Chapter 5; scale factor—Chapter 7; and non-proportional and qualitative change—Chapter 2.

Go to Chapters 2, 5 and 7 for discussions on missing value problems, scale factor, non-proportional problems and qualitative change problem types.

This discussion begins with examples of ratio problems and their structures that may influence the difficulty of the problem and the strategies students use to solve problems.

Ratio Problems and Problem Types

As was discussed in Chapter 1, ratio problems involve understanding multiplicative relationships between quantities in a ratio as well as working with different types of ratio relationships (e.g., part-to-part, part-to-whole).

To begin to understand these structures, study the ratios in the voting problem in Figure 6.3. What ratio relationships are explicitly stated in the table? What other ratio relationships can you determine from the data? What statements can be made about the results of the election?

Figure 6.3 The Voting problem

Dana and Jamie ran for student council president at Midvale Middle School. The data below represents the voting results for grade 7.

	Seventh-Grade Votes	
	Jamie	*Dana*
Boys	20	40
Girls	50	25

As briefly described in previous chapters, ratio relationships can be part-to-part or part-to-whole. These relationships can be *explicitly* stated or *implied* from the information provided in a problem. The data in the Voting problem allow for explicit part-to-part comparisons. Specifically, the quantities provided communicate the number of votes Jamie and Dana received from the seventh-grade girls and the seventh-grade boys. Some of the part-to-part relationships that are explicitly stated in this problem include:

- The ratio of boys who voted for Jamie to the boys who voted for Dana is 20 votes to 40 votes.
- The ratio of girls who voted for Jamie to the girls who voted for Dana is 50 votes to 25 votes.
- The ratio of boys who voted for Jamie to the girls who voted for Jamie is 20 votes to 50 votes.
- The ratio of boys who voted for Dana to girls who voted for Dana is 40 votes to 25 votes.

From these relationships, one can see that twice as many boys voted for Dana as voted for Jamie. You can also see that twice as many girls voted for Jamie as for Dana.

Part-to-whole ratios can be *implied* from part-to-part ratios. For example, one can use the data provided to create a part-to-whole ratio that compares the number of seventh-grade students that voted for Jamie to the total number of seventh-grade students in Midvale Middle School. This ratio, 70 votes to 135 votes, is an example of an *implied* ratio. The term *ratio referent* was introduced in Chapters 3 and 5. Ratio referents can be explicit or implied. You will find this term on the front page of the OGAP Proportionality Framework. Ratio problems that involve using an implied ratio tend to be more difficult for students than ratio problems with explicit referents.

The problem in Figure 6.4 is a variation of the voting problem in Figure 6.3. Read and solve the problem. What are the explicit ratios? What are the implied ratios?

Figure 6.4 Modified Voting problem

Dana and Jamie ran for student council president at Midvale Middle School. The data below represents the voting results.

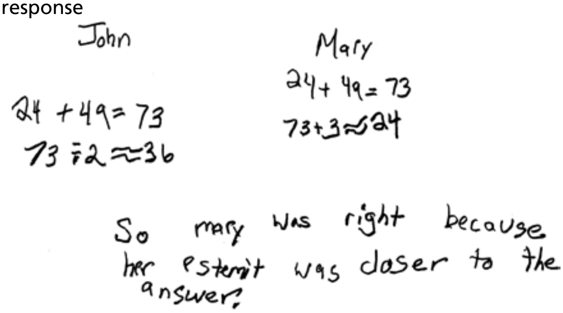
	Seventh-Grade Votes	
	Jamie	*Dana*
Boys	24	40
Girls	49	20

John said, "About half the students who voted for Jamie were boys." Mary said, "About one-third the students who voted for Jamie were boys." Who is correct?

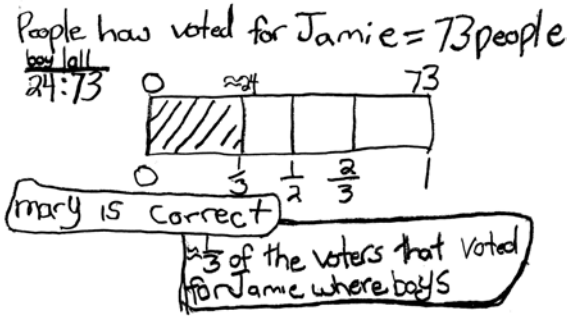
Figure 6.5 shows strategies Anna, Hanna and Marcus used to correctly apply the implicit relationship between the boys who voted for Jamie to all the seventh-grade students who voted for Jamie. Study the solutions. What strategies did the students use to solve the problem?

Figure 6.5 Three correct solutions to the Voting problem

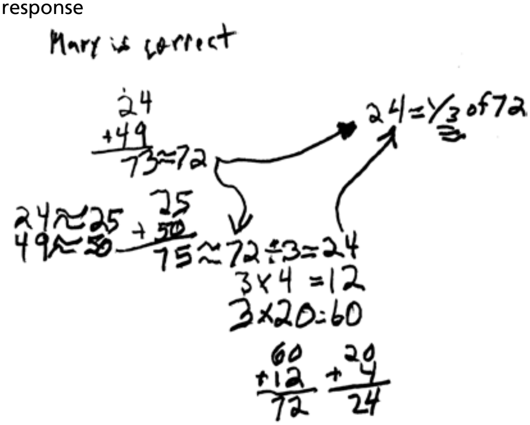
Anna's response



Hanna's response



Marcus's response



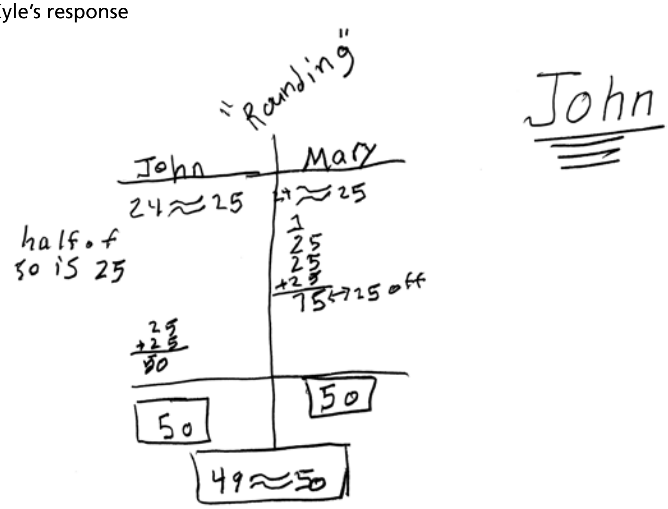
Notice how each student worked with the ratios differently. Anna directly used the implied ratio 24 boys to all the seventh-grade students. Anna divided 73 by

2 $(\times\frac{1}{2})$ and then divided by 3 to determine that Mary was correct. Hanna used a bar model to show that 24 votes is about $\frac{1}{3}$ of 73 votes. Marcus used estimates to simplify the problem. Even though each student used a different strategy, each found and applied the implicit ratio of 24 boys who voted for Jamie to 73 seventh-grade students who voted for Jamie.

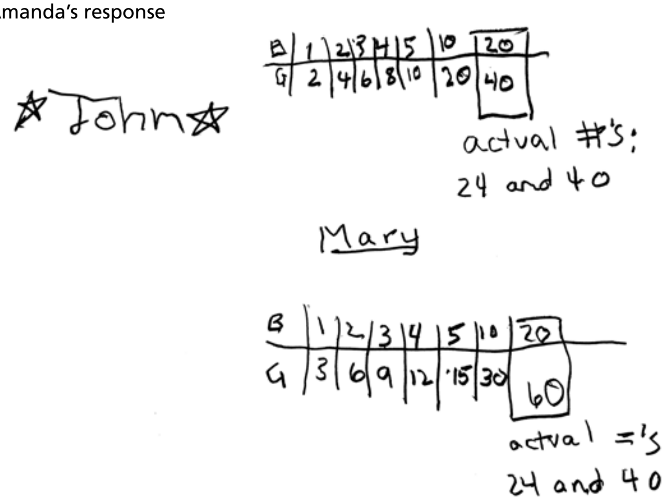
Now study Kyle's and Amanda's solutions in Figure 6.6 to the same problem. What ratios did the students use to solve the problem? What strategies did they use?

Figure 6.6 Kyle's and Amanda's solutions to the Voting problem (Figure 6.4)

Kyle's response



Amanda's response



Each of these students used the explicit ratio of boys' votes for Jamie to girls' votes for Jamie. Even though their solutions are incorrect because they used the wrong ratios to solve the problem, there is strong evidence that each had effective strategies to compare the explicit ratios. Kyle recognized that 2 times 24 is close to 50 votes and 3 times 25 votes would be 25 too many votes. Amanda's solution shows evidence of accurately using ratio tables to build up the ratio. In this case, Kyle's and Amanda's solutions were incorrect as a result of a misinterpretation of the question. That is, they did not interpret the ratio referent correctly using a part-to-part ratio not the part-to-whole implied in the problem. However, there was evidence of understanding the multiplicative relationships in the ratios they explored.

In contrast, there are misinterpretations of questions that are not useful. Sometimes students get the correct answer accidentally. Other times students have an incorrect answer that is "trivial or unhelpful in terms of growth in mathematical understanding" (Wells & Coffey, 2005, p. 204). This was not the case with Kyle's and Amanda's solutions.

The voting problem solved by the students in Figures 6.5 and 6.6 was given to 35 middle school students as part of an OGAP study. Of the 35 students, 15 students (43%) found and applied the correct part-to-whole ratio, 24 boys who voted for Jamie to 73 total votes for Jamie. Of the 35 students, 20 students (57%) incorrectly applied the ratio 24 : 49, which is a part-to-part ratio that compares the number of boys who voted for Jamie to the number of girls who voted for Jamie, similar to Kyle's and Amanda's solutions in Figure 6.6.

Case Study: Facilitate a discussion using a warm-up problem

Mrs. Sequeira, a seventh-grade math teacher, looked at the evidence in her students' solutions to the Voting problem and decided to engage her students in a warm-up problem that focused on explicit and implied ratios, recognizing that many students ignored the part-to-whole relationships in the problem.

Using partners and the think, pair, share strategy, she had the students do the following (see 1–4). Mrs. Sequeira had her students work on each question for a few minutes and then engaged students in a discussion listing the implied and explicit ratios.

1. List four ratios found in the table (e.g., 20 boy votes for Jamie: 40 boy votes for Dana).
2. List four ratios that are not stated in the table but can be found given the data in the table.
3. Write three questions that can be answered using the implied ratio relationship found in question # 2.
4. Solve one of the questions you wrote.

An activity like this warm-up can be used as a segue to solving other ratio problems involving both explicit and implied ratios.

Ratio Problems With More Than Two Ratios

Next, solve the Punch Bowl problem in Figure 6.7, which introduces us to other problem features that can impact the difficulty of ratio and proportion problems. How is this problem similar to the voting problem? How is this problem different from the voting problem? What difficulties might students encounter with this problem? What strategies might students use to solve this ratio problem?

Figure 6.7 The Punch Bowl problem

> A punch recipe calls for orange juice, ginger ale and cranberry juice in a ratio of 2 : 3 : 1.
>
> a. How many gallons of orange juice should you add to 4½ gallons of ginger ale?
>
> b. How many gallons of cranberry juice should be added?

Now study Michael's and Nick's solutions in Figure 6.8. Michael solved the problem using the multiplicative relationship between 3 and $4\frac{1}{2}$. What is the evidence? How did Nick solve the problem?

Figure 6.8 Michael's and Nick's responses to the Punch Bowl problem

> A punch recipe calls for orange juice, ginger ale and cranberry juice in a ratio of 2 : 3 : 1.
>
> a. How many gallons of orange juice should you add to 4½ gallons of ginger ale?
>
> b. How many gallons of cranberry juice should be added?

Michael's response

a. ga $3 \times 1\frac{1}{2} = 4\frac{1}{2}$ c

OJ $2 \times 1\frac{1}{2} = 3$ c

b. CJ $1 \times 1\frac{1}{2} = 1\frac{1}{2}$ c

Nick's response

1 1/2 gallons to get to 4 1/2

$$\begin{array}{r} 2 : 3 : 1 \\ +1\tfrac{1}{2}\ \ +1\tfrac{1}{2}\ \ +1\tfrac{1}{2} \\ \hline 3\tfrac{1}{2} : 4\tfrac{1}{2} : 2\tfrac{1}{2} \end{array}$$

Michael appropriately found and applied the multiplicative relationship between 3 gallons and 4½ gallons $(\times 1\frac{1}{2})$. Therefore, the amount of orange juice and cranberry juice will be $1\frac{1}{2} \times 2$ gallons = 3 gallons and $1\frac{1}{2} \times 1$ gallon = $1\frac{1}{2}$ gallons, respectively. Notice that Michael's solution suggests confusion with the units in the problem. This is something future instruction should address. Nick, on the other hand, found the absolute difference between $4\frac{1}{2}$ and 3, then used addition to create new ratios. Unlike Michael, Nick's strategy did not result in a mixture in the same ratio of orange juice to ginger ale to cranberry juice as the original punch.

The Punch Bowl problem and Michael's and Nick's responses in Figure 6.8 make three important points.

1. Ratios do not always involve just two quantities. In this case, we see a part-to-part-to-part relationship.
2. All students must be able to work flexibly with rational numbers when solving ratio and proportion problems, since working with quantities may involve working with fraction and decimal quantities, not just whole-number quantities.
3. It is vital that students understand that ratios and proportions involve multiplicative relationships, not additive differences.

The OGAP Ratio and Proportion Progression and Ratio Problems

The evidence in student solutions to ratio and proportional problems is categorized based on evidence of the use of multiplicative relationships between the quantities in ratios as well as the type of strategy used to solve the problem. In addition, ratio problems are classified based on whether the correct ratio referent was used to solve the problem.

One way to use the OGAP Ratio and Proportion Progression to classify evidence in student work involving ratio problems is by answering the following three questions:

1. Did the student apply the correct ratio referent given the problem situation?
2. Did the student solve the problem using the multiplicative relationships between the quantities in the ratio(s)?
3. Did the student use a visual model or ratio table that accurately represents the problem situation?

As you read each of the strategy levels that follow, refer to the OGAP Ratio and Proportional Progression on the inside of the OGAP Framework.

Proportional Strategy Level: If the answers to questions 1 and 2 are yes, then the evidence is located at the Proportional Strategy level on the progression, as evidenced by Michael's response in Figure 6.8. Michael's response shows evidence of finding and using the scale factor $(\times 1\frac{1}{2})$ based on the multiplicative relationship between the ratio 3 parts ginger ale to 4½ gallons ginger ale to determine how much orange juice and cranberry juice should be used if $4\frac{1}{2}$ gallons of ginger ale are used in the punch.

Transitional Proportional Strategy Level*:* Solutions at the Transitional Proportional Strategy level meet the criteria for questions 1 and 3. That is, the correct ratio referent was used, and a model was used to represent the solution as evidenced in Hanna's response in Figure 6.5. Hanna used the correct ratio referent and used a tape diagram to represent the solution.

Early Transitional Level*:* At the Early Transitional Strategy level, there is evidence of using a multiplicative relationship between quantities in the problem, but the correct ratio referents were not used. Amanda's response in Figure 6.6 is an example of a solution using a multiplicative relationship, but there is evidence that she used the part-to-part relationships, not the implied part-to-whole relationships.

Non-Proportional Level*:* Ratio solutions that fall in this category show evidence of not using the multiplicative relationship between the quantities in the problem situation even if it used the correct ratio referents. Nick's response in Figure 6.8 illustrates this type of strategy. While Nick did use the correct quantities, he found and applied the additive difference between $4\frac{1}{2}$ cups of ginger ale and 3 parts ginger ale to determine the amounts of orange juice and cranberry juice.

In the Looking Back section of this and other chapters, you will have more opportunities to locate evidence involving ratio situations on the OGAP Ratio and Proportion Progression.

This next section will focus on the structures in rate problems.

Rates/Unit Rates

In Chapter 1, rates were defined as *a special form of a ratio in which the compared amount is a unit amount* (e.g., 50 miles per hour, gallons per miles, dollars per ounce). This unit amount is referred to as the unit rate. In some problem situations, the unit rate is given, and other times, the unit rate has to be derived from the quantities given in order to solve the problem.

To understand how rate problems differ from the Voting problems in Figures 6.3 and 6.4 and the Punch Bowl problem in Figure 6.7, solve the problems in Figure 6.9. How are these problems similar to or different from the Voting problem and Punch Bowl problem? How are these problems similar to each other? How are they different from each other?

Figure 6.9 Some sample problems involving rates

1. A 20-ounce box of Toasty Oats costs $3.00. How much does Toasty Oats cost per ounce?
2. A train is moving at a constant speed. It has traveled 250 miles in 5 hours. If the train continues at this constant speed, how far will it travel in 12 hours?
3. A town has a population density of 50 people per square mile. If the town is 20 square miles, how many people live in the town?

Susan Lamon (2012) states that a "rate is an extended ratio, a ratio that applies not just to the situation at hand, but to a whole range of situations in which two quantities are related in the same way" (p. 235). In the case of the Voting problem, the election is completed, and there is no reason to extend the results to some arbitrary situation such as *How many votes would Dana receive if 200 students voted?* As more students vote, the ratio changes from 65 votes for Dana to 135 students who voted in the election to some unknown number of votes for Dana to 200 students who voted in the election. There is simply no logical reason to consider a rate in the Voting problem.

On the other hand, a solution involving reducing to a unit rate or extending to another equivalent ratio is a logical solution to the Punch Bowl problem, as well as to the three problems in Figure 6.9. Unlike in the Voting problem, these contexts suggest a rate that applies not just to the specific situation given in the problem but also to a whole range of situations in which two quantities are related in the same way.

Susan Lamon (2012) further states that "almost every rate context will require some discussion in the classroom to uncover the context-specific meaning" because of the possible nuances that can occur in different situations (p. 235). A valuable point here is the importance of teachers being sensitive to the nuanced differences between problems as they select appropriate problems for their students to engage in. The examples used thus far show that a rate or an extended equivalent ratio is useful and logical in some situations but not in others. Thoughtful problem selection that aligns with the teacher's specific instructional goals is an important component of effective ratio and proportion instruction.

Notice also that the rate problems in Figure 6.9 differ from the Voting problem because they involve three different types of quantities, while the Voting problem does not. That is, the quantities in the Voting problem are always votes. There are different types of votes such as votes for Jamie, votes for Dana or the number of votes girls cast for Dana. In contrast, the rate problems in Figure 6.9 involve three different quantities depending on the context: a rate and two other related quantities. See Table 6.3.

Table 6.3 Quantities in rate problems from Figure 6.9

Problem	*Three Quantities in the Rate Problems From Figure 6.9*		
	Rate	*Related Quantities*	
Box of Toasty Oats	Cost per ounce ($0.15/ounce)	Volume in ounces (20 ounces)	Cost in dollars ($3.00)
Train at a constant speed	(250 miles per 5 hours = 50 miles per hour)	Time (12 hours)	Distance (600 miles)
Town	People per square mile (50 people per square mile)	Area (20 square miles)	Population (1000 people)

Notably, a major difference between the three problems in Table 6.3 is the context in which each problem is situated. This difference is significant. Problem 1 is a buy/consume problem. Problem 2 involves constant speed, and Problem 3 is a density problem.

We learned earlier in the chapter that some researchers believe density and constant speed are less familiar contexts to students and thus more difficult. For this reason, the next two sections are dedicated to a discussion of speed and density.

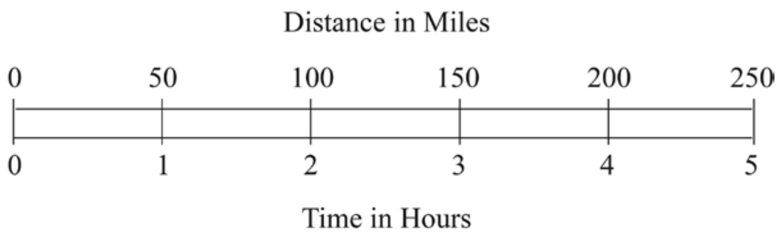
Constant Speed as a Rate

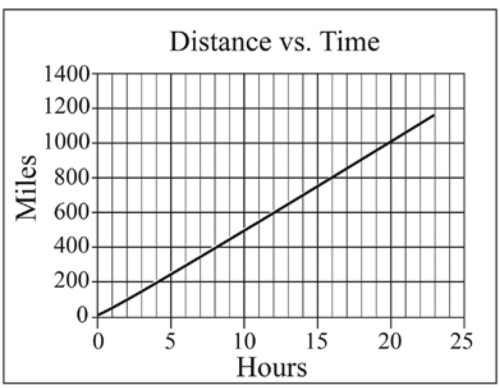
Even though "speed" is a commonly used term, researchers have found that speed is a difficult concept for students to understand because they do not understand that speed cannot be measured directly; rather, it is a measure of motion resulting from comparing two other quantities—the ratio of distance and time (Lamon, 2005; Hulbert et al., 2017). As described in Chapter 1, speed is an example of a ratio in which the two quantities involved are not explicitly stated. While more familiar contexts like buying and selling are easier for students to understand, students still need to realize that all rates are derived as the ratio of two quantities.

Chapter 2 included a discussion about how to use representations to develop ratio and proportion concepts. In that chapter, examples were given on how to use double number lines, ratio tables and coordinate planes to build understanding. Figure 6.10 illustrates each of these representations in the context of the Train problem.

Figure 6.10 illustrates each of these representations of the Train problem

A train is moving at a constant speed. It has traveled 250 miles in 5 hours. If the train continues at this constant speed, how far will it travel in 12 hours?





Time (hours)	Distance (miles)
1	50
2	100
3	150
4	200
5	250
...	...
12	600

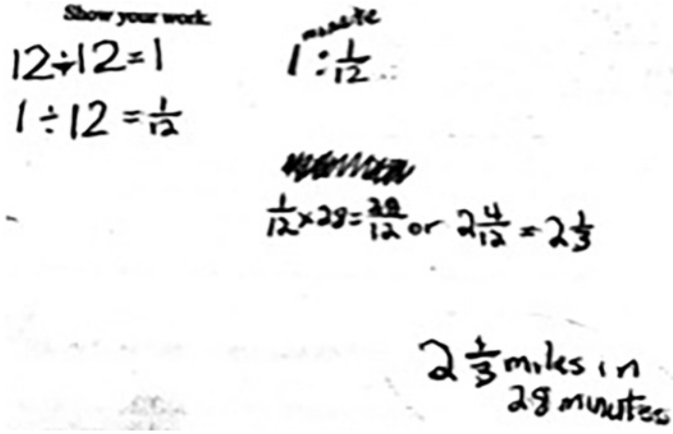
It is important to note that the term *speed* is not found on any of the representations except in the title. Rather, the quantities time (hours) and distance (miles) are explicitly stated, and the rate 50 miles per hour is derived from the ratio between miles and hours. No matter which ratio one chooses from either of the representations (e.g., 250 : 5; 200 : 4; 150 : 3; or, 100 : 2), the associated rate is 50 miles per 1 hour.

Study the problem and student solutions in Figure 6.11. What quantities are given and what quantity has to be derived in order to solve the problem? What difficulties might students encounter? What strategies did students use?

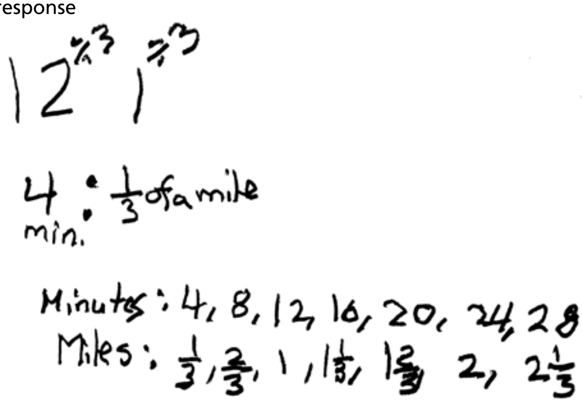
Figure 6.11 Donna's Run

Donna runs at an average rate of 12 minutes per 1 mile. At this rate, how many miles can she run in 28 minutes?

Kelyn's response



Ella's response



Christine's response

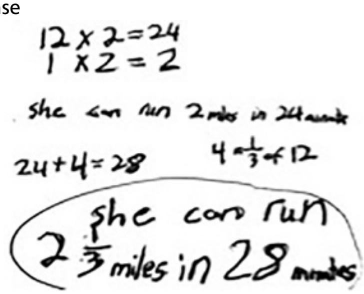
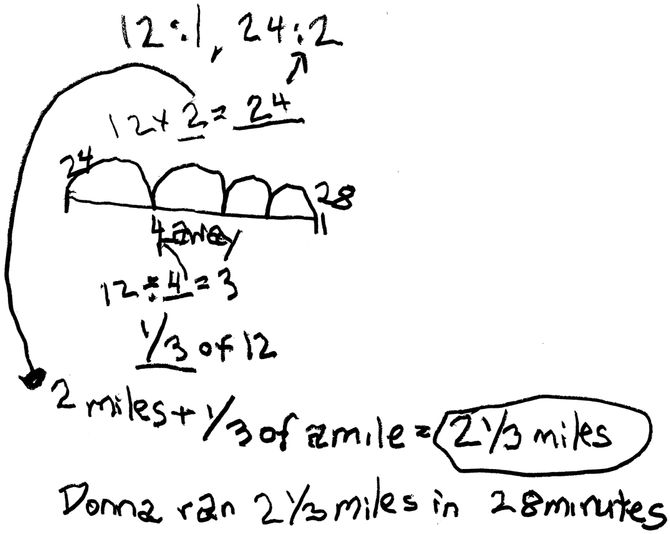


Figure 6.11 Continued.

Jason's response



Christine, Kelyn, Ella and Jason all successfully solved this problem using different strategies and relationships in the problem situation. Kelyn found and applied the unit rate of $\frac{1}{12}$ mile per minute $(\frac{1}{12}\text{ miles per minute} \times 28\text{ minutes} = 2\frac{1}{3}\text{ miles})$. Ella found the ratio of 4 minutes to $\frac{1}{3}$ mile. Ella represented this relationship in a ratio table, using skip counting by fours and thirds. The evidence in Jason's and Christine's solution indicates that they doubled the quantities 12 minutes to 1 mile to 24 minutes to 2 miles. They both then recognized that 4 minutes is one-third of 12 minutes and, therefore, one-third additional mile $(2\text{ miles} + \frac{1}{3}\text{mile} = 2\frac{1}{3}\text{ miles})$. In Chapter 4 Figure 4.10 Sophia, Max, Alex and Oliver all initially doubled the ratio like Jason and Christine but were unable to find how the 4 minutes, uncounted in their initial strategy, was related to the distance Donna ran in 28 minutes. It is the non-integral multiplicative relationship that made this problem challenging.

Go-To Go to Chapter 5 for a discussion on the impact of number types on student solutions.

Density as a Rate

Middle school students encounter the concept of density in a range of situations. For example, in science class, students might be asked to find the density of substances (grams per cubic centimeter). They learn in both science and social science about population density in ecosystems (e.g., deer per square mile), human population density (people per square mile), air quality measures and many others. In spite of these experiences, density is a difficult concept for students to understand (Xu & Clarke, 2012; Cramer et al., 1993).

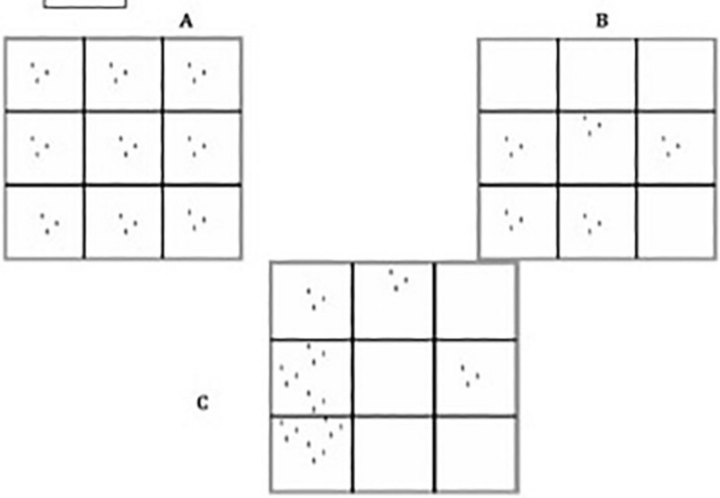
The problem in Figure 6.12 was given to 52 middle school students prior to instruction on population density as a rate. Review the problem. What difficulties might students have as they solve this problem?

Figure 6.12 Sample density problem

Students read an article that said that there were 3 raccoons per square mile in a local forest that has an area of 9 square miles.

Here are models that some students made to represent 3 raccoons per square mile.

Each ☐ represents 1 square mile. Each dot represents 1 raccoon.



Which model or models represent a population of 3 raccoons per square mile in a forest that is 9 square miles? Explain your choice (s).

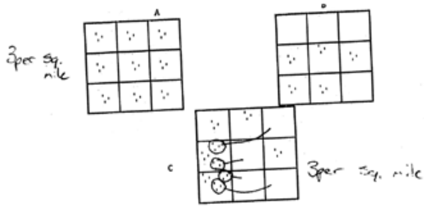
It is probably not surprising that of the 52 students who solved this problem, only 12 (23%) of the students chose the correct responses, A and C. Most of the correct responses provided evidence of understanding that the phrase *3 raccoons per square mile* refers to an average of 3 raccoons per square mile rather than the requirement that every square mile must contain exactly 3 raccoons.

While most people would say that in model C, "the density of raccoons in the western (left side) of the forest is greater than in other parts of the forest," the overall density of raccoons in the entire forest remains 3 raccoons per square mile. *Density* is an average rate per whatever whole is being considered and does not need to refer to a uniform distribution, in this case, raccoons in a forest.

Review the work of Azro, Frances, Ray and Joze in Figure 6.13. What understanding or misunderstandings are evidenced in their work? What is the evidence?

Figure 6.13 Azro's, Frances's, Ray's and Joze's responses to the Raccoon problem

Azro's response



A and C Represent 3 Racoons Per 1 Square mile of A forest with 9 Square miles.
How I figured this out is that for every 9 total Square miles there are 27 Racoons
$3 \cdot 9 =$ 27 Racoons Total
In A there Are 3 Racoons Per every Square, This is $3 \cdot 9 = 27$
And C] total is also A total of 27 Racoons.

Frances's response

(A) model a shows 9 boxes to represent 9 square miles, and 3 dots in each square to represent 3 racoons per square mile

(B) b only has 3 racoons in some squares. The problem says 3 racoons per some of the boxes so in every this does Not work.

(C) c not only doesn't have some boxes empty, but also has more than 3 dots per box in some boxes which means it does NOT have 3 racoons per sq. mile in a forest of 9 sq. miles

Figure 6.13 Continued.

Ray's response

A because the model has 9 square miles and each
mile has 3 racoons in it.
Model B does not have enough raccoons and Model
C has enough but they are not spaced out evenly
within the 9 square miles.

Joze's response

a Because b has 3 raccoons in a few square miles, but nearly half the
total land is unoccupied, as 4 sq. miles are totally empty. C is completely out of
the question because 2 sq. miles both contained 9 raccoons each, C is correct though IF
each mile doesn't have to hold the same # of raccoons, as it also
has 27 raccoons.

The responses are representative of the range of strategies seen in the 52 responses. As you can see, Azro's solution contains strong evidence of understanding the concept of population density. Frances's and Ray's solutions contain evidence typical of incorrect solutions in this set. That is, the evidence in their solutions supports the concept that in order to show the rate of 3 raccoons per square mile across 9 square miles, there must be three raccoons in each square mile. Joze's response shows evidence of uncertainty. At first, Joze indicated "C is completely out of the question" and then went on to say "C is correct though IF each mile doesn't have to hold the same number of raccoons."

Case Study—Mini-lesson: Facilitate a Discussion on Population Density Concept

Mrs. Sequeira realized the value of gathering information about how her students were conceptualizing the concept of density. She used her analysis of her students' solutions to design a mini-lesson focused on population density.

1. Mrs. Sequeira displayed Azro's, Frances's, Ray's and Joze's responses in Figure 6.13 for the class. She asked students to review each of the solutions and discuss with a partner which solution or evidence in the solution accurately represents the situation. As students were discussing the solutions, Mrs. Sequeira moved around the room, listening to the student discussion.

2. Next, Mrs. Sequeira displayed a list of the population densities for different states, as shown in Table 6.4. She asked the students to discuss with partners what the population densities of these states means. For example, in Alaska, are there 1.3 people in each square mile? She engaged them in a discussion focusing on the meaning of population density.

Table 6.4 Population densities of five U.S. states

State/District	*Population Density (people per square mile)*
Washington, D.C.	10,589
New York State	417
North Carolina	203
Maine	43
Alaska	1.3

3. Finally, she returned to the four original solutions. Which explanation(s)/solution(s) do you now think represents 3 raccoons per square mile? Why?

Sometimes a brief but targeted discussion that focuses on a concept students are struggling with can help address student confusion. In this case, Mrs. Sequeira asked the students to revisit the concept of density through a different, perhaps more familiar context. It may be more obvious in this context that people in a state or region are not evenly distributed in each square mile.

The next section focuses on ratio and rate comparisons.

Ratio and Rate Comparison Problems

In the beginning of this chapter, speed was used to exemplify a rate comparison problem (Sally walks 3 miles every day. It takes her 30 minutes. Ashley walks 5 miles every day, and it takes her 2 hours. Who walks faster?) Ratio and rate comparison problems involve comparing two or more ratios or rates. This type of problem can be posed in different contexts such as constant speed, density, buy/consume or concentration.

Read and solve the rate comparison problem in Figure 6.14. This problem involves finding and then comparing equivalent ratios (e.g., unit rate) to determine the best buy. What do you notice about the relationships between the quantities in the problem? What are some strategies students might use to solve the problems? Although this may be a familiar context for many students, what structures in the problem might cause difficulties for students when solving the problem?

Figure 6.14 Toasty Oats

A 20-ounce box of Toasty Oats cereal costs $3.00.

A 15-ounce box of Toasty Oats costs $2.10.

Which box costs less per ounce?

The first thing you may have noticed when solving this problem is that the information is given as ounces to dollars, while the question asks for cost per ounce. This is an example of a problem that utilizes a non-parallel structure, which was discussed in Chapter 5. Students often have difficulty with these types of problems. You also may have noticed that the multiplicative relationship between ounces and dollars is not integral.

Now study Armand's, Kari's and Kit's solutions in Figure 6.15. What is the strategy that each student used to solve the problem? What is the evidence of developing understanding?

Figure 6.15 Armand's, Kari's and Kit's solutions to the Toasty Oats problem

A 20-ounce box of Toasty Oats cereal costs $3.00.

A 15-ounce box of Toasty Oats costs $2.10.

Which box costs less per ounce?

Armand's response

20 ounce box:

20 ÷ 20 = 1 ounce   1: .15

3.00 ÷ 20 = .15   15¢ per 1 ounce

15 ounce box:

15 ÷ 15 = 1 ounce   1: .14

2.10 ÷ 15 = .14   14¢ per 1 ounce

The 15 ounce box is the better deal

Kari's response

20 oz box

3 ÷ 3 = 1

20 ÷ 3 = $6\frac{1}{3}$ 6.33 oz per dollar

15 oz box

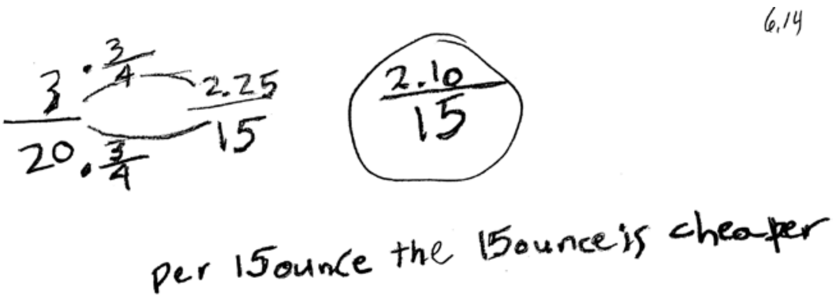
2.1 ÷ 2.1 = 1   7.14 oz per dollar

15 ÷ 2.1 = $7\frac{1}{7}$

The 20 oz box costs less

Figure 6.15 Continued.

Kit's response



Armand's response shows evidence of effectively finding and comparing unit rates. Armand's rates, \$0.15 per ounce and \$0.14 per ounce, are parallel in structure to the rates stated in the question. That is, the rate he used and the rates stated in the question are in the form of cost to weight not weight per cost indicated in the problem.

Kari's response shows evidence of determining correct unit rates but then misinterpreting which rate reflects the best deal. Perhaps this is due to the fact that Kari's ratios, 6.33 ounces per dollar and 7.14 ounces per dollar, were not parallel to the rates stated in the question. Kari could have used her ratios to correctly answer the question if she had correctly interpreted the meaning of a ratio in the form of weight to cost. This would have required her to reason that a 15-ounce box contains more ounces of cereal per dollar (7.14 ounces per dollar) than the 20-ounce box (6⅓ ounces per dollar). More ounces per dollar represents a less expensive price per ounce.

Kit determined that \$3 to 20 ounces were equal to the ratio of \$2.25 to 15 ounces. As these represent the price and weights as if both sizes of cereal cost the same per one ounce, he was able to conclude that \$2.10 for 15 ounces was the better deal.

Kari's response represents a typical error. That is, it determines unit rates that can be compared but misinterprets them.

Case Study: Sharing student solutions

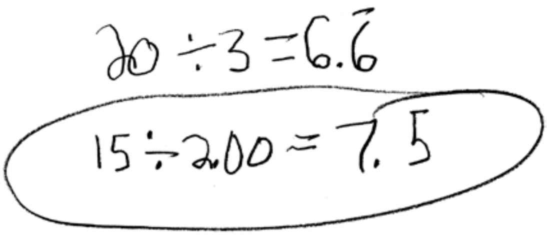
Mrs. Sequeira decided to engage the full class in a discussion of these three interpretations of the problem, hoping to help all students understand the problem, its structure, and different ways the problem can be solved and interpreted. The activity that follows is one way to build these ideas.

Mrs. Sequeira first displayed Amanda's, Kari's and Kit's responses and asked the following questions using a think, pair, share strategy.

1. I noticed that two of the solutions indicate that the 15-ounce box is the better deal, while one solution indicates that the 20-ounce box is the better buy. Which is the better buy? What is the evidence from these solutions?
2. How are Armand's and Kit's solutions similar? Different?

Next, Mrs. Sequeira showed Marissa's solution in Figure 6.16 to the students.

Figure 6.16 Marissa's response to the Toasty Oats problem



Finally, Mrs. Sequeira used the following questions in a full-group discussion:

3. How are Marissa's and Kari's solutions similar? Different?
4. Marissa found that the 15-ounce box is a better deal. Kari found that the 20-ounce box is the better deal. How might each of these students have used the same numbers yet interpreted the answer differently?
5. About a third of the students in the class solved the problem in the same way as Marissa and Kari and interpreted the results like Kari. All but a few of the remaining students solved the problem like Armand. How did those students interpret the problem?

The Toasty Oats problem was an example of a rate comparison problem in the context buy/consume.

Chapter Summary

- Problem contexts and problem types may appear similar but differ in important ways.
- Ratio and proportion problems can be the same context (e.g., constant speed) but structured as a different problem type with differing mathematical demands.
- Ratio relationships can be part-to-part or part-to-whole. These relationships can be *explicitly* stated or *implied* from the information provided in a problem.
- Rates are ratios that apply to more than just one situation and can be extended to a whole range of situations.
- Buy/consume problems are generally easier for students than problems involving constant speed and density.

Looking Back

1. Solve the Raccoon problem in Figure 6.17 and then answer the following questions:
   a. What is the context of this problem?
   b. What type of problem is it?
   c. What strategies do you think students will use to solve the problem?
   d. What errors might students make when solving this problem?

Figure 6.17 Raccoons in Town A and Town B

The chart below shows the population of raccoons in two towns and the area of the two towns.

Town A	*Town B*
60 square miles	40 square miles
480 raccoons	380 raccoons

Karl says that Town A has more raccoons per square mile than Town B. Josh says Town B has more raccoons per square mile than town A.

Who is right? Show your work.

2. Study David's and Mark's solutions to the Raccoon problem in Figure 6.17 and then answer the following:
   a. What strategy did David use to solve the problem? Locate this on the OGAP Ratio and Proportion Progression.
   b. What strategy did Mark use to solve the problem? Locate this on the OGAP Ratio and Proportion Progression.
   c. What instructional strategies might you use next to help students who used a strategy similar to David's?
   d. What instructional strategies might you use next to help students who used a strategy similar to Mark's?

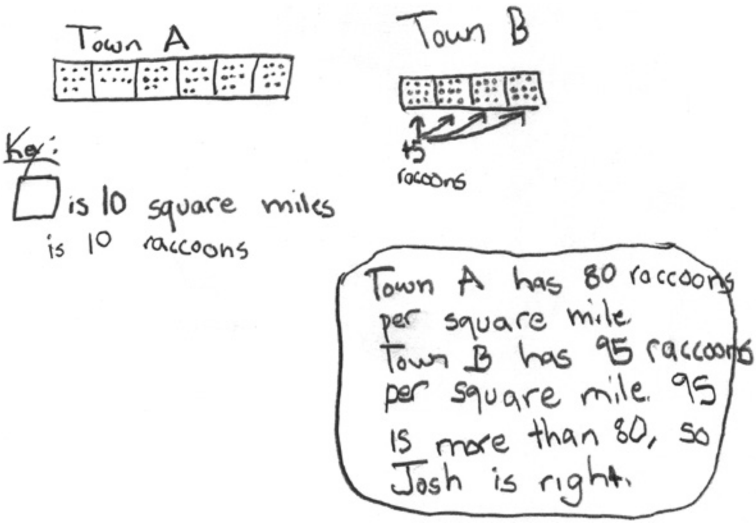
Figure 6.18 David's and Mark's responses

David's response

Karl is right
Because 480×60= 28,800 and
380×40= 15,200

Figure 6.18 Continued.

Mark's response:



3. Write or find in your math program materials/textbook three questions that are the same type of problem but use different contexts. For each problem, indicate difficulties that students may encounter.

4. Write or find in your math program materials/textbook three questions that are different types of problems but utilize the same context. For each problem, indicate difficulties that students may encounter.

5. For the following pieces of student work in this chapter, indicate the level along the OGAP Progression that the evidence is found. What is the evidence?
   a. Figure 6.11 Kelyn's response
   b. Figure 6.15 Armand's response
   c. Figure 6.15 Kari's response
   d. Figure 6.15 Kit's response

Instructional Link

1. To what degree do you or your instructional materials provide a range of opportunities for students to engage in proportion problems with different contexts in different problem types?
2. Based on this analysis, what adjustments do you need to make to your instructional materials to assure students are solving problems that vary the context and problem type to encourage students to make conscious decisions about the relationships of the quantities in the problems?

7
More Problem Contexts

Big Ideas

- The same problem type can be cast in different contexts.
- It is important for students to experience a range of contexts in order to generalize their understanding of proportionality, because this requires students to focus on the interaction between different quantities across different contexts.

This chapter continues the discussion of ratio and proportion problem contexts, with a specific emphasis on the importance of instructionally focusing on the meaning of the quantities in different ratio and proportion contexts such as similarity, scale factors, slope, measurement, monetary conversions, probability and sampling. Chapter 6 included a discussion about the differences between problem types and problem contexts, specifically about how different problem types can be cast in a variety of contexts. The example in Chapter 6 Figure 6.1 illustrated how problems involving the same context (e.g., constant speed) can include different problem types (e.g., rate comparison, unit rate or missing value). The opposite is also true. That is, the same problem type can be cast in different contexts.

The problems in Figure 7.1 are the same problem type cast in different contexts. Read and solve these problems. How are they alike, and how are they different? What difficulties might students encounter as they solve these problems?

Figure 7.1 Context and problem types

Problem 1: Sue has a bag of 3 black marbles and 5 red marbles.

Mary has a bag with 5 black marbles and 10 red marbles.

Whose bag would you choose to have the best chance of randomly picking a red marble?

Problem 2: A 20-ounce box of Toasty Oats cereal costs $3.00.

A 15-ounce box of Toasty Oats cereal costs $2.10.

Which box costs less per ounce?

Problem 3: The dimensions of four rectangles are given below. Which two rectangles are similar? Explain your choice.

Figure 7.1 Continued.

A. 2 inches × 8 inches

B. 4 inches × 10 inches

C. 6 inches × 12 inches

D. 6 inches × 15 inches

Problem 4: Train A travels from Newton to Derby, a distance of 300 miles in 5 hours.

Train B travels from Newton to Sudbury, a distance of 224 miles in 3 hours. Which train is traveling at the faster average speed?

Even though all three are ratio or rate comparison problems, situating the problem in different contexts changes the mathematical demand for a number of reasons. First, while the Toasty Oats problem cast in the context of comparing prices may be a familiar context for most students, both the probability aspect of the Marbles problem and the similarity aspect of the Four Rectangles problem may not be familiar unless students have received direct instruction on those topics. Second, it is unlikely that students will see all four problems as the same problem type because the quantities in the problems are different, as are the multiplicative relationships between the quantities. Review Table 7.1.

Table 7.1 Quantities in the problems 1–4 in Figure 7.1

Problem Context	Quantities Given	Comparison Units
Problem 1: Probability	Number of red and black marbles. (Note that the whole is implied.)	Ratio of red marbles to all the marbles in each bag.
Problem 2: Consume/Produce	Volume in ounces and cost in dollars	Cost per ounce
Problem 3: Similarity	Dimensions of rectangles in inches	Ratio of the length to the width of each rectangle
Problem 4: Constant speed	Distance in miles and time in hours	Miles per hour

By studying Table 7.1, one can see why it is important for students to experience a range of contexts in order to generalize their understanding of proportionality, because it requires students to focus on the interaction between different quantities across different contexts.

Solve the problem in Figure 7.2. What type of problem is this? What is the context of this problem? What are the quantities given? What is the resulting comparison unit? What difficulties might students encounter as they solve this problem?

Figure 7.2 Acres of pasture per horse

Big Horn Ranch raises 100 horses on 150 acres of pasture. Jefferson Ranch raises 75 horses on 125 acres of pasture. Which ranch has more acres of pasture per horse? Show your work.

This problem is a rate comparison problem set in the context of density that involves numbers of horses and acres of land. The comparison unit is acres per horse. Note the order in which the quantities are given in the problem (horses followed by acres) versus the order in which the quantities are stated in the question (acres per horse). This is an example of a non-parallel structure. This problem is designed specifically in this way to encourage students to pay close attention to the quantities they are given and the resulting comparison quantities (acres per horse).

Go To Go to Chapter 5 for discussion about parallel and non-parallel structures in ratio and proportion problems.

Study Hanna's, Kaitlyn's and Max's solutions in Figure 7.3 to the Acres of Pasture Per Horse problem. What is the evidence in their solutions that they misinterpreted the quantities in the problem? How does this impact their solutions?

Figure 7.3 Hanna's, Kaitlyn's and Max's responses

Hanna's response

Big Horn Ranch raises 100 horses on 150 acres of pasture. Jefferson Ranch raises 75 horses on 125 acres of pasture. Which ranch has more acres of pasture per horse? Show your work.

Big Horn Ranch→ Bighornhas 25 morethanJefersonand25 more acres thanjefferson

Jeferson Ranch Jefferson has 25 less horses and 25 less Achers

even

Kaitlyn's response

Big Horn Ranch raises 100 horses on 150 acres of pasture. Jefferson Ranch raises 75 horses on 125 acres of pasture. Which ranch has more acres of pasture per horse? Show your work.

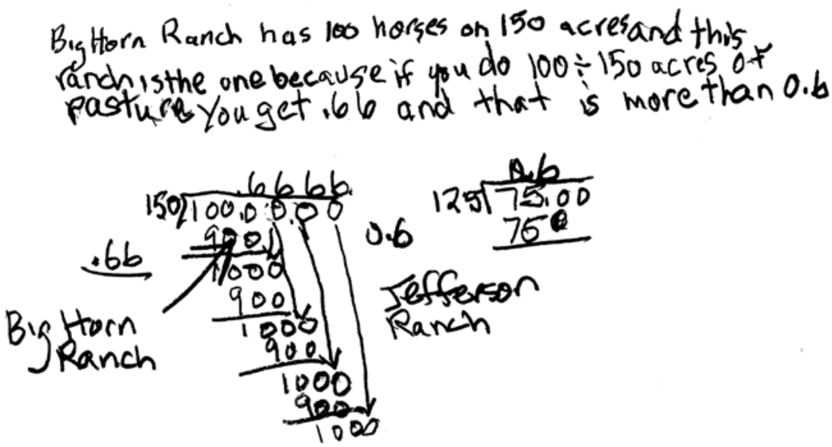
$$\begin{array}{r} 100 \\ +150 \\ \hline 250 \end{array} \qquad \begin{array}{r} 125 \\ +75 \\ \hline 200 \end{array}$$

Big Horn Ranch has more acres of Pasture because 100 Plus 150 equals 250.

Figure 7.3 Continued.

Max's response

Big Horn Ranch raises 100 horses on 150 acres of pasture. Jefferson Ranch raises 75 horses on 125 acres of pasture. Which ranch has more acres of pasture per horse? Show your work.



These three responses represent three different types of evidence that illustrate ways *the meaning of the quantities in the problem is ignored*.

- Hanna found the additive differences between horses and acres to be the same (25), concluding the acres to horses are the same for both ranches. This solution points also to another aspect of the design of this problem. Note that the additive differences between acres and horses in each ranch are the same (50). One might say that these design aspects are tricks, but they are not. Rather, they provide mechanisms for teachers to gather formative evidence of student understandings and test the strength of their proportional reasoning.
- Kaitlyn's response shows evidence that the meaning of the quantities were ignored in the problem when she added horses to acres.
- Max's response shows evidence of comparing 0.6 to 0.6666 without consideration of the meaning of the two quantities. That is, there is no evidence in Max's solution that he compared 0.6 horses per acre to 0.66 horses per acre. If Max had realized that Big Horn Ranch had more horses per acre, he would have been able to conclude that more horses per acre is actually fewer acres per horse.

Go to Chapter 6 for sample instructional responses to help build understanding of density as well as instructional responses to evidence similar to Hanna's and Max's responses.

The rest of this chapter focuses on ratio and proportion contexts, including similarity, scale factors, slope, measurement and monetary conversions, probability and sampling. Chapter 9 focuses on percent. Throughout the rest of this chapter, the emphasis will continue to be on the quantities and how they interact in different ratio and proportion problems involving these contexts and different problem types.

Similarity and Scale Factors

Similarity and scale factors are closely related. By definition, figures are similar if they are the same shape and their corresponding sides are proportional. Study the rectangles in Figure 7.4. Which rectangles are similar to each other? Which are not?

Figure 7.4 Rectangles

It is easy to see that Rectangle B is not the same shape as Rectangles A and C, and therefore, Rectangle B is definitely not similar to Rectangles A and C. However, in order to confirm that Rectangles A and C are similar to each other, one would have to determine if both dimensions are scaled in the same proportion. To test for similarity, one could find the ratio of length to width for each figure. If these ratios are equivalent, then the two figures are similar. Or one could determine the scale factor between the length of figure A and length of figure C. Then find the scale factor between the width of figure A and the width of figure C. If these two scale factors are the same, then the two figures are similar. Notice that this task is like Problem 3 in Figure 7.1. That is, students were given the dimensions of several rectangles and had to determine which rectangles were similar.

Study Armand's, Kim's and Tam's solutions to the similarity problem in Figure 7.5. What do you notice about their solutions? How are they alike? How are they different? What is the evidence that each student understands the quantities in the problem and the contextual situation?

Figure 7.5 Armand's, Kim's and Tam's solutions to the similarity problem from Figure 7.1

Armand's response

The dimensions to four rectangles are provided below. Which two rectangles are similar?

A. 2" × 8"

B. 4" × 10"

C. 6" × 12"

D. 6" × 15"

B+D because both of the first numbers go into the second number 2.5 times.

Figure 7.5 Continued.

Kim's response

A. 2" × 8"

B. 4" × 10"

C. 6" × 12"

D. 6" × 15"

The scale factor is 1.5

$4 \times 1.5 = 6$

$10 \times 1.5 = 15.$

Tam's solution

A. 2" × 8"

B. 4" × 10"

C. 6" × 12"

D. 6" × 15"

$\frac{2}{8}$ $2 \div 8 = \frac{1}{4}$

$\frac{4}{10}$ $4 \div 10 = \frac{2}{5}$

$\frac{6}{12}$ $6 \div 12 = \frac{1}{2}$

$\frac{6}{15}$ $6 \div 15 = \frac{2}{5}$

similar

Armand's, Kim's and Tam's solutions all show evidence of understanding similar rectangles. There is evidence in Armand's solution that he found the multiplicative relationship between the dimensions of Rectangle B (4 inches by 10 inches) and Rectangle D (6 inches by 15 inches) to have the same multiplicative relationship (× 2½). Tam's solution has evidence of finding the multiplicative relationship between both dimensions of the rectangles to be the same (× 1½). In other words, scaling the length and width of Rectangle B by a factor of 1½ results in rectangle D. Kim correctly indicated that the ratio of length to width for Rectangle B is equivalent to the ratio of length to width for Rectangle D. Each of the solutions shows evidence of understanding of similarity in this situation.

In contrast, study Rick's, Hanna's and Kasey's solutions in Figure 7.6. What is the evidence that they do not understand similarity in this situation?

Figure 7.6 Rick's, Hanna's and Kasey's solutions

The dimensions to four rectangles are provided below. Which two rectangles are similar?

Figure 7.6 Continued.

Rick's response

A. 2" × 8"
B. 4" × 10"
C. 6" × 12"
D. 6" × 15"

A and B because if you apply the rule (X+2, X+2) 2+2=4 8+2=10

They both go up by the same number meaning (4" x 10") is just an enlarged Version of (2" x 8")

Hanna's response

Explain your reasoning.

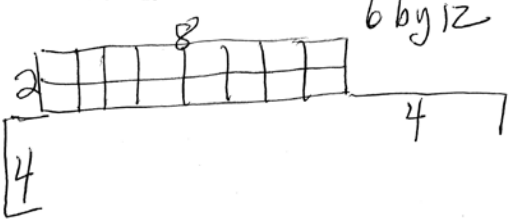
A. 2" x 8"
B. 4" x 10"
C. 6" x 12"
D. 6" x 15"

A and B because there dimentions like 2" x 8" are 6 apart from eachother and 4" x 10" are 6 apart from eachother

Kasey's response

A. 2" × 8"
B. 4" × 10"
C. 6" × 12"
D. 6" × 15"

A and C the 2 by 8 rectangle is 4 less square on both sides then the 6 by 12



Each of these solutions used additive differences between the dimensions instead of a multiplicative relationship to determine if the rectangles were similar. Instruction for these students needs to help them transition from additive to multiplicative comparisons.

In the Case Study that follows, a teacher responds to the evidence from a different classroom of students who solved the same similarity problem. In contrast, almost all the students in this classroom correctly used the scale factor between rectangles or the multiplicative relationship between the dimensions of each rectangle to identify the similar rectangles.

Case Study—Mini-Lesson: Facilitating a Discussion Focused on Making Connections to Slope and the Coordinate Plane to Extend Understanding

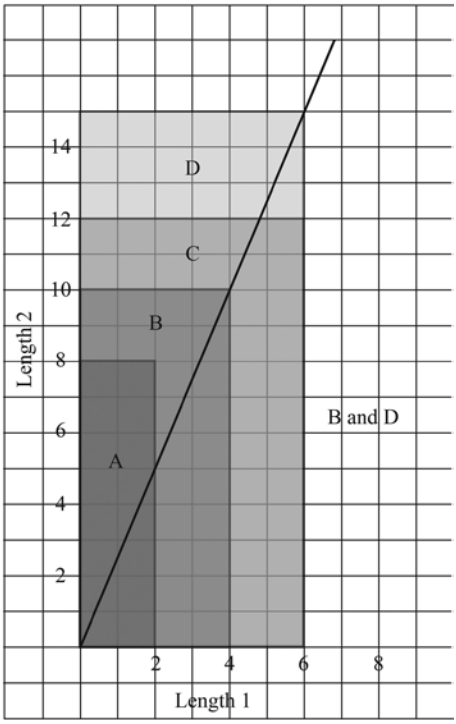
Chapter 2 contained discussions about the use of coordinate planes to help develop understanding of proportionality as well as to represent proportional situations. Ms. Austin, an eighth-grade math teacher, decided to lead a mini-lesson designed to extend the students' understanding of proportionality in similar figures by connecting their solutions to a representation of the problem on a coordinate plane and to slope, which was another topic the students had been studying. Notice that even though the students were successful on the assigned exit question about the similar rectangles in Figure 7.1, the teacher used the evidence to extend their understanding.

At the beginning of the class, Ms. Austin placed models of rectangles from the problem on the coordinate plane and gave the students a copy of the coordinate grid with the rectangles sketched on to it, as shown in Figure 7.7. She then asked a series of questions about the rectangles on the coordinate plane and how Tam's and Kim's solutions in Figure 7.7 were represented on the coordinate plane. The teacher used a think, pair, share strategy to engage all students in the discussion.

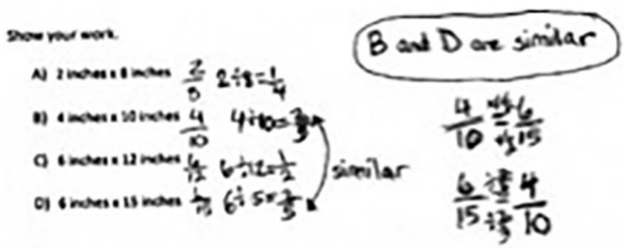
Here are a few questions Ms. Austin used during the full class discussion:

1. Is there a straight line through the origin that passes through the opposite corner of more than one rectangle? What does that line tell us about the relationship between the rectangles whose vertices fall on that line?
2. How are Tam's and Kim's solutions represented on the coordinate plane?
3. What are some dimensions of other possible rectangles that are similar to the two rectangles whose vertices fall on that line?
4. How many other similar rectangles can be represented on the coordinate grid?
5. What is the slope of the line that passes through the Rectangles B and D?
6. How is the slope of the line related to the scale factor?
7. If a line was drawn through the vertices of Rectangle A, what would be the slope of that line? What are the dimensions of the other possible rectangles similar to Rectangle A?

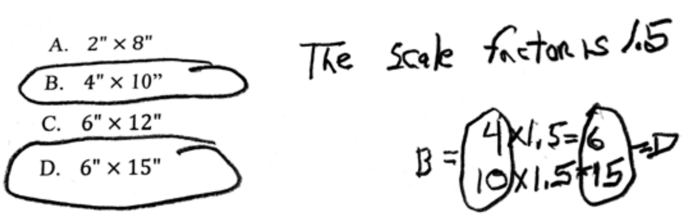
Figure 7.7 Similar figures



Tam's response



Kim's response



Ms. Austin designed her questions to help students make connections between the scale factor and the slope of the line that passes through the vertices of the similar rectangles. The line through the origin that passes through the opposite corners of both Rectangles B and D suggests a special relationship between these two rectangles. The slope of this line is 2.5 or $\frac{5}{2}$, which is the ratio of length to width for both Rectangles B and D. Thus, Rectangles B and D are similar because the ratio of their lengths to widths is equivalent.

Geometric similarity as well as scale factors is used in a variety of contexts. Read through the problems in Figure 7.8. Solve each of the problems. How are the problems alike, and how are they different?

Figure 7.8 Scale factor and similarity problems

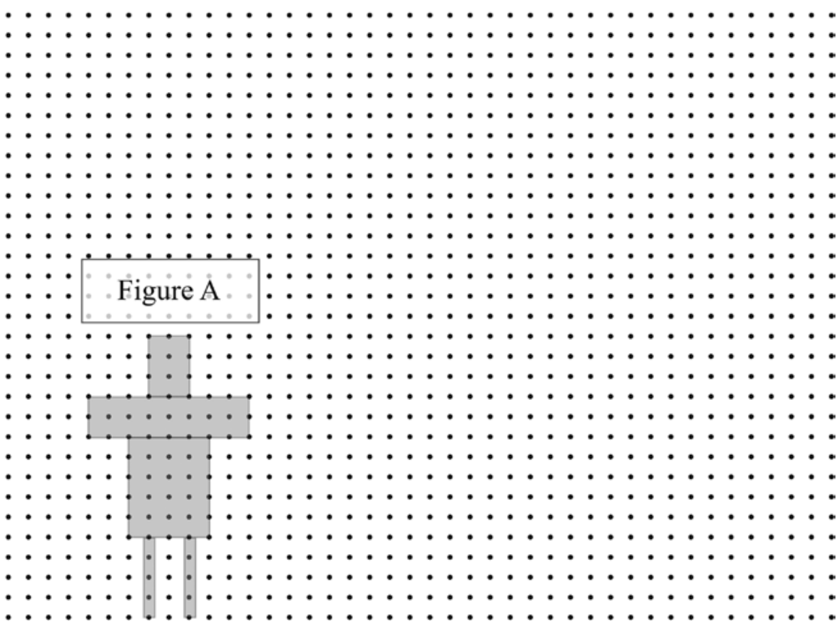
Problem 1: The distance between Ashville and Jackson is 25 miles. On a map, Ashville and Jackson are 4 cm apart. What is the scale of the map?

Problem 2: Holly has a photograph that is 8 inches by 10 inches. She scaled the photo down to 4 inches by 5 inches. What scale factor did she use?

Problem 3: Jack built a scale model of the Jackson Building. His model is 4 feet tall. The Jackson building is 1476 feet tall. One foot on Jack's model equals how many feet on the real Jackson Building?

Problem 4: A model of a farm tractor has a scale where 1 inch = 2 feet. The height of the model is 3 inches. What is the height of the real tractor?

Problem 5: Graphic computer programs are based on a coordinate grid created by the pixels on a computer screen. A part of a computer screen that is 30 × 40 pixels is shown below with an image a programmer plans to scale to twice its size. Sketch the new figure.



The main difference between these problems is the nature of the quantities given, what the student is asked to determine, and the different ways each problem requires students to apply scale factors, scales and similarity. Table 7.2 provides some of this information.

Table 7.2 Differences in the information given and what needs to be determined in Problems 1–5 in Figure 7.8

Problem 1: The distance between Ashville and Jackson is 25 miles. On a map, Ashville and Jackson are 4 cm apart. What is the scale of the map?

Problem Context	Quantities	Sample Solution
Determining scale of a map	Given the distance in miles (25 miles) and distance on map in cm (4 cm). Need to determine the scale of the map.	25 miles ÷ 4 cm = $6\frac{1}{4}$ miles/cm . Therefore the scale of the map is $6\frac{1}{4}$ miles = 1 cm

Problem 2: Holly has a photograph that is 8 inches by 10 inches. She scaled the photo down to 4 inches by 5 inches. What scale factor did she use?

Problem Context	Quantities	Sample Solution
Similarity cast in photo reduction	Given the dimensions in inches of two similar rectangles (photos). Need to determine the scale factor.	The scale factor is $\frac{1}{2}$. That is, the corresponding lengths of the desired 4” by 5” are half the lengths of the 8” by 10” photo.

Problem 3: Jack built a scale model of the Jackson Building. His model is 4 feet tall. The Jackson building is 1476 feet tall. One foot on Jack's model equals how many feet on the real Jackson Building?

Problem Context	Quantities	Sample Solution
Scale model—need to determine scale	Given the height of model (4 feet) and actual building (1476 feet). Need to determine the scale.	1476 feet ÷ 4 feet = 369; Therefore, the scale is 1 foot of model = 369 feet of actual building

Problem 4: A model of a farm tractor has a scale of 1 inch = 2 feet. The height of the model is 3 inches. What is the height of the real tractor?

Problem Context	Quantities	Sample Solution
Application of scale to a model	Given the scale (1 inch = 2 feet) and the height of the model (3 inches). Need to apply the scale to determine the height of tractor.	2×3 feet = 6 feet tall

Table 7.2 Continued.

Problem 5: Graphic computer programs are based on a coordinate grid created by the pixels on a computer screen. A part of a computer screen that is 30 × 40 pixels is shown below with an image a programmer plans to scale to twice its size. Sketch the new figure.

Problem Context	Quantities	Sample Solution
Similarity—applying a scale factor	Given a figure imposed on dot grid. Need to determine the dimensions of the figure and then scale the entire figure $(\times 2)$.	e.g., The head on the screen is 2×4 pixels; scaled $(\times 2)$ it would be 4×8 pixels.

This is only a sampling of a few ways that similarity and scale factors can appear in middle school mathematics instruction and everyday applications.

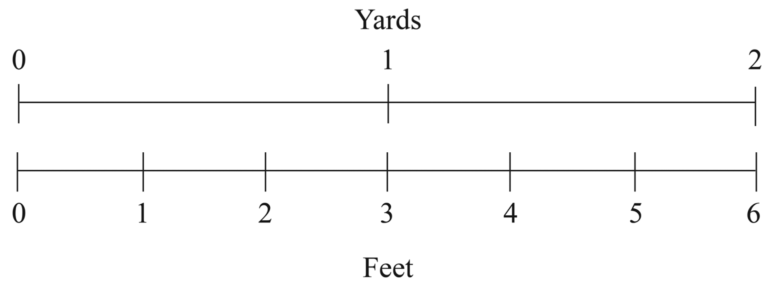
Another related but very different context is conversions. The next section focuses on this topic.

Measurement and Monetary Conversions

Measurement Conversions

Measurement conversions involve changing the size of a unit of measure using a conversion factor (e.g., 12 inches to a foot). This is different from a scale factor, which results in a larger or smaller quantity or size, as illustrated in the last section of this chapter. Figure 7.9 illustrates that converting from one unit to another does not change the actual size of an object.

Figure 7.9 Using the conversion factor 3 feet to 1 yard



Notice that the number of units increases by a factor of 3. Specifically, 2 yards converts to 6 feet. The size of the measurement unit also changes by a factor of 3; one yard is 3 times longer than 1 foot. Despite these changes in the number and size of the measurement units, a measurement conversion does not alter the size (in this case length) of the original object. That is, the absolute length does not change whether it is measured in feet or yards. At its core, a measurement conversion results in a statement of equality. In this case, 2 yards = 6 feet.

The example involved conversion within a measurement system, yet conversions between measurement systems behave in a similar way. For example, 2.2 pounds and

1 kilogram express the same weight because there are 2.2 pounds for every kilogram. Study Debbie's and Brice's solutions in Figure 7.10. What understandings are evidenced in their solutions? What misunderstandings are evidenced in their solutions?

Figure 7.10 Debbie's and Brice's responses

There are 2.2 pounds in a kilogram. A suitcase weighs 50 kilograms. How many pounds does the suitcase weigh?

Debbie's response

$50 \div 2.2 = 22.72$

22 pounds = 1 kilogram

$50 \div 2.2 =$ 22.72 pounds

The suitcase weighs 22.72 pounds

Brice's response

$1 \times 2.2 = 2.2$

$50 \times 2.2 =$ 110 pounds

Debbie's response contains evidence of not applying the conversion factor correctly, while Brice's response shows evidence of applying the conversion factor correctly.

Instruction should provide Debbie with experiences to help her understand that a kilogram is roughly twice as heavy as a pound. Then her teacher could ask questions such as, "Roughly how many pounds will 2 kilograms weigh?" and "Does the answer you calculated make sense?"

Monetary Conversions

Converting between currencies of different nations requires knowledge of monetary exchange rates. The exchange rate is the number of units of one currency one needs to buy a single unit of the other currency. Unless students travel to other countries or live near the U.S.–Mexican or Canadian border, they will probably have little experience converting money from one system to another.

For example, the exchange rates between the U.S. and Canada in 2019 were roughly \$1 Canadian = \$0.76 U.S. and \$1 U.S. = \$1.31 Canadian. Converting 100 Canadian dollars results in approximately 76 U.S. dollars. Converting 100 U.S. dollars results in approximately \$131 Canadian.

Solve the problems in Figure 7.11 and then study the student work. What understandings does a student need to solve this problem? What difficulties might students encounter? What is the evidence of understanding in Alec's and Tom's responses?

Figure 7.11 Conversion between U.S. dollars and Canadian dollars

1.31 Canadian dollars = 1 U.S. dollar

0.76 U.S. dollars = 1 Canadian dollar

Peter has 12 Canadian dollars. Mark has 9 U.S. dollars. Mark says that his money has more value than Peter's money. Is Mark correct? Why or why not?

Alec's response

Show your work.

$$\begin{array}{r} 1.31 \\ \times\ 9 \\ \hline 11.79 \end{array} \qquad \begin{array}{r} .76 \\ \times 12 \\ \hline 152 \\ .760 \\ \hline 9.12 \end{array}$$

Peter would have $9.12 in U.S. dollars meaning he has more than Mark Mark would have $11.79 in Canadian dollars meaning he has less than Peter

Tom's response

He is not correct because a Canadian dollar is worth more than a U.S. dollar and Peter has twelve of those, while Mark has 9 U.S. dollars.

Alec's response shows evidence of understanding that Peter's $12 Canadian is worth more than Mark's $9.00 U.S. Tom's solution shows no evidence of understanding conversions between the two systems. He strictly attended to the absolute values of the Canadian and U.S. dollars.

Probability

While the study of probability can be quite involved, this section will focus on simple probabilities—that is, the extent to which an event is likely to occur measured by the ratio of the favorable cases to the total number of outcome cases possible. Success with simple probabilities requires an understanding of ratios. Solve the problems in Figure 7.12. How are the problems similar and different? What strategies do you anticipate students using when solving the problems? What difficulties might students have as they solve the problems?

Figure 7.12 Marbles in a bag—two problems

> **Problem 1:** Sue has a bag containing 3 black marbles and 5 red marbles. Mary has a bag containing 5 black marbles and 10 red marbles. Whose bag would you choose to have the best chance of randomly picking a red marble? Show your work.
>
> **Problem 2:** Sue has a bag containing 8 marbles. She knows that there are 5 red marbles in the bag. Mary has a bag containing 15 marbles. She knows there are 10 red marbles in the bag. Whose bag would you choose to have the best chance of randomly picking a red marble? Show your work.

From discussions of structures of ratio problems in Chapter 6, one can see that Problem 1 has an *implied* whole (8 marbles in Sue's bag and 15 in Mary's bag), while Problem 2 *explicitly* provides the part of concern (red marbles) and the whole (the total number of marbles in the bag). One can also see that this is a ratio comparison problem—that is, the probability of picking a red marble from Mary's bag compared to the probability of picking a red marble from Sue's bag.

Two common errors are found in Marco's and Anna's responses in Figure 7.13 to Problem 1. What misunderstandings are evidenced in each response? What understandings are evidenced in each response that can be built upon?

Figure 7.13 Marco's and Anna's responses to Problem 1

> Sue has a bag containing 3 black marbles and 5 red marbles. Mary has a bag containing 5 black marbles and 10 red marbles. Whose bag would you choose to have the best chance of picking a red marble? Show your work.

Marco's response

Mary because there is a difference between 5-10 and that is five and the difference between 3-5 is two.

Anna's response

$$S = \overset{60\%}{3/5} \text{ chance}$$

$$M = 5/10 = \underset{50\%}{1/2} \text{ chance}$$

$$60\% > 50\%$$

There are a number of misunderstandings evidenced in the responses in regard to determining the probability of an event occurring. First, both responses compared the part-to-part relationships (red marbles to black marbles) not the part-to-whole relationships (red marbles to all the marbles in the bag). This is a fundamental understanding for probability. Additionally, Marco found the additive differences between the parts not the multiplicative comparison. Marco's response is classified at the Non-Proportional Level on the OGAP Ratio and Proportion Progression.

While these responses contain these common misunderstandings, there is evidence that can be built upon to help Marco and Anna develop understanding of simple probability. First, both responses show evidence of understanding that they were comparing ratios. Anna's response does show evidence of using the multiplicative relationships between the part-to-part ratios. This puts her response at the Early Transitional Strategy level on the OGAP Ratio and Proportion Progression. The next instructional steps for each of these students are very different. Anna needs focused instruction on the meaning of probability, while Marco needs more instruction on transitioning from additive differences to multiplicative comparison when dealing with ratios.

CCSSM and Probability

Grade 7 is the first time students engage in probability. The focus at grade 7 is on developing understanding of probability by engaging in activities like coin tossing or die rolling that help develop probability models in different situations. Additionally, students at grade 7 find "probabilities of compound events using lists, tables, tree diagrams, and simulations" (CCSSM/NGA, 2010). Fundamental to this understanding is that probability is based on part-to-whole ratios.

In seventh grade, students also "use random sampling to make inferences about a population" (CCSSM/NGA, 2010). The relationship between sampling and ratio and proportions is discussed in the next section.

Sampling

Gaining information about a larger population by examining a smaller portion of that group is called sampling. The large group is often called the *population*, and the smaller portion being examined is called a *sample* of the population.

How one samples a population depends upon the information one is seeking. For example, if I am interested in finding out how a large group of people feel about an issue (e.g., gun control), I might take a random sample of the whole population without consideration of the subgroups within that population. On the other hand, if one wants to know about how the different subgroups feel about the issue, one might draw a random sample from each of the subgroups. Sampling is a complicated topic. There are many ways to draw the sample (e.g., random, stratified random, clustered or convenience), the size of the sample to be drawn can vary with the size of the population and sampling involves inference. That is, "As soon as you draw some conclusion about a population when you only have information about a sample of it, you lose certainty, and you need to attach probabilities to your conclusion" (Rosenfeld, 2013, p. 90). We commonly see this reported in poll results typically reported as the margin of error.

Sampling is an important tool in many fields (e.g., science, economics, sociology, medicine). This chapter is not designed to explore this topic fully but to illustrate how sampling involves a ratio of targeted characteristics to the whole sample drawn. The example that follows illustrates how sampling can be used to estimate wildlife populations.

An example of sampling using ratios is commonly known as capture and release and is used by departments of natural resources to estimate wildlife populations. Let's say that we want to estimate the population of deer in a particular area. To do this, we begin by capturing in a random way a certain number of deer. In this example, we will use 97 deer. We then tag the deer and release them into the wild again. Sometime later, we return and capture 94 deer in a random way. We find that among the 94 captured deer, 14 of them have tags from the first capture. How can we use ratios to estimate the deer population in that particular area?

We create ratios of "tagged deer to deer population," both for the entire population and for the captured sample. The assumption is that given sufficient time, the tagged deer would be evenly distributed in the particular area so that the ratios for the sample and for the entire population would be equal.

$$\frac{\textit{total number of tagged deer}}{\textit{total deer polulation}} = \frac{\textit{number of tagged deer in the sample}}{\textit{sample deer population}}$$

That is,

$$\frac{97}{\textit{total deer polulation}} = \frac{14}{94}$$

Thus, we can estimate the total deer population to be 651 deer.

CCSSM

The CCSSM standards focus on two aspects of using sampling to make inferences about a whole population. First, the understanding that the relationship between a characteristic (e.g., favorite ice cream flavor) found in a random sample of the whole population is proportional to finding that characteristic in the whole population. Second, validity of the inference made from the sample depends upon how representative the sample is of the population as well as the size of the sample.

Study the problem and Jack's solution in Figure 7.14. What is the evidence that Jack understood the proportional relationship between the original sample and the prediction based on the sample of the full population? What is the evidence that Jack understood or misunderstood the characteristics of a random sample?

Figure 7.14 Jack's response to the Movie problem

Fifty girls and 50 boys at Maxwell Middle School were randomly selected to respond to a survey of the number of movies they watched during the week of September 16, 2019. The table below shows the results of the survey.

Results of movie survey—number of movies watched the week of September 16, 2019

Number of Movies Watched	*Number of Students*
0	18
1	19
2	23
3 or more	40

Figure 7.14 Continued.

a. Twelve hundred students attend Maxwell Middle School. Based on the random selection of students, predict how many students watched 3 or more movies during the week of September 23.

40:100
×12
480:1200

480

b. Based on the sampling method, how accurate do you think the prediction is? Explain your reasoning.

Not that aqurate because you would have to actually test 1200 kids to beacurate.

Jack's solution shows evidence of correctly predicting the number of students at Maxwell Middle School who watched three or more movies the week of September 23. This is based on the sample of the same population of movie watching the week of September 16 using the underlying proportional relationship between the sample and the full population. However, Jack's solution did not show evidence of understanding that inferences about a full population could be made based on a random sample of the full population by indicating that the full population needs to be sampled to be accurate.

Study Mia's solution in Figure 7.15 to a problem involving catch and release. What understandings or misunderstandings are evidenced in Mia's solution about the proportional relationship of a sample of a population to the full population?

Figure 7.15 Mia's response to Catch and Release problem

The U.S. Fish and Wildlife Department is using the catch-and-release method of sampling in order to estimate the population of deer in a 25-square-mile forest. They randomly captured 90 deer, sampling different sections of the forest. They tagged all 90 deer and then released them back into the forest. A month later, they captured another 90 deer using the same random sampling technique. Of the 90 deer they captured, 14 had tags. Based on the sample, about how many deer reside in the 25-square-mile forest?

$\frac{14}{90} \approx 15\%$

close

$\frac{14}{90} \quad \frac{15}{100}$

close

In contrast to Jack's response in Figure 7.12, the evidence in Mia's solution indicates she did not find the proportional relationship between the original sample and the full population. Rather, she found $\frac{14}{90} \approx 15\%$. Mia never made a prediction about the full population based upon the sample.

Concentrations

Another context that students encounter in middle school mathematics and in high school science is concentrations. Concentration problems involve comparing different concentrations or combining two or more mixtures together. Study the problem in Figure 7.16. What strategies do you think students will use to solve this problem? What difficulties might students have?

Figure 7.16 Sample mixture problems

Ann and Mary each made orange drinks for the class picnic.

- Ann used 2 cans of orange juice concentrate and 8 cans of water in her drink.
- Mary used 4 cans of orange juice concentrate and 12 cans of water.

Both girls used the same size cans and the same type of juice concentrate to make their drinks.

Whose drink had a stronger orange flavor?

Explain your answer.

Case Study: Providing actionable feedback

Ms. Austin had the students in her class solve the problem in Figure 7.16. She found that most of the students in her class added the cans together and compared the sum of the number of cans in each drink: (Ann) 2 cans + 8 cans = 10; (Mary) 4 cans + 12 cans = 16 cans. The students with this solution concluded that Mary had the stronger drink because she had more cans not because the ratio of cans of concentrate to cans of water was greater in Mary's solution. However, there was one student, Mason, who correctly solved the problem.

Ms. Austin decided that she wanted more information from Mason and also decided to challenge Mason by extending the problem situation. She decided to give Mason feedback.

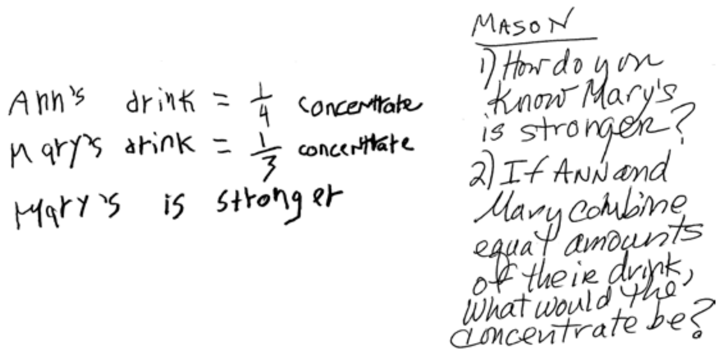
Ms. Austin understood that research has shown comments like good work or try again do not result in increased motivation and, therefore, do not result in increased student achievement. (Wiliam, 2011, p. 127).

> *In contrast, questions designed to engage students in thinking about the concepts and require students to respond have been shown to positively impact student motivation and achievement.*
>
> (Wiliam, 2011)

Therefore, providing written and oral feedback to individuals and to the whole class is a regular part of this teacher's practice. As a result, Mason knows he is expected to respond to feedback Ms. Austin provides him.

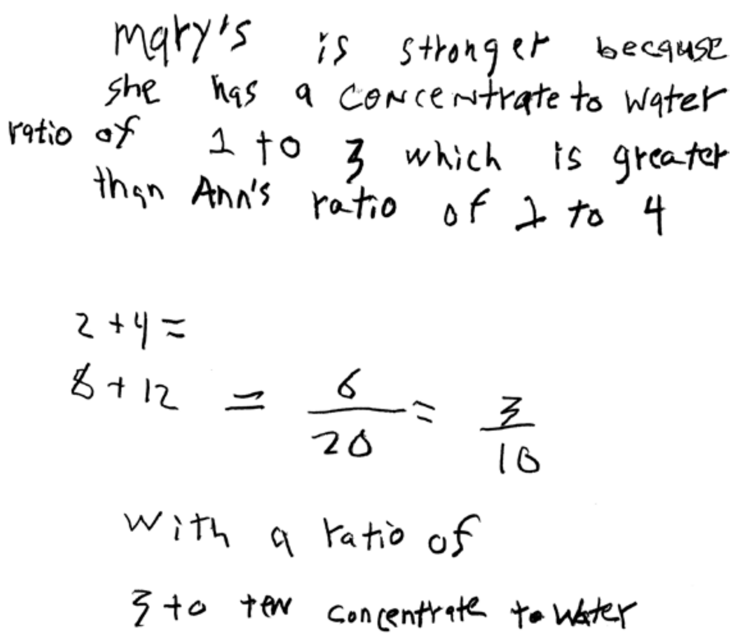
Study Mason's solution and Ms. Austin's comments in Figure 7.17. What additional information does the teacher expect to obtain from these questions?

Figure 7.17 Mason's response and the teacher's comments



Now study Mason's response to his teacher's questions in Figure 7.18. What additional evidence did the teacher obtain?

Figure 7.18 Mason's response to Ms. Austin's comments



As you can see, Mason's rationale for Mary's drink being stronger is based on comparisons of ratios involved in the problem. In response to the second question, one can see that Mason solved the problem—"What is the concentration if Mary and Ann combine their drinks?" not if they both contribute an equal amount of their drinks. However, the solution shows evidence of understanding how to combine ratios in a concentration context $\left(\frac{\text{cans of concentrate}}{\textit{cans of water}} = \frac{2\ \textit{cans} + 4\ \textit{cans}}{8\ \textit{cans} + 12\ \textit{cans}} = \frac{6}{20} = \frac{3}{10}\right)$ as opposed to inappropriately treating ratios as fractions and utilizing common denominators when combining them. This is important evidence of Mason's understanding of ratios.

Go To Go to Chapter 1 for an in-depth discussion of fraction–ratio confusion.

Chapter Summary

- In Chapter 6, we saw how problems involving the same context (e.g., constant speed) could be different problem types (e.g., a rate comparison problem, a unit rate comparison problem or a missing value problem). In this chapter, we now see how the opposite is also true. That is, the same problem type (e.g., scale factor) can be cast in different contexts (e.g., measurement conversions or similarity).
- Ratio and proportion problem context and the meaning of quantities used in different contexts are vitally important considerations during classroom planning and instruction.
- This chapter also considered other ratio and proportion contexts such as similarity, scale factors, measurement and monetary conversions, probability, sampling and concentrations.

Looking Back

1. Study Brooke's and Victoria's responses in Figure 7.19 to the Marbles in a Bag problem.
   a. What type of problem is it?
   b. What is the context?
   c. Where along the OGAP Ratio and Proportion progression are Brooke's and Victoria's solutions to the problem? What is their evidence?
   d. What is the evidence of Brooke's understanding or misunderstanding the concept?
   e. What did Brooke do differently than the other student work examples of the Marbles in a Bag problem found in this chapter?
   f. What is the evidence of Victoria's understanding or misunderstanding the concept?

   Sue has a bag containing 3 black marbles and 5 red marbles. Mary has a bag containing 5 black marbles and 10 red marbles. Whose bag would you choose to have the best chance of picking a red marble? Show your work.

Figure 7.19 Brooke's and Victoria's responses

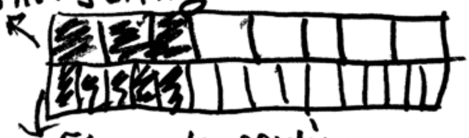
Brooke's response

Sue has the chance of nor picking it → $\frac{3}{8}$

Mary 5/15 is less than 3/8, so she has a better chance at getting a red marble

Mary has the chance of not picking it → $\frac{5}{15}$

3/8 not getting



5/15 not getting

$3/8 > 5/15$

Victoria's response

Sue: $\frac{3}{5}$ Mary: $\frac{5}{10}$

Sue

$\frac{3}{5}$ = more then $\frac{1}{2}$

$\frac{5}{10} = \frac{1}{2}$

mary has a lower chance of finding a red marble because $\frac{5}{10} = \frac{1}{2}$ but $\frac{3}{5}$ is more than $\frac{1}{2}$ so Sue has a better Chance of finding a red marble.

2. For each of the problems that follow:
   a. Identify the context of the problem.
   b. Identify the problem type.
   c. What strategies do you anticipate students would use to solve the problem?
   d. What difficulties do you anticipate students may encounter when they solve the problem.

Figure 7.20 Problem 1: Playground

A town enlarged a rectangular playground. The dimensions of the new playground are proportional to the old playground. Using the information in the table below, what is the length of the new playground? Show your work.

	Old Playground	*New Playground*
Width	50 feet	150 feet
Length	80 feet	*x*

Figure 7.21 Problem 2: Lakeview to Hillside

A road travels in a straight line between Hillside and Lakeview. What is the distance between Lakeview and Hillside?

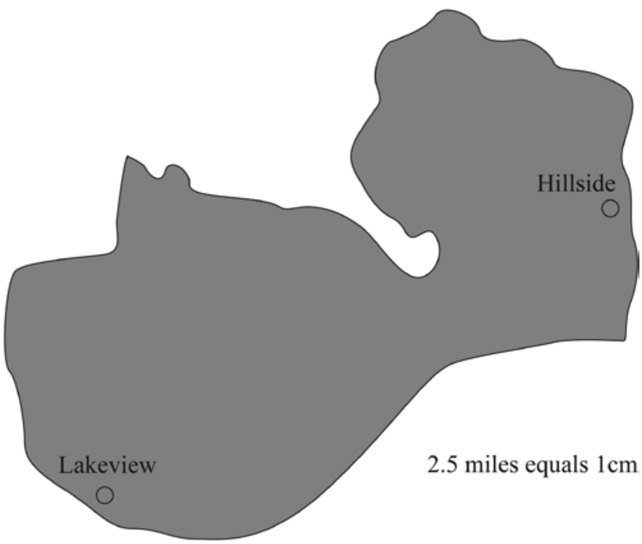


Figure 7.22 Problem 3: Car A and B

Car A travels 240 miles on 12 gallons of gasoline. Car B travels 180 miles on 9 gallons of gasoline.

- San says Car A travels more miles per gallon of gas because it traveled 240 miles and Car B traveled only 180 miles.
- Tracy says Car B travels more miles per gallon of gas because it used less gas than Car A.
- Tom says Car A and Car B use the same number of miles per gallon.

Who is correct? Explain why.

Figure 7.23 Problem 4: Belmont Forest

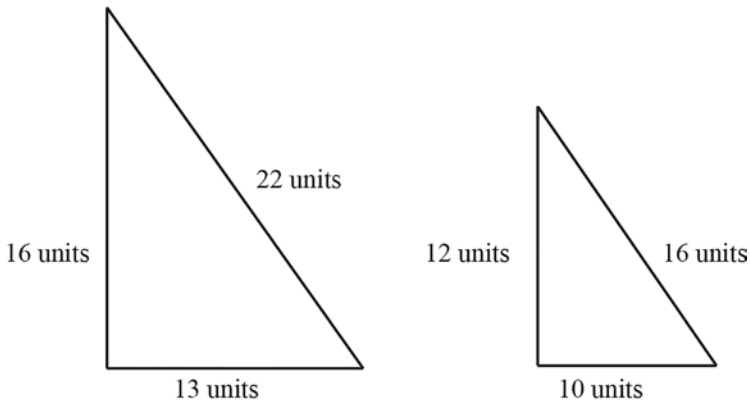
Belmont Town Forest has an area of 10 acres. There are about 30 maple trees in Belmont Town Forest. Douglas Town Forest has an area of 15 acres. There are about 38 maple trees in Douglas Town Forest. Which town forest has more maples trees per acre?

Figure 7.24 Problem 5: Mexican pesos

Erin is exchanging U.S. dollars for Mexican pesos. Four U.S. dollars are worth about 78 pesos. How many pesos should Erin get for $10?

Figure 7.25 Problem 6: Triangles

John says that the two triangles below are similar. Is he correct? Why or why not?



3. Review the student work in Figures 7.3, 7.5, 7.10, 7.11 and 7.13.
   a. Where along the OGAP Ratio and Proportion Progression is each piece of work?
   b. What is the evidence?
4. For the following pieces of student work, how might the context, problem type and/or other structure have influenced their strategy?

   a. Figure 7.3 Max
   b. Figure 7.6 Rick
   c. Figure 7.10 Debbie

8
Understanding the Cross-Products Procedure

Big Ideas

- Premature introduction or limiting students to only using the cross-products procedure may interfere with developing proportional reasoning.
- The cross-products procedure is not a trick. Rather, it has a mathematical basis.
- It is not a question of *if* students should be introduced to the cross-products procedure but *when* is the optimal time in their experience with ratios and proportions so that use does not interfere with the development of ratio and proportion concepts.

The cross-products procedure, also known as the cross-multiplication and the proportion procedure, is an efficient proportional strategy. However, it is often "poorly understood by students, seldom a naturally generated solution method, and often used to avoid proportional reasoning rather than facilitate it" (Lesh et al., 1988, p. 93). Since students rarely naturally generate the cross-products procedure, it follows that they may believe the cross-products algorithm is a trick rather than based on sound mathematical understanding. This chapter is included to help demystify this procedure and explain the most effective ways to use it in instruction.

This is not to say, of course, that students who use the cross-products procedure do not have a strong sense of proportionality. Rather, *premature introduction* of the procedure or *limiting* students to only the use of the cross-products procedure has been found by researchers to interfere with building strong proportional reasoning (Cramer, 2017; Lamon, 2005; Lesh et al., 1988; and others).

Cramer (2017) makes two very important and connected statements about the importance of an instructional emphasis on strategies that build and strengthen the understanding of the multiplicative relationships in ratio and proportion situations.

> When students solve proportional problems using a unit rate or a scale factor, they are using strategies based on the multiplicative relationships inherent in proportional situations. It follows then that students who are able to reason proportionally have a much better understanding of those relationships than those who are limited to using the standard cross-products procedure.
>
> (Cramer, p. 25)

To understand this idea study, Leo's, Olivia's and Harry's solutions in Figure 8.1. What do the strategies they use suggest about their understanding of proportionality in each of the problem situations?

Figure 8.1 Leo's, Olivia's and Harry's responses

Leo's response

Carrie is packing apples. It takes 3 boxes to pack 2 bushels of apples. How many boxes does Carrie need to pack 7 bushels of apples?

$$7 \div 2 = 3.5$$
$$3.5 \times 3 = 10.5$$

Olivia's response

Use the data in the table below to determine the number of cups of pancake mix to make 400 pancakes.

Pancake Mix					
Number of Pancakes	12	24	36	120	400
Cups of Pancake Mix	1¾	3½	5¼	17½	$57\frac{3}{4}$
Milk	1¼	2½	3¾	12½	$41\frac{1}{4}$

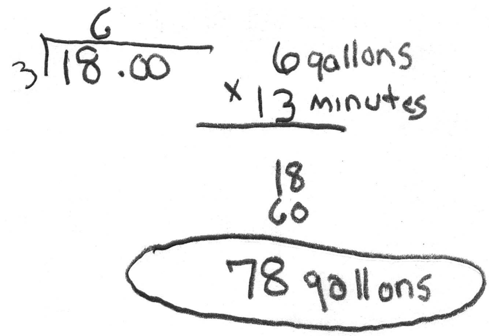
$$120 \times 3 = 360$$
$$120 \times \frac{1}{3} = 40$$
$$120 \times 3\tfrac{1}{3} = 400$$

$$17.5 \times 3\tfrac{1}{3} = 57.75 = 57\tfrac{3}{4}$$
$$12.5 \times 3\tfrac{1}{3} = 41.25 = 41\tfrac{1}{4}$$

Harry's response

Bob's shower uses 18 gallons of water every 3 minutes. How many gallons of water does Bob use if he takes a 13-minute shower?

Figure 8.1 Continued.



Leo found the multiplicative relationship between 2 bushels of apples and 7 bushels of apples $(7 \div 2 = 3.5)$ and then applied this relationship (3.5×3) to determine the number of boxes needed to pack 7 bushels of apples. Olivia correctly found and applied the multiplicative relationship between 120 pancakes and 400 pancakes $(\times 3\frac{1}{3})$. Notice there is a calculation error $\left(17.5 \times 3\frac{1}{3} \neq 57\frac{3}{4}\right)$. Harry found and applied the multiplicative relationship between gallons and minutes ($\times$ 6). The evidence in these solutions suggests that all three students used the multiplicative relationships between the quantities in these problem situations even given the non-integral feature in each of these problems.

In Chapter 2, there was an in-depth discussion of strategies proposed to help build proportional reasoning evident in these student solutions.

1. Using representations to build understanding of ratios and proportions
2. "Interpreting the effects of change on variables without numerical values" (Cramer, 2017, p. 27)
3. "Discriminating between examples and non-examples of proportional situations" (Cramer, 2017, p. 27)
4. "Reasoning using a wide variety of problem solving strategies and connections" (Cramer, 2017, p. 28)

It is suggested you read Chapter 2 before continuing with this chapter.

The rest of this chapter focuses on the mathematical connections between the strategies evidenced in Leo's, Olivia's and Harry's solutions and the cross-products procedure, how the multiplication property of equality can bring meaning to the cross-products procedure and finally advice from researchers on when best to introduce the cross-products procedure so as not to interfere with students' developing proportional reasoning.

What Are Mathematical Connections Between the Multiplicative Change, Unit Rate and Cross-Products Procedure?

Not only is the cross-products procedure rarely generated naturally by students, but the multiplicative relationships represented in proportional relationships is not overtly transparent in this procedure as it is in other strategies such as scaling up or down, using the multiplicative relationships between quantities or applying the unit rate. That being said, the cross-products procedure is not a trick. Rather, it is founded on logical mathematics principles. This section focuses on how the unit rate strategy, the multiplicative change strategy and the cross-products procedure are mathematically connected.

Study the problem and the three solutions in Figure 8.2. In what ways are these solutions similar, and in what ways are they different?

Figure 8.2 Solution strategies to. . .

Twenty-four apples cost $8.00. How many apples can be bought for $32?

Unit Rate Strategy

$8 per every 24 apples = 3 apples per dollar

32 · 3 = 96 apples

Multiplicative Change Strategy

$$\overset{\times 4}{\frac{24}{\$8} = \frac{x}{\$32}} \quad x = 96 \text{ apples}$$

$$\times 4$$

Cross-Products Procedure

$$\frac{24}{8} = \frac{X}{32}$$

$$8x = 768$$

$$X = 96 \text{ apples}$$

Looking closely at these solutions hints at why the cross-products procedure is not usually naturally generated and does not have the same potential to support students' development of proportional reasoning. In the unit rate strategy, one uses the solution to how many apples one can purchase with $1 (3 apples per $1) to find out how many apples can be bought with $32. In the multiplicative change strategy, the scale factor between $8 and $32 is identified and applied to scale 24 apples to 96 apples. Both these strategies reflect the problem situation, are founded on reasoning and utilize the multiplicative relationships between the quantities, in both cases scaling up multiplicatively. One can see why these strategies are more likely to be naturally generated by students.

In contrast, it is hard to see the same multiplicative relationships in the cross-products procedure when you multiply \$32 × 24 apples. What does the resulting product of \$32 and 24 apples mean? It doesn't mean 768 apples per dollar or \$768 per apple. Also, 768 dollar-apples has no apparent connection to the context. Importantly, unlike the unit rate and the multiplicative change strategy, in which one can visualize the scaling up (or down) multiplicatively, the multiplicative change is not evident in the cross-products procedure.

In spite of this confusion, the three strategies are mathematically connected. The equations that represent each of these solution strategies are in Figure 8.3. Study these equations. What is similar, and what is different?

Figure 8.3 Equations for different strategies

Unit Rate Equation: (24 apples ÷ \$8) × \$32 = 96 apples

Multiplicative Change Equation: (\$32 ÷ \$8) × 24 apples = 96 apples

Cross-Products Procedure: (24 apples × \$32) ÷ \$8 = 96 apples

Now study the equation in Figure 8.4, which represents the expressions created for each solution strategy above. Is this equation linking the three different strategies correct? Why or why not?

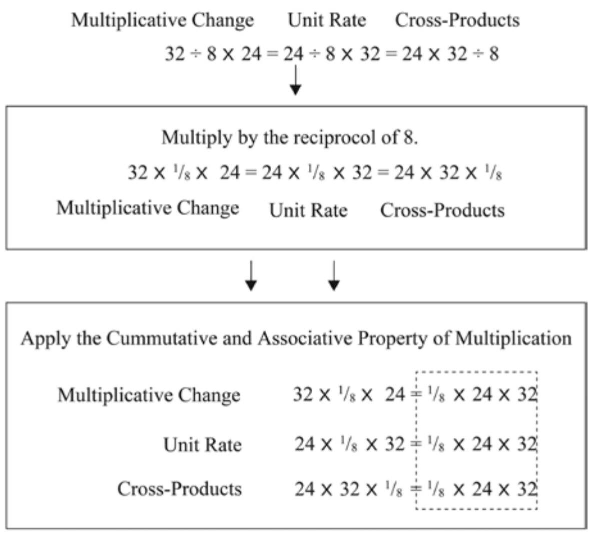
Figure 8.4 Are these expressions equivalent? Why or why not?

$$\underbrace{(32 \div 8) \times 24}_{\text{Multiplicative Change}} = \underbrace{(24 \div 8) \times 32}_{\text{Unit Rate}} = \underbrace{(24 \times 32) \div 8}_{\text{Cross Products}}$$

Now study Figure 8.5. How are the relationship between division and multiplication and the commutative property of multiplication used to prove that the three expressions $(32 \div 8) \times 24$, $(24 \div 8) \times 32$ and $(24 \times 32) \div 8$ are equivalent?

Figure 8.5 Mathematical explanations for the equivalence of the expressions

Twenty-four apples cost \$8.00. How many apples can be bought for \$32?

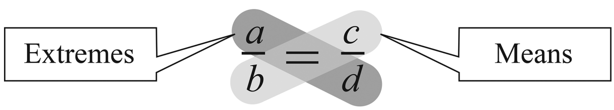


The key to understanding that the equation in Figure 8.4 is true is illustrated in Figure 8.5. Note that the first equation in Figure 8.5 consists of expressions representing the multiplicative change strategy, the unit rate strategy and the cross-products procedure for the apple problem. Next is understanding that dividing by 8 is equivalent to multiplying by $\frac{1}{8}$ shown in first big box in Figure 8.5. Finally, in multiplication the order in which the terms are multiplied does not change the product of the terms (associative and commutative properties of multiplication). Thus, the terms of each expression can be multiplied in any order, resulting in the equivalent expressions for each strategy found in the dashed box. This allows us to see that the three expressions representing the three different solution strategies are equivalent.

Multiplication Property of Equality and Cross-Products Procedure

The cross-products procedure is based on the means-extremes property that states: In any equation that includes two equal ratios, the product of the extremes equals the product of the means. The third solution in Figure 8.2 is an example of a solution based on this property. Figure 8.6 shows the algebraic representation of the means-extremes property in the context of the cross-products procedure. This relationship provides an efficient method to solve proportion problems.

Figure 8.6 The product of the extremes equals the product of the means



$$ad = bc$$

The means in a proportion are the two terms that are closest to each other when the proportion is written with colons. That is, the proportion in Figure 8.6 can also be represented as $a : b = c : d$. The means, therefore, are b and c. The extremes are the two terms in the proportion furthest from each other or, in our example, a and d. But what is the mathematical basis for the means-extremes property? For many, this procedure was taught as a short cut rule. As Lesh et al. (1988) indicated it is "seldom a naturally generated solution method" and is applied without understanding proportionality. The multiplication property of equality is key to understanding the mathematical basis for this property.

The multiplication property of equality states that if $a = b$, then $a \times c = b \times c$. As you read this, you might remember that the multiplication and addition properties of equality can be used to solve equations. For example, when solving the equation $\frac{x}{3} = 12$ we can multiply both sides of the equation by 3 to find that $x = 36$ $(3 \times \frac{x}{3} = 12 \times 3)$.

Using variables, one can use this property to illustrate that the product of the means equals the product of the extremes. Study Figure 8.7. In what ways is the multiplication property of equality being applied to illustrate that the product of the extremes equals the product of the means?

Figure 8.7 The product of the extremes equals the product of the means

$$\frac{a}{b} = \frac{c}{d} \qquad \text{Original proportion}$$

$$\cancel{b} \bullet \frac{a}{\cancel{b}} = b \bullet \frac{c}{d} \qquad \text{Multiply both sides of the equation by } b.$$

$$\downarrow$$

$$a = \frac{bc}{d}$$

$$a \bullet d = \frac{bc}{\cancel{d}} \bullet \cancel{d} \qquad \text{Multiply both sides of the equation by } d.$$

$$\downarrow$$

$$ad = bc \qquad \text{The product of the extremes equals the products of the means.}$$

Study Figure 8.8. How is the application of the multiplication property of equality applied to solve the problem, and what is the relationship to the cross-products procedure?

Figure 8.8 Application of the multiplication property of equality

Twenty-four apples cost \$8.00. How many apples can be bought for \$32?

$$\frac{24 \text{ apples}}{\$8} = \frac{n}{\$32} \qquad \text{Original proportion}$$

$$\cancel{\$8}\left(\frac{24 \text{ apples}}{\cancel{\$8}}\right) = \frac{n}{\$32} \times \$8 \qquad \text{Multiply both sides by \$8}$$

$$\downarrow$$

$$24 \text{ apples} = \frac{8n}{32}$$

$$32 \times 24 \text{ apples} = \frac{8n}{\cancel{32}} \times \cancel{32} \qquad \text{Multiply both sides by 32}$$

$$\downarrow$$

$$\frac{768}{8} = \frac{\cancel{8}n}{\cancel{8}} \qquad \text{Divide both sides by 8}$$

$$n = 96 \text{ apples}$$

While the multiplicative relationships central to ratios and proportions is not apparent in the cross-products procedure, one can see by using the multiplicative property of equality that the procedure is founded on mathematical properties. Using the multiplication property of equality, one can also see mathematically why 24 is multiplied by 32 even though the resulting quantity, 768, does not have a meaningful unit. In addition, one can see in Figure 8.8 the product of the extremes (24×32) and the product of the means ($8n$).

When Do Researchers Suggest Introducing the Cross-Products Procedure?

Students develop their proportional reasoning over time, bridging from their experiences with multiplication and division in the elementary grades to solving unit rate problems in grade 6. They then build on these early experiences as they interact with proportional situations using graphs, ratio tables and equations, leading to understanding of functional relationships. Key to this development is having the opportunity to interact with a range of problem contexts (Chapter 7) and problem structures (Chapters 6 and 7) and making decisions about the best proportional strategy to use given the problem situation and structures.

With this said, it is not a question of *if* students should be introduced to the cross-products procedure but *when*. Ellis (2011) indicates that students are positioned to understand the "value and efficiency" of the cross-products procedure if they have met the following:

> Students have gained experience with (a) learning how to simultaneously attend to two quantities, (b) comparing quantities multiplicatively rather than additively, (c) forming a ratio, either as a multiplicative comparison or as a composed unit, and, (d) understanding a proportion as equivalent ratios.
>
> (p. 4)

Procedural fluency is characterized by more than simply knowing how to carry out a procedure. It also includes the knowledge of when to use a particular procedure appropriately. With this in mind, one hopes the cross-product strategy is taught in a conceptual way so students understand that the cross-products procedure is a non-obvious shortcut which can be proven. Importantly, students should understand the cross-products procedure as another strategy they can use to solve proportion problems when it is applicable. It should not be seen as the only strategy for solving proportion problems.

For example, the cross-product strategy may not be the most efficient strategy for solving the Apple problem, because the quantities are relatively small, the unit rate can be easily calculated due to the integral relationship between the 24 apples and their cost, $8.00 and the context is familiar. A unit rate strategy seems more appropriate for this problem.

The OGAP Ratio and Proportion Progression

Study the strategies at the Proportional Strategies level on the OGAP Ratio and Proportion Progression. Notice that the cross-products procedure is one strategy at this level in addition to use of $y = kx$, comparing fractions, ratios and rates, applying the multiplicative scale factor and applying a unit rate.

Chapter Summary

- Premature introduction or limiting students to only using the cross-products procedure may interfere with developing proportional reasoning.
- The cross-products procedure is founded on the multiplication property of equality.
- The cross-products procedure seems to be instructionally appropriate after students understand how to simultaneously deal with the two quantities in a ratio, understand the multiplicative nature of ratios and proportions and understand a proportion as equivalent ratios.

Looking Back

1. Answer questions a and b using the following problem situation.
   On Monday, Francis walks 6 miles in 2 hours. If Francis walks at the same rate on Tuesday, how long should it take him to walk 8 miles?
   a. Solve the problem using multiplicative change, unit rate and cross-products strategies. Write an equation representing each strategy.
   b. Describe how the three strategies are mathematically related.

2. Use the multiplication property of equality on the equation you created in question 1a for the cross-products procedure to illustrate the mathematical basis of the cross-products procedure.

3. Case study: You have just started your proportion unit with your sixth-grade students. A parent contacts you, wondering why their daughter is not using the cross-products algorithm to solve proportion problems. Write a description of what you would say to help the parent understand the importance of building understanding of ratio and proportions before introducing the cross-products procedure. What research would you cite? How might you use student work samples or other examples to make your case?

Instructional Link

Use the questions that follow to analyze ways your math instruction and program provides students opportunities to build fluency and understanding of important ratio and proportion concepts and skills.

1. To what degree do your math instruction and program focus on a range of strategies to build understanding of proportions?
2. If you or your math program introduce the cross-products procedure into instruction, to what degree is the procedure introduced through understanding rather than a series of steps?
3. Based on your analysis of your math program for questions 1 and 2, are there modifications you would make to your instruction?

9
Percents—Building Understanding, Flexibility and Fluency

Big Ideas

- Researchers have found that percent concepts are difficult for students and adults alike even though the application of percents is prevalent in all aspects of everyday life.
- Percent instruction should build on students' intuitive sense of percents and on benchmark percents and fractions.
- Researchers suggest using visual models, percent tables, and mental calculation to build understanding and fluency with the relationships among fractions, decimals and percents, and to focus on the relationships among the quantities in percent situations.

Introduction

Percents are found in all aspects of our daily life. In spite of this, researchers have found both students and adults alike, including in-service teachers, have difficulty understanding percents and using them to solve everyday problems (Cockcroft, 1982; Parker, 1994; Ginsburg, Gal, & Schuh, 1995; Koay, 1998).

Interestingly, researchers have found that students often enter middle school having experience with percents in the world around them. Students are often familiar with benchmark percents such as 25%, 50% and 75% and the equivalent fractions $\frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$, although that familiarity does not extend to understanding that percent is a ratio expressed per 100 (Koay, 1998). Researchers, however, have also found that "formal instruction on percent tends to restrict pupils' creativity and flexibility in their approaches to percent problems. Formal instruction tends to ignore the relationships underlying the referents inherent in percents and makes pupils more reliant on the procedural knowledge and less on the proportional relationships" (Koay, 1998).

Recognizing the complexity and importance of students' deep understanding of and ability to flexibly use percents, this chapter focuses on the following:

1. Definition of percents
2. Understanding different percent situations and their impact on instruction
3. Grades 6, 7 and 8 CCSSM expectations for percents
4. Building on intuitive sense of percents

5. Using visual models to help develop understanding of the meaning of percents, the relationships between fractions, decimals and percents and the relationships among the quantities in percent problem situations
6. Using percent tables and mental calculation to develop flexibility when solving percent problems
7. Understanding percent increase and decrease
8. Common errors or misconceptions that interfere with students' understanding of percents or solving problems involving percents

What Is a Percent?

A percent is a ratio expressed per 100 $\left(\frac{N}{100}\right)$ and is written with a percent symbol (%). A key feature of percents is the use of an independent unit to *norm to a standard*; in this case 100 (Wu, 2011). The use of percents as a norming standard facilitates comparisons. To begin to understand what this means, study the labels in Figure 9.1 for the fat content of ground beef. We are familiar with these labels and have confidence that we can make a comparison about the fat content in various kinds of ground beef because they are compared using percents. That is, both have the same denominator of 100. Imagine if the labels were expressed as a fraction of the weight of each package. One package might indicate that $\frac{4}{5}$ of the weight is fat and the other that $\frac{7}{12}$ of the weight of the package is fat. While one could compare these two fractions, perhaps using a common-denominator strategy, differences in fat content between the two packages is not obvious and requires computation. Instead, norming to 100 creates fractions with denominators equal to 100, making comparisons easier and more obvious. Notice how much easier it is to compare the fat content in the two packages of beef when the portions of fat in the two packages, $\frac{20}{100}$ and $\frac{10}{100}$, utilize the same denominator.

Figure 9.1 Ground beef labels

What Are Some Different Percent Situations and the Implications for Instruction?

There are different ways percents are used in everyday life. These differences have implications for instruction. Solve the problems in Figure 9.2. How do these percent situations differ? What are implications for instruction?

Figure 9.2 Different percent situations

1. Eighty percent of the students in the class completed the assignment on time. There are 30 students in the class. How many students completed the assignment on time?

2. The school population was 1200 in 2019. It increased by 20% in 2020. What is the school population in 2020?

3. The discount on the purchase of an iPad was 20%. The original cost was $800. How much did the iPad cost after the discount?

4. The school nurse expects that for every two students in a school classroom who get the flu, eight students will not get the flu. What percent of students in a classroom is expected to get the flu?

5. The CD at the bank yields 3% interest per year. If Max has a one-year $1000 CD, how much will it be worth after 1 year?

You probably noticed that each of these problems involves a different percent situation. Each situation forces one to reason differently about the quantities in the situation, and each situation is represented by a different equation. Study the sample equations in Table 9.1. How do these different percent situations impact the related equations? Why?

Table 9.1 Percent situation and sample equation associated with each problem in Figure 9.2

Problems from Figure 9.2	*Percent Situation*	*Sample Related Equations*
Problem 1: Eighty percent of the class. . . .	Percent of a Whole	80% × 30 students = 24 students
Problem 2: The school population was 1200 in 2019. . . .	Percent Increase	1200 students + (20% × 1200 students) = 1440 students
Problem 3: The discount on an iPad was 20%. . . .	Percent Discount	$800 – (20% × $800) = $640
Problem 4: The school nurse expects that for every two students in a school classroom that get the flu. . .	Probability	2 students ÷ 10 students = 20%
Problem 5: The CD at a bank yields 3% interest per year. . .	Simple Interest	$1000 + ($1000 × 3%) = $1030

The equations to solve these percent situations differ, because in everyday usage of percents, quantities vary. Researchers suggest placing emphasis on the quantities and their relationship in instruction instead of set formats or procedures for solving percent problems (Van de Walle, Bay-Williams, & Karp, 2013; De Bock, Dooren, Janssen, & Verschaffel, 2007). That is, instead of thinking about the quantities, "students perceive them to be one of four numbers used to compute" (Collins & Dacey, 2010, p. A24) when setting up a proportion, or, in the case of the use of a 'part = *n*% × whole' template (______is ______ percent of _____), students think about putting two numbers in the blanks and not about the relationships among the quantities in a problem.

CCSSM and Percent Problem Situations in the Middle School

Building off the study of ratios and proportions, students at grade 6 are introduced to percent problems that involve the part, the whole and the percent. This is extended at grade 7 to a range of problem situations such as tax, markups, tips, fees, percent increase and decrease, simple interest and percent error (CCSSM/NGA, 2010). In addition, students in middle school will begin to apply their understanding of percents to other disciplines (e.g., population increase or decrease, percent error in an experiment). This suggests the importance of working closely with the teachers of other disciplines so instruction on percents is consistent across a student's experience.

Researchers suggest a number of ways to help students focus on the quantities rather than on a set of procedures (Van de Walle et al., 2013; Collins & Dacey, 2010) to build understanding, fluency and flexibility with percents.

1. Build on students' intuitive sense of percents using benchmark percents.
2. Use visual models to build understanding of the relationships between fractions, decimals and percents and to extend to solving percent problems.
3. Engage students in mental calculation of percent situations using benchmark percents and compatible numbers.
4. Engage students in a range of percent problems in context, focusing on patterns and relationships among the quantities.

Each of these strategies is examined in this chapter.

Building on Students' Intuitive Sense of Percents

Students arrive at middle school with an intuitive sense of percents (Lembke & Reys, 1994). While some teachers may treat percents as a new topic in middle school mathematics classrooms, most students encounter percents in their everyday lives. Figure 9.3 contains some examples of the ways students encounter and develop an intuitive sense of percents that instruction can build on. Study Figure 9.3 and think about how you can use these and other examples as an instructional tool.

Figure 9.3 Everyday percents

One might begin percent instruction by asking students to explain what they think each of the percents in Figure 9.3 means. The information you might gather from an activity like this can help engage students in the importance and utility of percents as well as provide you with important formative information about your students understanding of percents. For example, what is their intuitive sense of 100% or 50%. Do students understand what 2% and 10% fat content means? Do students understand % Daily Value?

An activity such as this has the potential to give you a starting place to support students' understanding of the relationship among the whole, the part and the percent and between fractions and percents. For example, building the idea early in percent instruction that 100% means all is important. Researchers have identified that understanding that 100% means all is a difficult concept for students (Ginsburg et al., 1995).

As mentioned earlier in the chapter, percent concepts are difficult and poorly understood by students and adults alike. From an initial activity like this one, instruction can build upon the formative information gathered to extend and deepen students' understanding, flexibility and fluency with the range of percent topics. This can be done by using visual models and mental calculation and engaging students in problems in context while focusing on patterns and relationships among the quantities.

The next section focuses on how to use visual models to extend understanding of the meaning of percents as well as the relationships among fractions, decimals and percents.

Using Visual Models to Extend and Deepen Understanding, Flexibility and Fluency With Percents

One way to help students develop understanding of percents is through using visual models. Some visual models suggested in the research for developing percent concepts include percent bars, 10×10 grids, tables and double number lines. This section will describe the role each of these visual models plays in developing understanding of percent concepts.

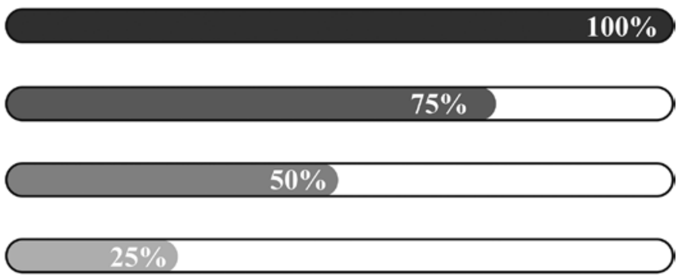
Percent bars and double number lines can be used effectively to help develop students' understanding of and fluency with two major percent concepts: (1) equivalence between benchmark fractions and percents and (2) the relationships among quantities in percent problem situations (part, whole and percent). A key to capitalizing on percent bars and double number lines is engaging students in word problems in which benchmark fractions and percents are used along with compatible numbers. Benchmark percents include 1%, 10%, 25%, 50%, 75% and 100%. Compatible numbers are numbers that are easily calculated mentally or are not complicated calculations (e.g., 10% of 80 compared to 12.5% of $75.12). Benchmark percents and their fraction equivalents in initial instruction help students develop understanding of important percent concepts without interference from complicated calculations (Van de Walle et al., 2013) as well as to develop percent number sense (Lembke & Reys, 1994). That is, "Familiarity with benchmark percents provides conceptual anchors to use when percents are encountered" (Reys, Lindquist, Lambdin, & Smith, 2009, p. 303).

Throughout the chapter, there are examples of benchmark percents and compatible numbers and discussion on how their use along with visual models can help students build a strong foundational understanding of percents.

Using the Percent Bars and Double Number Lines to Focus on Equivalence Between Benchmark Fractions and Percents

Starting with the percent bar model provides a link to experiences students have had downloading apps or movies or charging cell phones. Study Figure 9.4, which shows the download progress of four apps. What questions could you ask students to link the percent bar model to a fraction bar model?

Figure 9.4 Percent APP download completed

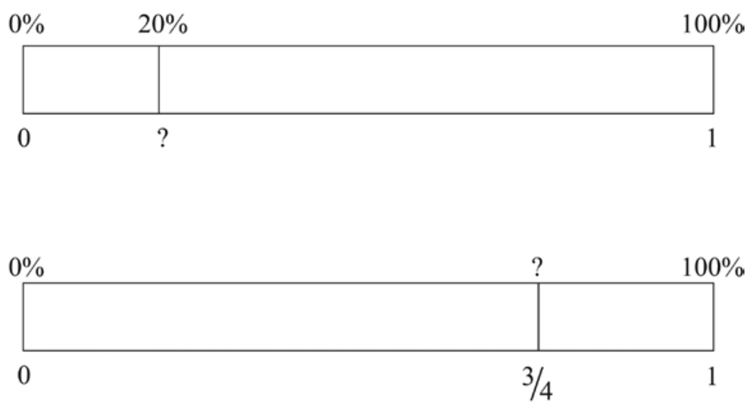


One question a teacher might ask is, "What fractional part of each download is complete?" In addition, one might ask how one can use the percent bars given to determine the fractional part of each download. This question asks students to link their developing ideas of percent to fraction concepts they have worked with in previous grades.

Figure 9.5 is an example of a percent bar designed to help students understand fraction and percent equivalence. Notice that the length of the bottom edge of each bar is labeled with a 0–1 scale. The top edge of each bar utilizes a 0% to 100% scale. The first example focuses students' thinking on the connection between a given percent and its fraction equivalent. The second example asks students to consider the percent equivalent for a given fraction.

What are the missing quantities on the percent bars in Figure 9.5?

Figure 9.5 Missing fractions and percents

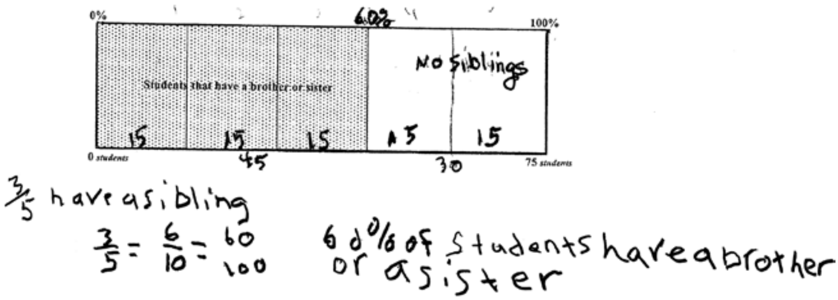


While this activity is similar to the activity in Figure 9.4, it utilizes a percent bar that integrates percent and fraction scales to specifically focus on equivalent fractions and percents.

Look at Shawn's work in Figure 9.6. How does Shawn use the percent bar and his understanding of equivalence between fraction and percents to answer the question?

Figure 9.6 Shawn's response

Judy surveyed the 75 students at her middle school to find the number of students that have a brother or a sister. She used the model below to show her findings. Use Judy's model to estimate the percent of students who have a brother or sister.



Shawn partitioned the whole bar into fifths and equally distributed the 75 students in each fifth (15 students). He found that $\frac{3}{5}$ of the students had siblings and then restated that this means 60% of the students have a brother or a sister.

While the percent bars and double number lines are useful for helping students develop understanding of equivalence between benchmark fractions and percents, these models lose their utility for developing understanding of equivalence between fractions, percents and decimals for non-benchmark percents (e.g., 73%, $43\frac{1}{2}$ %, 25.6%) and for understanding percents less than 1% (e.g., 0.25%). The 10 × 10 grid, discussed later in this chapter, has more utility when extending understanding to non-benchmark percents and percents less than 1%.

Using the Percent Bars and Double Number Line to Focus Relationships in Percent Situations

A major use of the percent bar and double number line is to help students focus on the relationships among the part, the whole and the percent quantities in percent problem situations when the problems involve benchmark percents and compatible numbers.

Solve the Sneaker problem that follows using a percent bar. What do you notice about the numbers in the problem? What do you notice about the percent in the problem? What important idea is introduced with this problem?

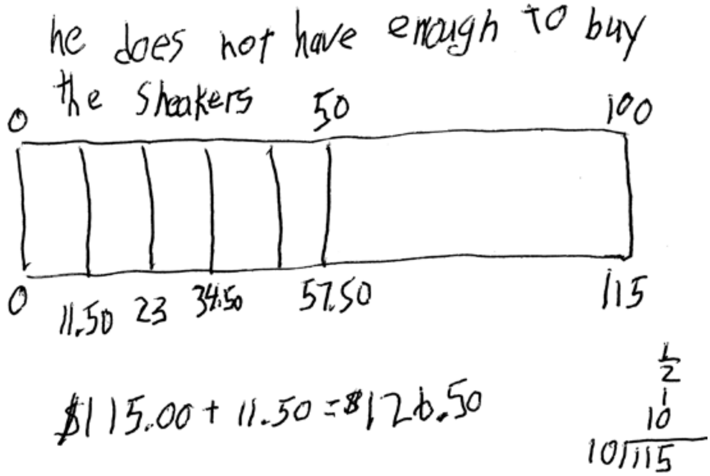
> John has $125.00 to buy sneakers. Is this enough money to buy sneakers that cost $115.00 plus tax of 10%?

You probably noticed that the percent is the benchmark 10%. Importantly, this problem introduces the idea that the percent bar can represent a whole greater than one. In this case, the whole is represented by $115.00.

Study Margaret's solution to the cost of the sneakers in Figure 9.7. How is the amount of the whole—$115—reflected in the percent bar? How does the percent bar help build student understanding of the relationship between the cost of just the sneakers to the cost of the sneakers with 10% tax added?

Figure 9.7 Margaret's response

John has $125.00 to buy sneakers. Is this enough money to buy sneakers that cost $115.00 plus a sales tax of 10%?

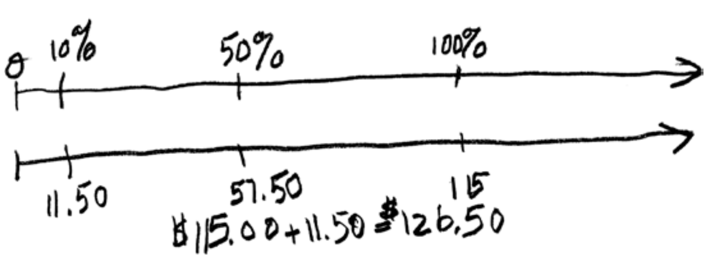


The percent bar requires attention to the relationships among the quantities. That is, identifying that the whole is $115.00 and equal to 100% then finding the part equal to 10%. In this case, the student partitioned the whole into tenths and determined that a tenth of the cost of the sneaker is $11.50; thus the scale on the bottom edge of the bar increases by $11.50. Although not explicitly stated, the evidence in the solution suggests Margaret partitioned the bar into tenths using fractional reasoning.

This same problem can be solved using a double number line. Study the solution to the cost of sneakers using a double number line in Figure 9.8. How is the double number line solution the same or different than using a percent bar?

Figure 9.8 Double number line solution

John has $125.00 to buy sneakers. Is this enough money to buy sneakers that cost $115.00 plus tax of 10%?

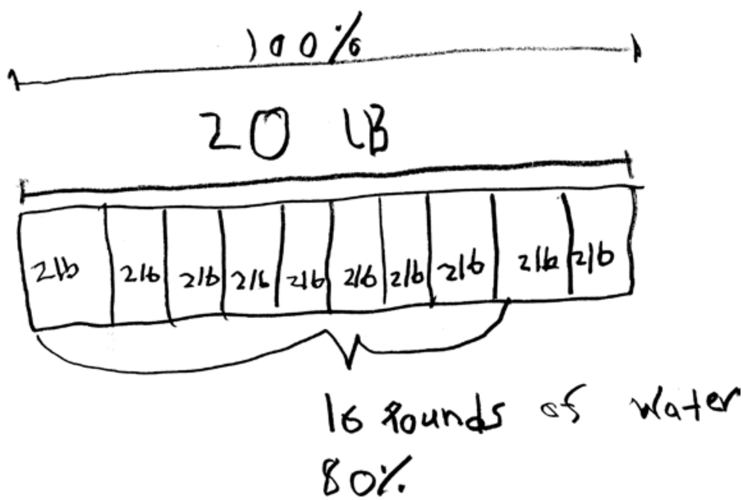


Some aspects of the solutions appear similar. That is, the students partitioned both the percent bar and the double number lines into tenths as well as determined the quantity each tenth represents ($11.50). In the solution in Figure 9.8, the distance from 0 to 100% and the distance from 0 to $115.00 is labeled. One difference is the arrow on a number line implies that there are amounts and percents greater than 100%.

This difference between a percent bar and double number line isn't major, nor is one model better suited than the other for the purposes identified in this section. In fact, in many cases, student-generated models are a hybrid of a percent bar and a double number line. Study Arnold's solution to the problem in Figure 9.9. What is the strategy used in the situation?

Figure 9.9 Arnold's solution to Baby's Water Weight

The average baby at age 1 weighs 20 lbs. About 80% of a baby's weight is water. What is the weight of the water in the baby?

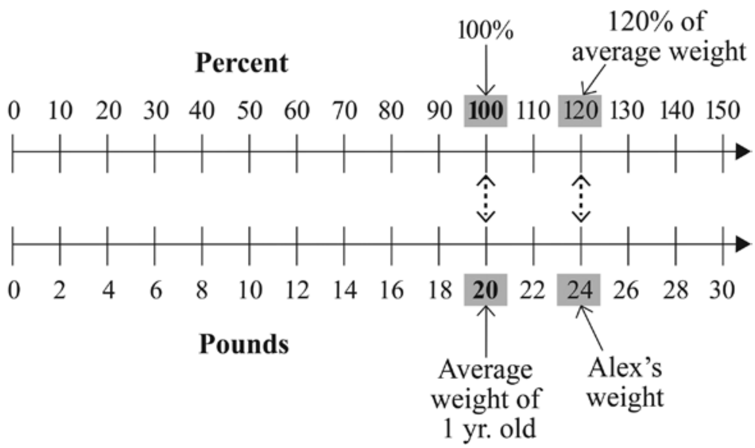


It appears that the visual model in Arnold's solution is a hybrid of a percent bar and a partitioned area model. He partitioned the area of the bar but, importantly, indicated that the 20 lbs. equals 100% and 16 pounds equals 80%. The solution focused on the relationships between the quantities in the problem situation, which is the important mathematical point. Early OGAP Fraction studies (OGAP, 2005, 2007) found that some teachers focused their instruction on the visual models themselves rather than using visual models to build mathematical concepts. The nuance seems small but is actually large. The important lesson is to be flexible and realize that student-generated visual models might be hybrids. *The most important instructional consideration is that the visual model and the student solution should reflect the underlying mathematical concept central to the problem.*

A double number line is an effective visual model for making percents greater than 100% visible. Study the solution in Figure 9.10. In what ways does the double number line make percents greater than 100% visible?

Figure 9.10 Double number line

The average weight of a one-year-old is 20 pounds. If Alex is one year old and weighs 120% of the average weight of a one-year-old, how much does Alex weigh?



While this problem can be successfully solved using other visual models or strategies, the fact that the number line extends beyond 100% visually illustrates how 120% is greater than the original 20 pounds.

Now solve the problems in Figure 9.11 using percent bars or a double number line. How are the problems the same or different? How do the percent bar and double number line help build student understanding of the relationships among the various quantities (part, whole and percent) in each of the problems?

Figure 9.11 Solve using percent bars and/or double number lines.

1. In one minute, approximately 60,000 barrels of oil are used in the world. The U.S. uses almost 15,000 of those barrels. What percent of the worldwide use of barrels of oil does the U.S. use per minute?
2. A python sleeps 75% of each day. How many hours per day does a python sleep?
3. A Utah ski area claims its average yearly snowfall is about 500 inches. In 2019, the snowfall was 125% of the average yearly snowfall. About how much snow fell at this Utah ski area in 2019?
4. A leopard bullfrog's life span is about 6 years. That is about 20% of the life span of a lobster. What is the approximate life span of a lobster?

Each of these problems involves a different relationship among three distinct quantities: the percent, the part and the whole. Table 9.2 illustrates how the given quantities vary in these problems. Notice that within a given problem, each quantity is in the text of the problem or in the question.

Study your own solutions. Make note of how the actual construction of the percent bars and double number lines focused your attention on these three quantities in each problem rather than on identifying key words or applying a formulaic procedure. Researchers suggest students engage in a variety of story problems that help students focus on finding patterns and relationships in their solutions to build a conceptual framework for understanding the relationships among the percent, the part and the whole.

Table 9.2 How do the quantities in problems 1, 2, 3 and 4 in Figure 9.11 vary?

Percent	*Whole*	*Part*
1. What percent of the worldwide use of barrels of oil does the U.S. use per minute?	60,000 barrels per minute	15,000 barrels per minute in the U.S.
Python sleeps 75% of the day.	24 hours	**2. How many hours per day does a python sleep?**
125% of the average snowfall	Average snowfall 500 inches	**3. How much snow fell in the Utah ski area in 2019?**
A leopard bullfrog's life is 20% of a lobster's life.	**4. What is the life span of a lobster?**	Life span of a leopard bullfrog is 6 years.

As mentioned, the use of a percent bar or double number line focused attention on the relationships between the quantities. For example, question 1 involved the ratio of 15,000 barrels of oil used per minute in the U.S. to 60,000 barrels of oil used per minute worldwide: $\frac{15{,}000 \text{ U.S.}}{60{,}000 \text{ world wide}} = \text{percent U.S.}; \frac{15{,}000}{60{,}000} = \frac{1}{4} = \frac{25}{100} = 25\%$.

In contrast, OGAP facilitators have observed that some teachers have students set up a proportion (e.g., $\frac{15{,}000}{60{,}000} = \frac{n}{100}$) and use the cross-products procedure to solve proportion problems. The cross-products procedure for this situation generates the equation $1{,}500{,}000 = 60{,}000n$. Notice how the value of 1,500,000 is not easily explained in the context of the problem. In addition, plugging values into the 'part = n% x whole' template results in the equation 15,000 = n% of 60,000. Students, of course, should be able to solve this equation, but notice how it hides the fact that this is a ratio problem instead turning it into a calculation.

Additionally, other strategies such as key words are not encouraged. Clement and Bernhard (2006) found that "students become over reliant on key words and used them in lieu of reasoning about or understanding the situation" (p. 363). *Focusing on the quantities provides students with an "explicit process through which they can begin to make sense of mathematical situations"* (Clement & Bernhard, 2006, p. 363).

Students should solve different types of percent problems, continually considering the part, the whole, and the percent. They should use visual models and study patterns that help focus their understanding of the relationships among the quantities and should not be encouraged to memorize procedures or use key words. The case study that follows provides one possible instructional approach to help students focus on the quantities and the relationships among the quantities in percent problems.

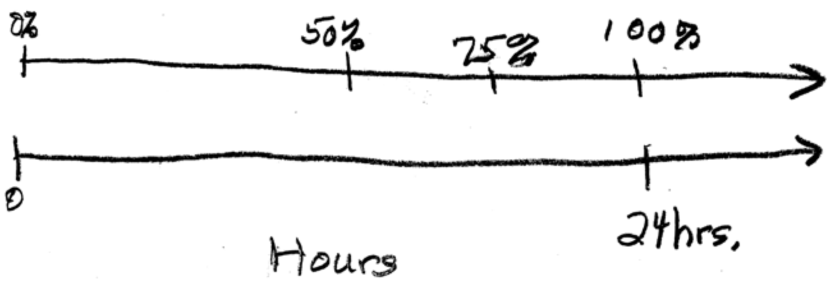
Mini-Lesson: Engage Students in a Range of Percent Problems in Context, Focusing on Patterns and Relationships Among the Quantities

Ms. Huntoon has just analyzed a set of exit questions involving percent problems in context in which different quantities (i.e., part, whole, percents) are given in the problems. This was the first time she had given more than one type of percent problem at the same time. She realized that many of her students were not discriminating among the quantities given in the text and the quantity contained in the question. In addition, few of her students used a percent bar or a double line to make sense of the problem situation even though that had been the focus of instruction. She decided to lead a mini-lesson in which she created incomplete visual models for the two exit questions (Questions 2 and 4 in Figure 9.11).

Study the two incomplete visual models in Figures 9.12. How might you use these incomplete visual models to help students focus on the relationships among the quantities in the problem?

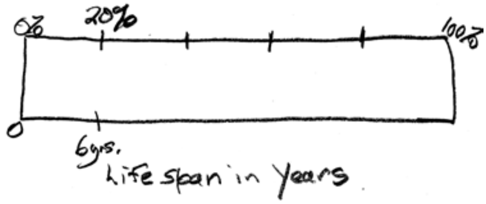
Figure 9.12 Partially completed visual models to questions 2 and 4 in Figure 9.11

Q2: A python sleeps 75% of each day. How many hours per day does the python sleep?



Q4: A leopard bullfrog's life span is about 6 years. That is about 20% of the life span of a lobster. What is the approximate life span of a lobster?

Figure 9.12 Continued.



Using a think, pair, share strategy, Ms. Huntoon engaged her students in the problems and the incomplete visual models. She had students study the Python problem first then the Life Span problem, recognizing that the Life Span problem is often more difficult for students.

She asked for each problem:

1. What are the quantities that are given in the problem?
2. How are they reflected in the visual model?
3. What are the relationships among the quantities shown in the visual model?
4. Where is the solution on the visual model?
5. Use the relationships among the quantities and solve the problem.

She then asked:

1. How are the problems alike, and how are they different?
2. How are the incomplete number lines and percent bar alike? Different?

Ms. Huntoon then had students write two additional word problems that could be solved using the partial visual models and be prepared to discuss why their problem situation fit the visual representation. Students then shared their problems with a partner. After a discussion, Ms. Huntoon asked the students to identify any patterns or relationships they discovered during the lesson.

Ms. Huntoon gave students three problems for exit questions that involved different quantities and relationships. In reflection, Ms. Huntoon realized that her instruction prior to this mini-lesson had not focused intentionally on the quantities or varied the problem situations, as it should have.

The next section illustrates how a 10 × 10 grid can be used effectively to help students develop understanding and fluency in three major ideas: (1) deepening understanding of the meaning of a percent; (2) developing understanding of equivalent fractions, decimals and percents for benchmark as well as non-benchmark percents and percents less than 1%; and (3) further strengthening understanding of the relationships among quantities in different percent situations.

Using 10 × 10 Grids to Develop Understanding and Fluency With Equivalent Fractions, Decimals and Percents

Because a percent is a ratio expressed per 100, base-10 models, including 10 × 10 grids, are better suited than number lines and can be effectively used to help develop percent concepts and build fluency between fractions, percents and decimals when extending understanding past benchmark percent equivalences to non-benchmark percents and for understanding percents less than 1%.

The importance of using 10 × 10 grids to develop understanding of the meaning of percents as well as equivalence among fractions, decimals and percents is reflected in Richard's, Tom's, Maria's and Elizabeth's solutions in Figure 9.13. Study the problem and the student solutions. What misunderstandings are reflected in the solutions?

Figure 9.13 Richard's, Tom's, Maria's and Elizabeth's responses

Trevor ordered the following numbers from smallest to largest. Is Trevor correct? Why or why not?

Trevor's Order

0.8 9% 0.55

Richard's response

he is incorrect the correct order is .55, .8, .9 the 8 and the 9 are in the 100th place thats why

Tom's response

no because 9% is bigger than .8 and .50 because it is not bebw zero

Maria's response

$\boxed{\text{yes}}$ I think the . takes place of %

Elizabeth's response

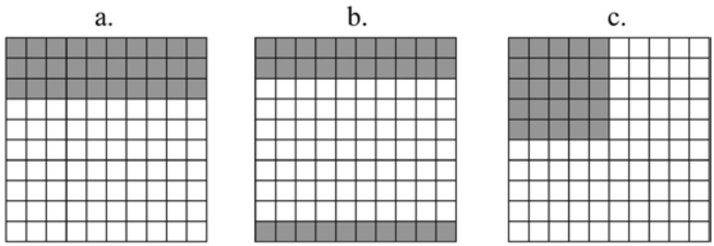
I Know it's not because decimals are smaller than percent and then I knew in decimals the bigger the # is the smaller it is

None of these responses show evidence of understanding the meaning of percents or the relationship between decimals and percents in the problem. It is easy to imagine the difficulty students with these misunderstandings will encounter when engaging in percent problem situations, particularly percent situations involving fractional percents or percents less than 1% or more than 100%. The 10 × 10 grid can play an important role in developing understanding of percents as a ratio expressed per 100. Also, a 10 × 10 grid can be used to build student understanding of the relationship among fractions, decimals and percents.

Because we are discussing a percent as a ratio, it is important to use 10×10 grids in conjunction with a logical context and to initially engage students with benchmark percents and fractions. Solve the problem in Figure 9.14. Explain the features of the 10×10 grid that make it a good visual model to use to build understanding of the meaning of a percent and for developing equivalence ideas. What other questions can be answered with these grids and the problem situation?

Figure 9.14 Students who walk to Middletown Middle School

There are 100 students in the seventh grade at Middletown Middle School. Thirty percent of the students in each grade walk to school. Which grid(s) represents this problem situation?



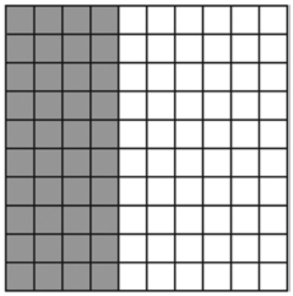
Grids a and b both represent 30% of students in the school that walk to school. Ten-by-ten grids are not new to most middle school math students, and therefore understanding that each box represents 1% of the grid or 1% of the students is generally obvious to students. This may or may not be true for your students. Questions that can be asked using this problem or other problem situations like it fall into two categories: (1) equivalence and (2) understanding the relationships among the quantities in a problem. See Table 9.3 for some questions for each of these categories.

Table 9.3 Sample questions

Equivalence	*Relationships among quantities in problems*
1. What fractional part of the students walk to school? 2. What is the decimal equivalent to 30%? 3. What percent of the students do not walk to school? 4. What fractional part of the students don't walk to school? 5. What is the decimal equivalent? 6. What percent of the whole does each box represent? 7. What fraction of the whole does each box represent?	1. How many students walk to school? 2. If there are 200 students in the school, how many walk to school? 3. If only 10% of the 100 students at Maxwell Middle School walk to school, how many students walk to school?

It is important that students learn to think flexibly about relationships among fractions, decimals and percents. The 10 × 10 grid can also be used to help students develop this flexibility. Study the 10 × 10 grid in Figure 9.15. How does this model facilitate understanding of equivalence among fractions, decimals, and percents?

Figure 9.15 Forty percent represented on a 10 × 10 grid



Because the 10 × 10 grid is partitioned in 100 equal boxes, one can see that 40 of those boxes are shaded. The shaded portion, therefore, is $\frac{40}{100} = \frac{4}{10} = \frac{2}{5} = 0.40 = 40\%$.

In addition, students should also be provided opportunities to identify quantities that are less than 1%, fractional percents and percents greater than 100%. These include quantities such as 0.5%, $66\frac{2}{3}\%$ and 210%. The goal of these activities is for students to understand that percents are not limited to whole numbers between 0 and 100. Ten-by-ten grids can also be used to have students identify different percents, including fractional percents and percents less than 100. Consider the following series of questions for students.

Using your 10 × 10 grid, represent each of the following:

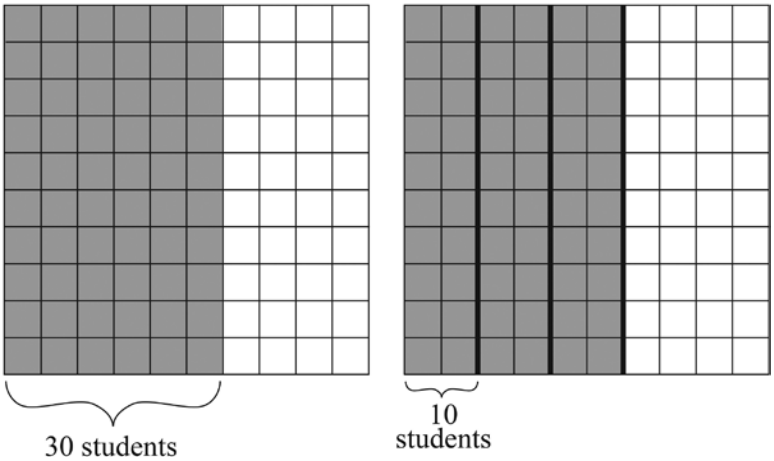
1. 1%
2. 13%
3. 45%
4. 0.5%
5. $25\frac{1}{2}\%$
6. If one box in a grid represents 3 trees, how many trees are represented in the grid? (100 × 3 trees = 300 trees)
7. If one box in a grid represents 0.5 pounds, how many pounds are represented by the grid? (100 × 0.5 lbs. = 50 lbs.)
8. If one box in a grid represents 1.5 lbs., how many pounds does 15 boxes represent? (15 × 1.5 lbs. = 22½ lbs.)

Notice how one box represents two quantities simultaneously. For example, in number 6, one box represents both 1% and 3 trees. In number 7, one box represents 1% and 0.5 lbs. Beckman (2014) refers to this as "going through 1%." This strategy is discussed in more detail later in this chapter.

In the example that follows in Figure 9.16, the strategy is extended to using 20% to scale up to 100%. Study the solution in Figure 9.16. What are the two values of 20 boxes as shown in the example in Figure 9.16? How were the values of 20 boxes determined? How were they used to solve the problem?

Figure 9.16 Mrs. Goodhall's Class solution using 10 × 10 grids

In a class poll, 60% of Mrs. Goodhall's class reported that they play a sport. How many students are in Mrs. Goodhall's class if 30 students reported that they played a sport?



In this situation, 20 squares are equal to both 20% and 10 students. To find 100% of the class, students may subdivide the 60% into three sections of 20%, with each section representing 10 students. The total class has 5 × 10 students or 50 students.

Final Thoughts on Using Visual Models to Develop Understanding and Fluency

You will find other visual models in your instructional materials that are not discussed in this section. They may include the rational number circle (circle partitioned into 100ths) and others. All of these can be used effectively, as discussed in this section, as long as the focus is on developing understanding of the meaning of a percent, the relationships between fractions, decimals and percents, including percents less than 1%, fractional percents and percents more than 100%, and the relationships among the quantities in the problems.

Using Percent Tables and Mental Calculation to Reason Flexibly

Beckman (2014) suggests that an effective strategy for strengthening reasoning with percents is using a percent table including using a "general method which we call *going through* 1%" (p. 81). One percent is the unit rate that can then be scaled to find any other percents.

Go To

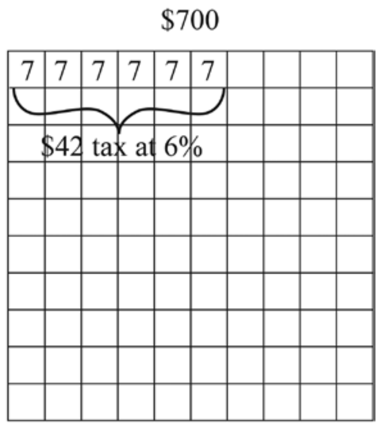
Go to Chapter 1 for a discussion of unit rates.

To understand this idea of the *going through* 1% strategy for solving percent problems and to extend it to other benchmark percents, study the tables and corresponding 10 × 10 grids in Figures 9.17 to 9.19. Both percent tables and 10 × 10 grids are shown to illustrate how both tools can enhance student understanding and flexibility when working with percents. How is the going through 1% strategy illustrated and/or extended in Figures 9.17 to 9.19? What is the big idea in the *going through* method?

Figure 9.17 Example of "going through" 1%

How much is the sales tax on a $700.00 cell phone if the sales tax is 6%?

Percent	Cost
100%	$700.00
1%	$7.00
6%	$42.00



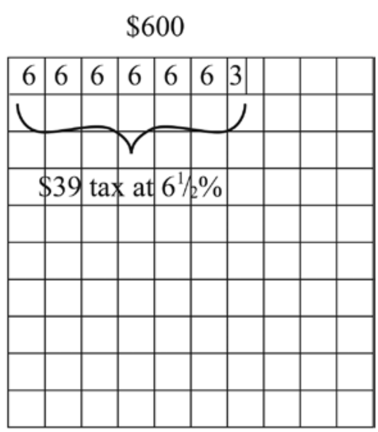
In the example in Figure 9.17, notice that 1% of $700 is $7.00. That is, one box represents both 1% and $7.00. Therefore, 6% of $700.00 is 6 × $7.00 or $42.00. Knowing 1% allows us to find any percent. The thinking behind this strategy can be extended to other percents of 10 × 10 grids.

How is the same thinking extended to finding $6\frac{1}{2}\%$ in Figure 9.18?

Figure 9.18 Example of "going through" 1%

How much is the sales tax on a $600.00 cell phone if the sales tax is $6\frac{1}{2}\%$?

Percent	Cell Phone
100%	$600.00
1%	$6.00
6½%	$39.00



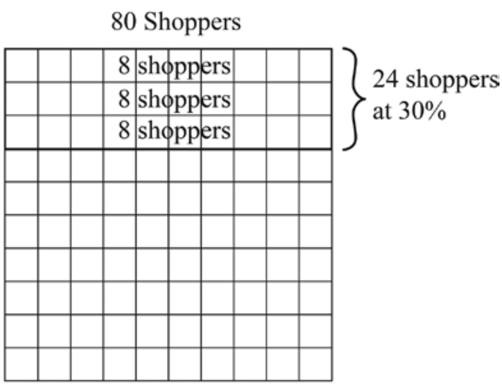
In Figure 9.18, 1% of \$600 is \$6.00. Notice how easily one can visualize $\frac{1}{2}\%$ being equal to \$3.00 because we know that 1% is \$6.00. Therefore, $6\frac{1}{2}\%$ of \$600.00 is \$6 × 6½ = \$39.00.

Study Figure 9.19. How is this concept of "going through 1%" extended to "going through 10%"?

Figure 9.19 Example of "going through" 10%

In a recent poll of 80 shoppers, 30% stated that they would like the mall to remain open on Sunday. How many shoppers wanted the mall to remain open on Sunday?

Percent	Shoppers
100%	80
10%	8
30%	24



In this case, 10% of 80 shoppers is 8 shoppers. Therefore, 30% of the shoppers is 3 × 8 shoppers since 30% is 3 × 10%.

The important idea illustrated is that students will understand that if they find 1% of a quantity, they can extend that to any other percent (e.g., 23%, 23.5%, 340%). Notice that benchmark percents and compatible numbers were used in these examples.

Solve the problems and study Kerry's and Nat's work in Figure 9.20. What is the evidence that Kerry used the go through 1% strategy? How does Nat use benchmark percents and a different *go through* strategy to solve the problem?

Figure 9.20 Kerry's and Nat's responses using "go through" strategies

Kerry's response

A survey of 10,000 residents of a large city found that 0.5% of the residents own a pet lizard. About how many people in this city own a pet lizard?

10,000 ÷ 100 = 100

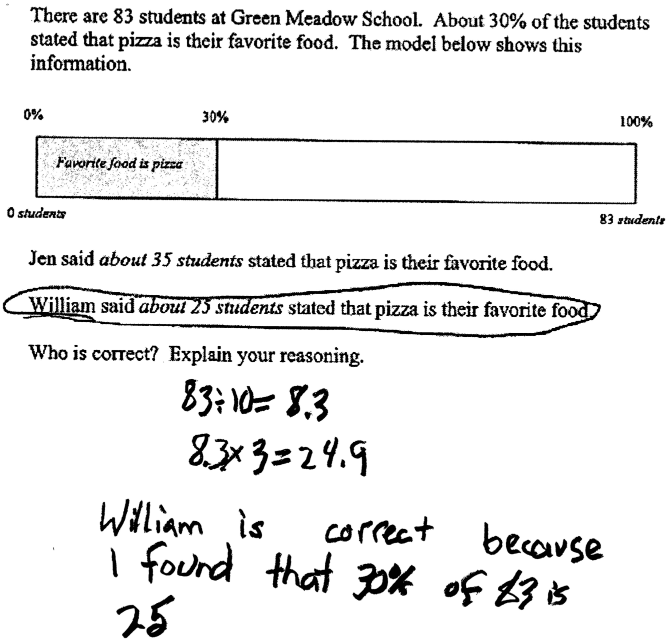
0.5 of 100 × 100

0.5 has 50

Figure 9.20 Continued.

Nat's response

There are 83 students at Green Meadow School. About 30% of the students stated that pizza is their favorite food. The model below shows this information.


There are 83 students at Green Meadow School. About 30% of the students stated that pizza is their favorite food. The model below shows this information.

Jen said *about 35 students* stated that pizza is their favorite food.

William said *about 25 students* stated that pizza is their favorite food.

Who is correct? Explain your reasoning.

83÷10= 8.3
8.3x 3=24.9

William is correct because I found that 30% of 83 is 25

Both Nat and Kerry used a going through strategy to solve the problems. The evidence in Kerry's solution suggests the going through 1% strategy was used. Kerry found 1% of 10,000 $(10{,}000 \div 100 = 100 \text{ people})$. Even though the problem Nat solved involved 30%, the evidence in his work suggests that he used his benchmark understanding that 30% is a multiple of 10% to solve the problem. In this way, he used a going through 10% strategy. Nat found 10% of 83 students $(83 \text{ students} \div 10 = 8.3 \text{ students})$. Nat then multiplied 8.3 students representing 10% of the class by 3 to find 30% of the class.

Warm-Up—Percent Mental Computation

Ultimately one should strive to have students use mental calculations for many of the problems presented to this point in the chapter. One strategy Ms. Huntoon uses to reinforce and build mental calculation is to conduct warm-ups on a regular basis. Study the set of problems in Figure 9.21. Solve the problems using mental calculation. Why are these good problems for mental calculation? How do mental strategies help build flexibility?

Figure 9.21 Mental calculations

1. What is 18% of the cost of a blouse worth $40.00?
2. What is 98% of 150 lbs.?
3. What is 45% of the price of a $600 item?
4. What is 12% of 500 people?
5. What is a 20% tip on $30.00?
6. About what is a 20% tip on a meal costing $38.75?

It is easy to see that each of the problems can be solved without paper and pencil using mental images of visual models and strategies such as the *going through* methods discussed earlier in the chapter. None of the problems involve complex calculations, and all can be solved working through benchmark percents. For example, finding 18% of $40.00 can be solved a number of ways. One could find 20% or $\frac{1}{5}$ of the $40 dollars ($8.00), and then subtract 2% of $40.00 ($0.80) to determine the solution ($7.20). Following this line of thought, one could solve each of the problems in this set with practice.

To help strengthen this idea students, might generate a table that explores a range of benchmark and non-benchmark percents or find missing values in a percent table like the one in Table 9.4.

Table 9.4 What are the discounts on $750 for each of the values?

Percent Discount	*Amount of Discount*
1%	$7.50
2%	?
?	$30.00
10%	$75.00
15%	?
25%	?

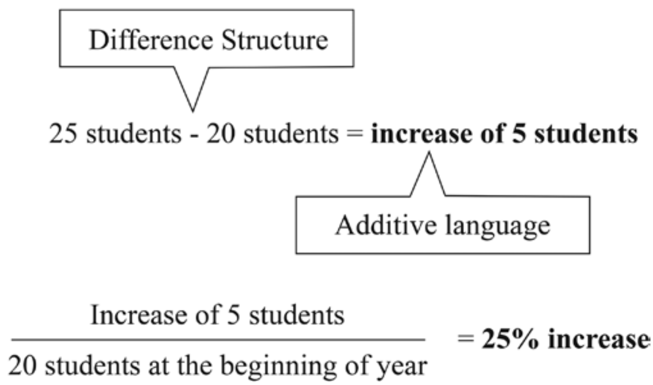
Notice that 1% can be extended to any of these percents. Also, once it is established that 2% (or 2 boxes) is $15.00, then 4% is worth twice that amount ($30). Students who are flexible can mentally calculate the savings at a 30%-off sale or a 20% tip.

Percent Increase and Decrease

Percent increase and decrease is a difficult concept for students because it involves a difference quantity and additive language in a multiplicative relationship (Parker, 2004). To understand these concepts, study the problem and graphic in Figure 9.22. Notice the difference between 25 students and 20 students is an increase in the number of students (5). Also notice that the percent increase is based on the original amount (20 students).

Figure 9.22 Difference quantity and additive language in a multiplicative relationship

The number of students in a classroom in the beginning of the year was 20 students. There were 25 students in the class at the end of the school year. What is the percent increase in the number of students from the beginning of the year to the end of the school year?



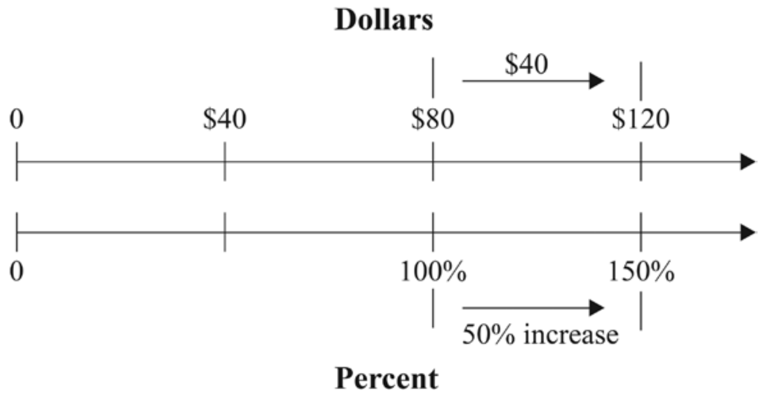
The beginning amount is called the *referent*—that is, the starting point or the whole when considering a percent increase or decrease. The *referen*t in this situation is the original amount from which the percent increases or decreases. Identifying the correct *referent* is key for solving percent increase and decrease problems. In the problem in Figure 9.22, the *referent* is the number of students who started the school year. To understand why defining the referent is important, consider the following situation.

> A store raises its price of a radio on Monday from $80.00 to $120.00. On Friday, the store decreases the price back to $80.00. The change in both situations is $40.00. Is the percent increase and decrease the same?

The answer is no. That is, even though the change in both situations is $40.00, the percent increase and decrease are not the same because the difference in price is compared to different referents: the different starting amounts for the two price changes.

Study the double number line in Figure 9.23 with particular attention to the percent increase aspect of this problem.

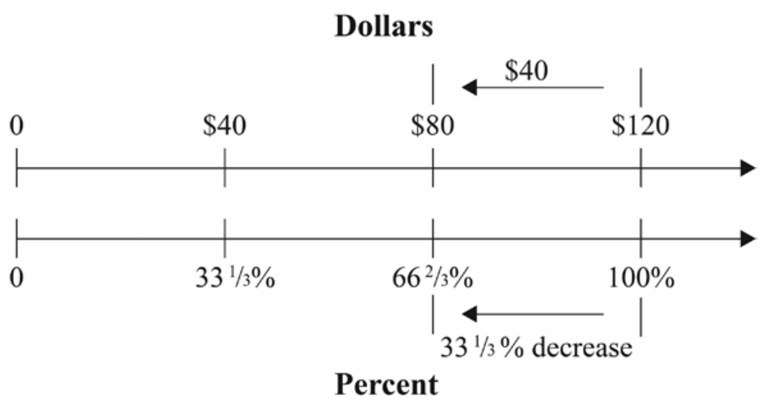
Figure 9.23 Percent increase from $80.00 to $120.00



Notice the referent is the original price of the radio ($80.00). The increase from the original price ($80.00) to the new price ($120.00) is $40.00. Forty dollars is half of $80.00 (the original price). Therefore, the increase is 50% of the original cost of the radio, and the price is 150% of the original cost.

Now study Figure 9.24 representing the percent decrease from $120.00 to $80.00.

Figure 9.24 Percent decrease from $120.00 to $80.00



Notice the referent is now the increased price of the radio ($120.00). The decrease in this price to the reduced price ($80.00) is again $40.00, but now the $40.00 is one-third of the $120.00. Therefore, the decrease is $33\frac{1}{3}\%$ of the $120.00 price of the radio.

Students have a difficult time understanding the impact of percent increase and decrease in the same situation like those in Figures 9.23 and 9.24. Solve the problem in Figure 9.25. Study Katya's response to the situation in Figure 9.25. This is a non-numerical problem that requires reasoning rather than calculation.

Figure 9.25 Katya's response

Ralph has a picture of his dog. He enlarged the picture by 20%. He then shrank the enlarged picture by 20%. Did Ralph end up with a picture the same size as his original? Explain why or why not.

Yes because he enlarged his Picture by 20% then Shrunk it at 20%

Katya's response in Figure 9.25 is an example of a student ignoring the referents and focusing solely on the percent. Katya does not reason that once the original picture is enlarged by 20%, the decrease of 20% is now determined from the enlarged picture and not the original.

An additional challenge when dealing with percent increase and decrease problems is the language used. In Figure 9.23, the problem involved a percent increase. Examine the problem and Naressa's response in Figure 9.26. What is the understanding evidenced in Naressa's response?

Figure 9.26 Naressa's response

Last year, 650 students at Diamond High School reported having a pet. This year, the number of Diamond High School students that reported having a pet was 110% of last year's number.

Which statement below is true?

A. The number of Diamond High school students that reported having a pet this year is the same as last year.

B. 110 Diamond High school students reported having a pet this year.

C. **About 710 Diamond High School students reported having a pet this year.** (student circled this response)

D. 110 fewer students reported having a pet this year.

Naressa calculates 110% of 650 students and also explains why the other choices for responses are not reasonable.

$650 \times 1.10 = 715$

which is closest
it can't be less and it isnt the same

Some Additional Common Errors and Misconceptions

This section will identify and illustrate additional common errors or misconceptions not already discussed in this chapter that may interfere with students learning new percent concepts or solving percent problems.

Earlier in the chapter, you studied solutions to Trevor ordering .8, 9% and .55. In those solutions, a number of common misunderstandings were evidenced. For example, one student thought decimals were less than zero. Another student ordered the numbers by the magnitude of the digits using whole-number reasoning, not the magnitude of the numbers. Both of these are common misconceptions.

Percents that end in fractions are especially difficult for students (Maxim, 1982; Payne & Allinger, 1984). Study the student work in Figure 9.27. What is the misunderstanding evidenced in Max's solution?

Figure 9.27 Max's solution

A survey of 10,000 residents of a large city found that 0.5% of the residents own a pet lizard. About how many people in this city own a pet lizard?

$0.5\% = 50\% = \frac{1}{2}$

$10{,}000 \div 2 = 5{,}000$

$10{,}000 \times \frac{1}{2} = 5{,}000$

Max's solution shows evidence of confusing 0.5% with 50%. Max is correct that 0.5 is equivalent to $\frac{1}{2}$. However, 0.5% is not equal to $\frac{1}{2}$ but to $\frac{1}{2}\%$. The fact that 50% can also be written as 0.5 makes 0.5% especially difficult.

Another common error involves students treating a percent as a whole number and not as a ratio. Students may remove the percent symbol and add or subtract (Ginsburg et al., 1995).

Look at Joy's solution in Figure 9.28. What is the evidence that Joy treated 25% as a whole number, not a ratio?

Figure 9.28 Joy's solution

There are 80 students in the ninth grade at Union High School. Twenty-five percent of those students play a fall sport. Jason said that 25 of the students in the ninth grade at Union High School play a fall sport. Is he correct? Explain why or why not.

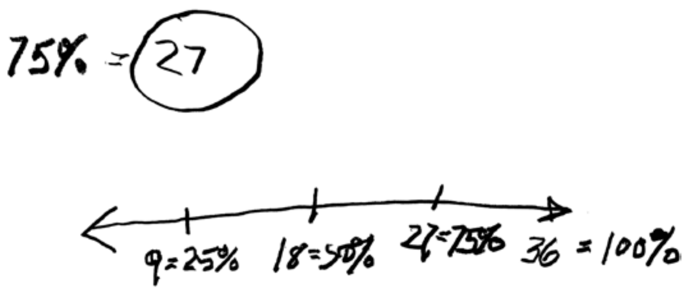
No because
25% is .25 so 80-25=55 not 25

From the evidence, it appears that Joy removed the percent sign and then subtracted 25 from 80 instead of finding 25% of 80.

Students often have a difficult time finding the whole when they are given just a part (Behr & Post, 1992). How does Marty's solution in Figure 9.29 illustrate this misunderstanding? What is the evidence that Marty does understand benchmark percents?

Figure 9.29 Marty's response

In a school poll at Riverside Middle School, 75% of the seventh-grade students reported that they play a sport. If 36 students reported that they play a sport, how many students are in the seventh grade?



The evidence in Marty's response shows the use of benchmark percents and successfully finding 25%, 50% and 75% of 36 students. There was no indication in Marty's work, however, of understanding that 36 students represented only 75% of the class, and the task was to use that information to determine how many students were in the whole class.

Final Thoughts

In the beginning of this chapter, some research was cited indicating that percents are difficult for both students and adults alike. From your own experiences with percents as well as teaching percents, this was probably not new information. The chapter has provided a number of strategies to focus on building understanding as well as a flexibility when solving percent problems. In particular, the chapter provided strategies to help students focus on the quantities in percent situations, building on students' intuitive sense of percents and using visual models and related strategies (i.e., benchmark percents, going through 1%). In addition, the authors intentionally included only percent problems in this chapter that involved a context. This is consistent with recommendations from math education researchers. The authors recognize that some percent situations involve complex calculations. However, when students have a deep understanding of the relationships among the quantities in different percent situations, those calculations hopefully will become routine.

Chapter Summary

- Researchers suggest that percent instruction focus on building understanding and fluency of the equivalence among fractions, decimals and percents as well as focus on the relationships among the quantities in percent problem situations by:

- Building on students' intuitive sense of percents and benchmark percents and fractions.
- Using visual models.
- Using percent tables.
- Engaging students in a range of percent situations, with a focus on the relationships between the quantities in the problems.

• Percent increase and decrease is a difficult concept for students because it has a difference structure and additive language in a multiplicative situation.

Looking Back

1. Look at Kai's and Cathleen's work in Figure 9.30.
   a. What strategy was used to solve the problem? Locate this strategy on the OGAP Ratio and Proportion Progression.
   b. What instructional strategies might you use next to help these students?

Figure 9.30 Kai's and Cathleen's solutions

Kai's response

A book has a price of $15.00. How much will it cost to buy the book if the sales tax is 10%? Show your work.

15 ÷ 10 = 1.5 $15.00 - $1.50 = $13.50

Cathleen's response

A book has a price of $15.00. How much will it cost to buy the book if the sales tax rate is 6%? Show your work.

15 ÷ 6 = 2.5
$15.00 - 2.5 = $12.50

2. Look at Joy's solution in Figure 9.28. Joy states that 25% = .25¢
   a. What is the underlying assumption that must be true for Joy to be correct?
   b. What are the models and contexts you might use to help Joy build understanding about decimal and percent equivalents?

3. Look at Figure 9.27, Max's solution. Max writes that 0.5% = 50%.
   a. What models would help Max build understanding of the difference between 50% and 0.5%?
   b. What are the implications for instruction when working with students on decimal and percent equivalence?

4. Look at the structure of the problem in Figure 9.31.
   a. How are benchmark percents used in the problem?
   b. Locate the strategy used to solve the problem on the OGAP Ratio and Proportion Progression.
   c. Look through your program materials. Do students have the opportunity to reason with benchmark fractions and percents?

Figure 9.31 Union High School

There are 80 students in the ninth grade at Union High School. Twenty-five percent of those students play a fall sport. Jason said that 25 of the students in the ninth grade at Union High School play a fall sport. Is he correct? Explain why or why not.

No he is not 25% of 100 is 1/4 and 1/4 of 80 is 20 not 25

5. For the following student work in this chapter, indicate the level along the OGAP Ratio and Proportion Progression that the evidence is found. What is the evidence?
   a. Figure 9.6 Shawn's response
   b. Figure 9.7 Margaret's response
   c. Figure 9.13 Maria's response
   d. Figure 9.20 Nat's response

Instructional Link

1. To what degree do you or your instructional materials engage students in solving a range of percent problems using visual models?
   a. To develop understanding of the meaning of percents?
   b. To develop understanding and fluency involving the relationships between fractions, decimals and percents, including percents less than 1%, fractional percents and percents more than 100%?
2. To what degree do you or your instructional materials engage students in studying patterns and relationships among quantities in percent problems as opposed to directly teaching rote procedures?
3. Based on this analysis, what adjustments do you need to make to your instructional materials?

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