MARJORIE M. PETIT, ROBERT E. LAIRD,
CAROLINE B. EBBY AND EDWIN L. MARSDEN

A FOCUS ON FRACTIONS

Bringing Mathematics Education Research to the Classroom

Third Edition

SUPPORT MATERIAL

ROUTLEDGE

Studies in Mathematical Thinking and Learning

A Focus on Fractions

The third edition of this book offers a unique approach to making mathematics education research on the teaching and learning of fraction concepts readily accessible and understandable to pre-service and in-service K-8 mathematics teachers.

Revealing students' thought processes with extensive annotated samples of student work and vignettes characteristic of classroom teachers' experience, this book provides teachers a research-based lens to interpret evidence of student thinking, inform instruction and ultimately improve student learning. Based on research gathered in the Ongoing Assessment Project (OGAP), and updated throughout, this engaging and easy-to-use resource also features:

- Two new chapters dedicated to understanding the OGAP Fraction Framework and Progression—based on research conducted with hundreds of teachers—to gather and interpret evidence of student learning along a learning progression, referenced throughout the book so readers can apply the concepts to their instruction;
- A close focus on student work, including 180+ annotated pieces of student work, to help teachers improve their ability to recognize, assess, and monitor their students' errors and misconceptions, as well as their developing conceptual understanding;
- A discussion of decimal fractions, also new to the third edition;
- In-chapter sections on how Common Core State Standards for Math (CCSSM) are supported by math education research;
- End-of-chapter Looking Back questions to allow teachers to analyze student thinking and consider instructional strategies for their own students;
- Instructional links to help teachers relate concepts from each chapter to their own instructional materials and programs;
- Accompanying online Support Material includes an answer key to Looking Back questions, as well as a copy of the OGAP Fraction Framework and Progression.

A Focus on Fractions is part of the popular *A Focus on . . .* collection, designed to aid the professional development of pre-service and in-service mathematics teachers. As with the other volumes on addition and subtraction, ratios and proportions, and multiplication and division, this updated new edition bridges the gap between what math education researchers know and what teachers need to know in order to better understand evidence in student work and make effective instructional decisions.

Marjorie M. Petit has provided leadership in the development and ongoing implementation of the Ongoing Assessment Project (OGAP) and is a partner in OGAPMath LLC.

Robert E. Laird is Co-director of Teaching, Learning, and Research for the Vermont Mathematics Institute (VMI), USA. Bob is a OGAP National Facilitator and a partner in OGAPMath LLC.

Caroline B. Ebby is Senior Researcher at the Consortium for Policy Research in Education (CPRE) and Adjunct Associate Professor at the Graduate School of Education at the University of Pennsylvania, USA. Caroline is a National OGAP Facilitator.

Edwin L. Marsden was Professor of Mathematics and Former Chair of the Department of Mathematics at Norwich University, USA.

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A Focus on Fractions

Bringing Mathematics Education Research to the Classroom

Third Edition

**Marjorie M. Petit, Robert E. Laird,
Caroline B. Ebby and Edwin L. Marsden**

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The third edition of *A Focus on Fractions: Bringing Mathematics Education Research to the Classroom* is dedicated to Beth Hulbert, whose tireless leadership is instrumental in ensuring OGAP professional development and implementation continually meets the needs of all our school and district partners. This edition is also dedicated to the OGAP National Facilitators who, guided by Beth's leadership, have provided high-quality OGAP professional development and support to thousands of educators.

Contents

Preface

A Focus on Fractions: Bringing Mathematics Education Research to the Classroom

The first edition of *A Focus on Fractions: Bringing Research to the Classroom*, published in 2010, was written to communicate important mathematics education research about how students develop an understanding of fraction concepts, common errors students make, and preconceptions or misconceptions that can interfere with students learning new fraction concepts and solving problems. The book grew out of the Ongoing Assessment Project (OGAP), a successful formative assessment project based on mathematics education research related to the teaching and learning of fractions.

In the 11 years since the publication of the first edition, there have been several developments that have influenced this latest revision:

- Knowledge about research-based learning progressions and the importance of using them to guide instructional decision-making and instructional materials has continued to proliferate in the field of mathematics education.
- In the US, standards and instructional resources have been developed to reflect current research on student learning, resulting in a more coherent curricular progression for teaching and learning fractions in K-8 schools.
- Readers have asked for more explicit emphasis on the *OGAP Fraction Progression*, unit fraction reasoning, discussion of decimals, and connections between equivalence and comparing and ordering fractions.

Response to the second edition of *A Focus on Fractions*, published in 2016, continues to be enthusiastic. In particular, readers have found the authentic student work samples and the clear link between mathematics education research and classroom instruction to be invaluable in supporting effective fraction instruction and thoughtful analysis of evidence in student solutions.

In addition, readers have found that the content of *A Focus on Fractions* fosters:

- more effective instruction, particularly in the way that new topics are introduced to students
- a clearer understanding of the purpose of activities in mathematics programs, thus maximizing the instructional potential of these instruction materials; and
- a deeper understanding of fraction concepts

A Focus on Fractions: Bringing Research to the Classroom, Third Edition

This third edition of *A Focus on Fractions* maintains many of the core features of the previous editions that have been so well received: a focus on fraction content knowledge, mathematics education research, hundreds of samples of student work, a focus on developing understanding, questioning to monitor and support student understanding, connections to the Common Core State Standards in Mathematics (CCSSM), and examples of ways curricular materials reflect the research.

You will notice, however, that the order of the existing chapters has changed and there are some additional chapters. The first four chapters now focus on the concepts that are foundational to understanding fractions and rational numbers: understanding fractions as numbers, the use of visual models, understanding fractions in relation to the whole, and equipartitioning.

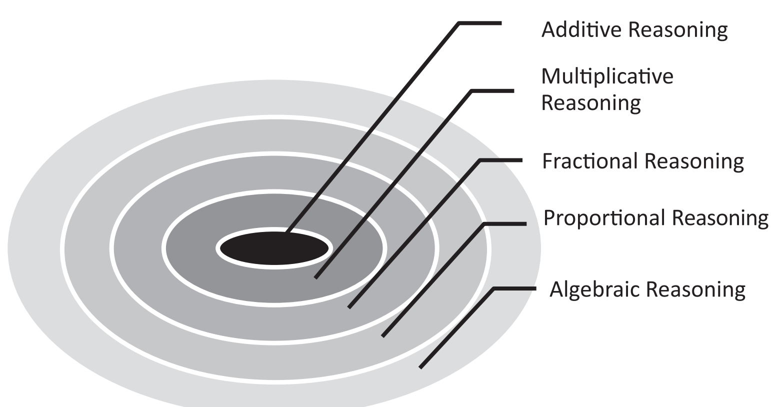
The *OGAP Fraction Framework* and the *OGAP Fraction Progression* are introduced in Chapter 5, which also includes a copy of the progression that can be referenced throughout the rest of the book and when applying the ideas in this book to your instruction. Note: The *OGAP Fraction Progression* with additional student work examples can be found online at: www.routledge.com/9781138816442

Chapter 6 is a new chapter devoted to using the OGAP formative assessment system to inform instruction. The chapter includes a framework for thinking about levels of instructional responses and selecting specific instructional strategies based on evidence of student thinking. These different instructional responses are explored through case studies in subsequent chapters. Some case studies appear in the Looking Back sections as opportunities to practice using the progression to analyze student work (Chapter 7, 8), and some are integrated into the discussion of the content in the chapter (Chapter 9, 11, and 12).

Chapters 7, 8, and 9 now include discussion of the relationships between fractions and decimals, both mathematically and in terms of developing conceptual understanding of the equivalence between fractions and decimals, comparing and ordering decimals, and extending the base-ten number system. These chapters include illustrations of how students use their understanding of the foundational concepts explored in Chapters 1 through 4 to develop understanding of decimals as well as fractions.

Even though we believe these additions are important and improve the quality of *A Focus on Fractions*, we do not want them to detract from what we believe to be the heart and soul of our work: communicating to teachers the mathematics education research related to the teaching and learning of fractions, coupled with a sharp focus on student work analysis. *A Focus on Fractions: Bringing Research to the Classroom, Third Edition* remains a book primarily dedicated to understanding how students make sense of fraction concepts and how teachers can use mathematics education research to improve the learning of fractions for all of students.

Since the publication of the first edition of *A Focus on Fractions* in 2010, three additional books have been published on additive, multiplicative, and proportional reasoning, creating a complete series that provides a comprehensive roadmap for developing mathematical reasoning in grades K-8. These core content areas build on each other to create mathematical proficiency. Additive reasoning lays the foundation for multiplicative reasoning; multiplicative reasoning lays the foundation for fractions; fractional reasoning lays the foundation for proportional reasoning; and all of these are important for algebraic reasoning in the secondary school years



Over the last decade, learning progressions, or learning trajectories as they are often called in mathematics education research, have had an influence on current mathematics standards, instructional models, and curriculum and assessment materials (Daro et al., 2011; Lobato & Walters, 2017; Sztajn et al., 2012). The use of learning progressions in professional development and classroom instruction has also been found to impact teacher learning, instructional practice, and student learning (Carpenter et al., 1989; Clements et al., 2011; Clements et al., 2013; Edgington, 2012; Supovitz et al., 2021). A recent study of the implementation of the OGAP formative assessment system in multiplication and fractions in grades 3–5 showed significant impacts on teacher knowledge and student performance in a large urban school district (Supovitz et al., 2018, 2021).

A Book Designed for Classroom Teachers, In-Service Use, and Pre-Service Training

Like the first two editions, the third edition of *A Focus on Fractions: Bringing Research to the Classroom* is first and foremost for classroom and pre-service teachers. The case studies used throughout the book are based on real issues teachers face as they search for ways to teach fraction skills and concepts to all students. The student work samples, the Instructional Link sections, and the Looking Back sections provide teachers with rich opportunities to analyze student thinking, to consider instructional strategies for their own students, and to link important concepts with their own instructional textbooks and materials. Groups of math teachers from the same school, teachers involved in fraction professional development, and grade-level teaching teams have told us that working together with ideas presented in Instructional Links and Looking Back sections are particularly powerful. Answers to the Looking Back questions can be found at www.routledge.com/9781138816442.

Instructors working with pre-service teachers will find the numerous samples of student work to be valuable in bringing authentic student thinking into pre-service class discussions. In addition, pre-service teachers will be introduced to important educational research related to fractions, and they will be provided with many opportunities to "see" the research in authentic student work, discuss research with peers, and consider the important instructional decisions central to effective mathematics teaching,

Bridging the Gap between Researchers and Practitioners

We have found that educators are hungry for the mathematics education research described in this book regarding how students develop understanding of fraction concepts. Teachers are often amazed at how readily the evidence in their students' work aligns with the findings in the research. *A Focus on Fractions: Bringing Research to the Classroom, Third Edition*, like its predecessors, is a bridge between what mathematics education researchers have discovered about the learning of fraction concepts and the knowledge teachers need to make effective instructional decisions.

Note: The sections throughout this book focused on the CCSSM are based on CCSSM (CCSSO/NGA, 2010) and the CCSSM Progressions (Common Core Standards Writing Team, 2013a, 2013b).

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1
Understanding a Fraction as a Number

Big Ideas

- Like whole numbers, fractions are numbers, each with a specific magnitude.
- Fractions are a natural extension of whole numbers.
- Fractions are composed of unit fractions.
- Students often use inappropriate whole number reasoning when solving problems involving fractions.
- Instruction matters. Teachers can utilize specific strategies to address inappropriate whole number reasoning.

You may wonder why this book starts with a concept that seems obvious; of course, fractions are numbers. However, research has shown that difficulties students have with fractions often stem from the fact that they do not always understand fractions as numbers. Students might see $\frac{3}{4}$ as a 3 and 4, not a number between 0 and 1. You may have seen students add fractions by simply adding numerators and denominators, for example $\frac{3}{4}+\frac{3}{8}=\frac{6}{12}$. Why do you think they would do that? Teachers may react to these issues by reteaching the definition of numerators and denominators or common denominators louder and slower. What is really going on here? Study the problem and student work by a fifth-grade student in Figure 1.1. What surprises you about this work? What evidence suggests that the student does not understand a fraction as a number?

Figure 1.1 Student accurately added the fractions then used the magnitude of the denominator or numerator in the sum to determine that $\frac{23}{24}$ is closest to 20.

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20
B. 8
C. $\frac{1}{2}$
D. 1

Explain your answer.

$\frac{1}{12}+\frac{7}{8}=\frac{2}{24}+\frac{21}{24}=\frac{23}{24}$ is closest to 20.

A teacher would probably be pleased that the student accurately used a common denominator to calculate the correct sum but alarmed that the student did not interpret the fraction $\frac{23}{24}$ as closest to 1. As you work through this and other chapters in this book you will be introduced

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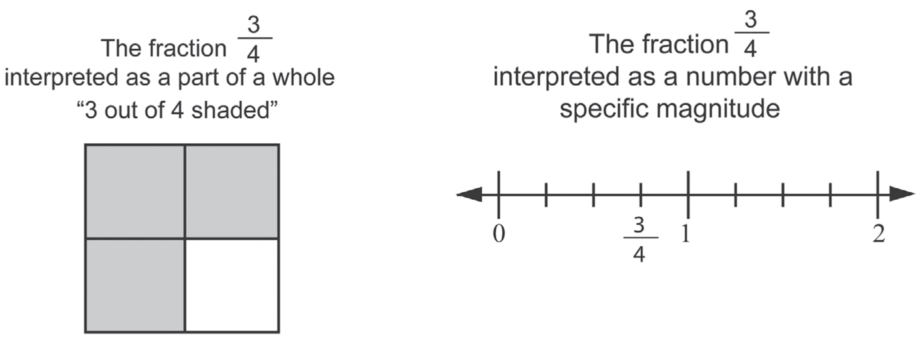
to mathematics education research and content focused on how to help students build a strong foundational understanding of fractions as numbers as they compare and order fractions, find equivalent fractions, and operate with fractions.

Research has long supported the notion that many students have difficulties learning and applying fraction concepts and skills. Even in countries where student achievement in fractions is relatively high, such as Japan and China, fractions is considered a difficult topic to both learn and teach (NCTM, 2007). There are several reasons for this, many or which will be discussed throughout this book, yet a common difficulty for students and teachers is the fact that certain truths for whole numbers do not hold for fractions (Fazio & Siegler, 2011). For example, experience with whole numbers falsely supports the idea that multiplication always "makes larger" and division always "makes smaller."

Fractions have historically been taught as a way to represent a part of a whole. For example, to illustrate $\frac{3}{4}$ students might partition a rectangle into 4 subsections of equal area and shade 3 of the sections as shown in Figure 1.2. They might even describe this representation as, "3 out of 4." Although an important interpretation, a fraction as a part of a whole does not communicate the vital idea that a fraction is a number with a specific magnitude. Siegler et al. (2010) indicated that instruction focused solely on part-whole relationships when introducing fractions may "leave unclear how fractions are related to whole numbers" (p. 19). A rich understanding of a fraction as a number is foundational to the notion that fractions, like whole numbers, can be represented on a number line, can be compared, can be added and subtracted, and can be multiplied and divided.

Figure 1.2 The fraction $\frac{3}{4}$ interpreted as a part of a whole and as a number.



Fractions are natural extensions of whole numbers, and like whole numbers they are composed of units. The next two sections discuss these concepts.

Fractions as a Natural Extension of Whole Numbers

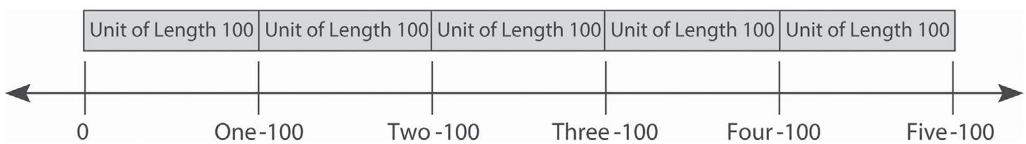
Building a foundation for understanding that fractions are numbers involves understanding that fractions have magnitude just as whole numbers have magnitude. While textbooks may contain many different definitions of fractions (e.g., part to whole, fractions as quotients), Wu (2010) suggests that using the definition of a fraction as a point on a number line allows one to visualize the magnitude of both whole numbers and fractions relative to each other as well as understand that fractions are a "natural extension of the whole numbers" (p. 136). Figure 1.3 shows the position of $\frac{5}{4}$ relative to the whole numbers 0 to 5, but also shows that $\frac{5}{4}$ is $\frac{1}{4}$ greater than 1.

Figure 1.3 The fraction $\frac{5}{4}$ is a point on a number line.

Fractions are Composed of Unit Fractions

Whole numbers are composed of the units that make up our base-ten number system: ones, tens, hundreds, thousands and so on. Five hundred, for example, is made of 500 units of magnitude 1, 50 units of magnitude 10, and 5 units of magnitude 100. This process of composing the number 500 through the iteration of a 100 unit is illustrated in the number line model in Figure 1.4.

Figure 1.4 The number 500 represented on a number line by iteration of the 100 unit.



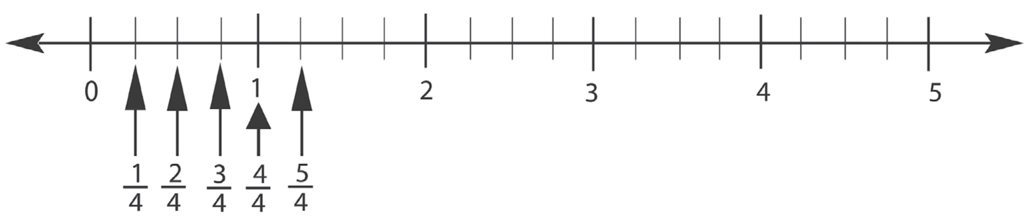
This interpretation of the number 500 in Figure 1.4 communicates several important ideas:

- The magnitude of 500 can be represented on a number line using units of 100.
- 500 can be interpreted additively as 100 + 100 + 100 + 100 + 100.
- 500 can be interpreted multiplicatively as 5 × 100.
- Whether interpreted additively or multiplicatively, 500 can be defined as, "5 one hundreds" or "5 iterations of one hundred."

The idea that whole numbers are composed by iterating units is an important concept as it allows one to work with numbers flexibly by decomposing a number into its foundational units. Many calculations apply this concept (e.g., 420 + 340 = 400 + 300 + 20+ 40 = 760).

Fractions are composed by iterating *unit fractions*. Unit fractions, each with a numerator of 1 such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{10}$, $\frac{1}{12}$ are the building blocks of all fractions that allow one to decompose fractions into their unit parts which, as we will see in the examples throughout this book, can provide students valuable strategies for comparing and ordering fractions, finding equivalent fractions, and operating with fractions. Figure 1.5 shows how the fraction $\frac{5}{4}$ can be composed by iterating the unit $\frac{1}{4}$ five times.

Figure 1.5 The fraction $\frac{5}{4}$ can be located on a number line by iterating the $\frac{1}{4}$ unit 5 times.



Just like the number 500, the number $\frac{5}{4}$:

- has magnitude that can be represented on a number line,
- can be interpreted additively $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$,
- can be interpreted multiplicatively as $5\times\frac{1}{4}$, and
- can be defined as, "Five one-fourths" or "Five iterations of one-fourth."

This concept that a fraction is composed of units that can be represented on a number line supports the important understanding that a fraction, like all numbers, has magnitude and is composed of units. For example, "5 pieces of size $\frac{1}{4}$" suggests size just like "5 pieces of size 1000" or "5 pieces of magnitude 1." We will see in subsequent chapters how this interpretation of fractions supports the concepts related to comparing fractions.

It is interesting to note that the concept of numbers being created by iterating units is not unique to fractions and whole numbers. For example:

- 0.05 is created by iterating the unit 0.01 five times.
- $5x$ is created by iterating the unit x five times
- $5(x+1)$ is created by iterating the unit $(x+1)$ five times.

Finally, the similarities in the ways numbers and expressions are created and defined extend to the way one speaks about a number in the English language. The first word or syllable spoken is generally the number of iterations and the second word is the size of the unit. Table 1.1 provides a few examples

Table 1.1 Examples of Units and Iterations in the Way We Commonly Speak About Numbers in the English Language

Number	*Number Spoken*	*Interpreted Through the Lens of Units and Iterations*
1600	"Sixteen hundred"	• Sixteen: number of iterations • Hundred: size of the unit
$\frac{7}{8}$	"seven-eighths"	• Seven: number of iterations • Eighths: size of the unit
0.008	"8 one-thousandths"	• Eight: number of iterations • One-thousandths: size of the unit
$5\,(x+10)$	"Five times x plus 10"	• Five: number of iterations • x plus 10: size of the unit

Thus, even the language we use to communicate fractions supports the idea that fractions are numbers constructed from units, just like whole numbers, decimal fractions, and algebraic expressions.

Mathematics education researchers support the use of number lines to build an understanding that fractions are numbers with specific magnitudes just like whole numbers have magnitude (Behr & Post, 1992; Saxe et al., 2007). As a matter of fact, teachers in OGAP studies found that number lines facilitated students' understanding of fractions as numbers, allowing them to order and compare fractions and find equivalent fractions (OGAP, 2005, 2007) and move away from whole number reasoning. Chapter 9: Number Lines provides an in-depth discussion of the use of number lines.

CCSSM

Because of the foundational role unit fractions play in the understanding of fractions as numbers, it is not surprising that the Common Core State Standards for Mathematics (CCSSM) places considerable emphasis, particularly in grades 3 and 4, on unit fractions and their vital role in understanding fractions. There are a number of standards in these grades that emphasize unit fractions. The two standards that follow are specific to the concepts we examined in this section:

Grade 3

Developing understanding of fractions as numbers.

3 NF.2b Understand a fraction as a number on a number line; represent fractions on a number line diagram.

- Represent fraction a/b on a number line diagram by marking off a lengths of $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Grade 4

Build fractions from unit fractions by applying and extending previous understandings of operations with whole numbers

4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$

4.NF.4a Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

4.NF.4b Understand a fraction a/b as a multiple of $1/b$.

The remainder of this chapter focuses on student work samples and discussion that provide the basis for understanding the importance of seeing fractions as numbers. The student work illustrates some common errors and misconceptions that arise when students have an incomplete understanding of fractions as numbers. This chapter also provides a brief introduction to instructional strategies that address common errors and misconceptions and that support students understanding of fractions as numbers. These instructional strategies will be examined in depth throughout the book.

Inappropriate Whole Number Reasoning

How do many students understand this number?

$$\frac{3}{4}$$

As we examined in the previous pages some students may interpret $\frac{3}{4}$ as "three out of four," yet according to research, some students see a fraction as composed of two whole numbers, in this case 3 and 4. This interpretation often results in students inappropriately applying whole number reasoning to fraction situations instead of reasoning with a fraction as a single quantity (Behr et al., 1984; OGAP, 2005; Saxe et al., 1999). We refer to this inappropriate interpretation of a fraction as *inappropriate whole number reasoning*.

Inappropriate whole number reasoning often results in students making errors when they:

- locate fractions on a number line
- compare fractions
- identify fractional parts of wholes
- estimate the magnitude of fractions
- operate with fractions

Many teachers in the 2005 and 2007 OGAP studies were surprised at how readily their students applied inappropriate whole number reasoning to many aspects of fractions. In fact, a preliminary analysis of a sample of 39 OGAP fourth grade pre-assessments $\left(\frac{39}{229}\right)$ illustrates this point. About 44% $\left(\frac{137}{308}\right)$ of all incorrect responses analyzed on the 39 OGAP pre-assessments sampled were attributed to the use of inappropriate whole number reasoning (OGAP, 2005).

These teachers found that some students focused on just the numerators or on just the denominators of the fraction when comparing fractions, finding the sums or differences of fractions, or finding a fractional part of a whole. Figures 1.6 to 1.9 provide some examples of the ways in which inappropriate whole number reasoning is evidenced in student work.

Figure 1.6 Inappropriate whole number reasoning example: used the magnitude of the denominator to locate the fractions on the number line.

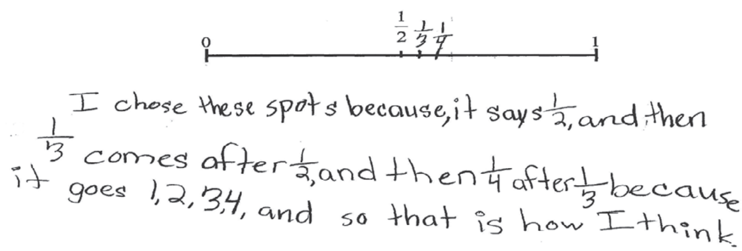


Figure 1.7 Inappropriate whole number reasoning example: used the magnitude of the numerators and the denominators to compare the fractions.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate

$\frac{3}{10}$ of the candies are peppermint

Are there more chocolate or peppermint candies in the dish?

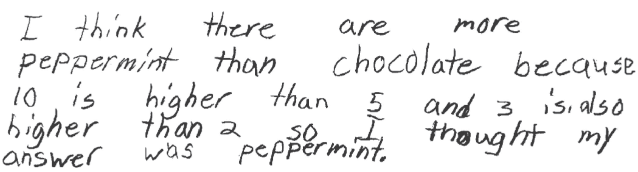


Figure 1.8 Inappropriate whole number reasoning example: circled the number of suns equal to the sum of the numerator and denominator (13 suns), not $\frac{5}{8}$ of the suns (10 suns).

Circle $\frac{5}{8}$ of the suns.

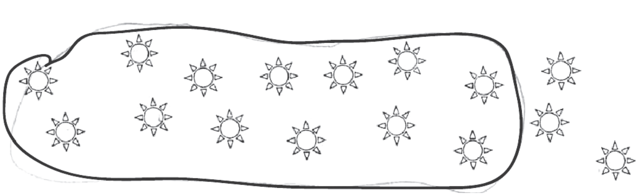


Figure 1.9 Inappropriate whole number reasoning example: added numerators and denominators to find the sum.

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20
B. 8
C. $\frac{1}{2}$
D. 1

Explain your answer.

$\frac{1}{12} + \frac{7}{8} = \frac{8}{20}$ is close to $\frac{10}{20}$ whic his half.

The student work from the pre-assessments for the 2005 and 2007 OGAP studies, which contained solutions like the examples in Figures 1.6–1.9, provided teachers with compelling evidence that many students were not thinking about fractions as quantities, rather seeing them as composed of two whole numbers. As you can see, this impacted the ways students solved fraction problems. What was not clear was whether the inappropriate whole number reasoning was an artifact of previous instruction on fractions or was a result of students incorrectly generalizing their understanding of whole numbers to fractions. In either case, teachers in the studies had critical information to use as they prepared and implemented their fraction instruction.

To help students understand fractions as numbers, these teachers placed a greater emphasis on comparing and ordering fractions and on the use of number lines. These instructional decisions are backed by current research-based recommendations supporting the use of the number lines, beginning in the early grades, to help students develop an understanding of fractions as numbers (CCSSO/NGA, 2010; Siegler et al., 2010).

Teachers in the OGAP studies also recognized that instruction might have unintentionally reinforced inappropriate whole number reasoning. An example of this is providing students with opportunities to only solve problems involving finding the fractional part of a whole in which the number of objects in a set or the number of parts that make up the whole in an area is equal to the denominator of the fraction given (OGAP, 2005).

What do you notice about the answer to the following three tasks in which the number of items in the set is equal to the denominator?

- $\frac{3}{4}$ of 4 children is 3 children
- $\frac{2}{5}$ of 5 pounds is 2 pounds
- $\frac{3}{8}$ of 8 hours is 3 hours

You undoubtedly recognized that the answer to these types of tasks is equal to the numerator of the fraction in the problem. Consistent use of only tasks like these may inadvertently support students focusing on one part of a fraction, in this case the numerator. This can unintentionally reinforce inappropriate whole number reasoning

This observation made by teachers points out the importance of students having experience, even in the early grades, of physically partitioning sets of objects or areas in which the number of objects or parts in the whole is a multiple of the denominator. The Dinosaur problem in Figure 1.10 is an example of this type of fraction task. To solve tasks structured in this way students need to use fractional reasoning to reflect on the relationship between the numerator and the denominator.

Figure 1.10 Young students can physically divide sets of objects into equal groups to find a fractional part of a set of objects. Later, they can transition this understanding to paper-and-pencil tasks.

How many dinosaurs is $\frac{1}{3}$ of a set of six dinosaurs?

As we examined earlier in the chapter, instruction that overemphasizes fractions as part–whole relationships and using language such as "3 out of 4" can be problematic and inadvertently encourage students' inappropriate whole number reasoning when working with fractions. The use of multiple visual models, particularly area models such as the example in Figure 1.11 and number lines as shown in Figure 1.12, can help students internalize the concept that the fraction $\frac{2}{3}$, for example, is composed of 2 pieces each of size $\frac{1}{3}$ (or 2 one-thirds) rather than 2 out of 3 equal parts.

Figure 1.11 Two-thirds represented as 2 one-thirds of the area of the rectangle. The entire rectangle is composed of 3 one-thirds.

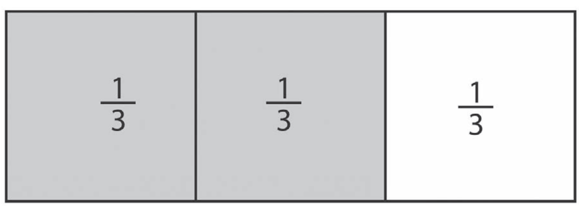


Figure 1.12 Two-thirds represented as 2 one-thirds on a number line.

Go-To See Chapter 8, Comparing and Ordering, and Chapter 9, Number Lines, for more information on using number lines and comparing and ordering fractions.

Chapter Summary

Understanding a fraction as a number with a specific magnitude is foundational to students' work with fraction equivalence, comparing and ordering fractions, and operations. Like whole numbers, fractions are built from or composed of units. To emphasize the importance of this interpretation of a fraction, one of the lead writers of the CCSSM states, "The ultimate underlying principle you want kids to understand is that fractions are numbers. They're new, but they're not in a different galaxy" (McCallum as cited in Loewus, 2014).

Understanding the role that unit fractions play in building and describing fractions is central to students seeing them as numbers not simply as parts out of a whole. Inappropriate whole number reasoning is a common misconception for students and can be inadvertently reinforced by instruction that is not first and foremost focused on presenting fractions as numbers.

Although the preponderance of inappropriate whole number reasoning was overwhelming to teachers in the 2005 and 2007 OGAP pre-assessment, they too realized that it was not inevitable. Data from the OGAP study (2005) supported these observations. In the post-assessments, 18% $\left(\frac{27}{152}\right)$ of errors were attributed to the use of inappropriate whole number reasoning compared to 44% $\left(\frac{137}{308}\right)$ in the pre-assessment (see Table 1.2). It was obvious their instruction mattered.

Table 1.2 OGAP 2005 Study—Use of Inappropriate Whole Number Reasoning in OGAP Grade 4 Pre- and Post-Assessments (OGAP, 2005)

	Percentage of students (n = 39)	*Percentage of incorrect responses*	*Average number errors attributed to inappropriate whole number reasoning (only students who made error are included)*
Pre-assessment	85% (33/39)	44% (137/308)	4.1 (33 students)
Post-assessment	18% (7/39)	18% (27/152)	1.8 (7 students)

Further results from the 2005 and 2007 OGAP studies indicated that teachers utilized specific instructional actions to address the inappropriate whole number reasoning students were applying to their work with fractions. Their instruction intentionally and systematically focused on three major strategies:

- seeing fractions as numbers,
- the use of visual models, particularly number lines,
- comparing and ordering fractions.

Chapter 2 provides a detailed discussion of visual models central to fraction instruction, student use of visual models, and why they are an essential part of effective fraction instruction. Chapter 8 offers a close look at the foundational concepts related to comparing and ordering fractions, and Chapter 9 examines the vital role number lines play in students' development of fraction concepts and skills.

Looking Back

1. Figures 1.13 and 1.14 show Michael's pre- and post-assessment solutions. Review both responses, and answer the following questions.
   a. What evidence in Michael's pre-assessment response suggests that Michael inappropriately used whole number reasoning when placing $\frac{1}{3}$ and $\frac{1}{4}$ on the number line?
   b. What was Michael able to do on the post-assessment that was not shown in his response on the pre-assessment?

c. Michael's post-assessment response is different from his pre-assessment response. What is one instructional focus that might have helped Michael to move from inappropriate use of whole number reasoning to treating each fraction as a single quantity?

Figure 1.13 Michael's pre-assessment response to placing $\frac{1}{3}$ and $\frac{1}{4}$ on a number line from 0 to 1.

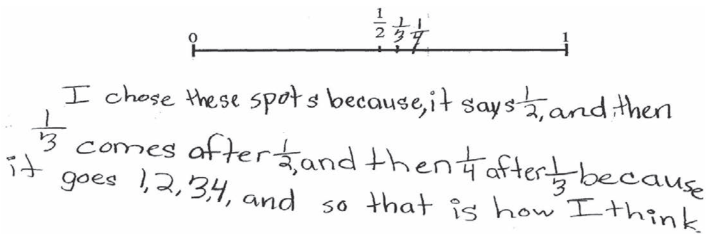
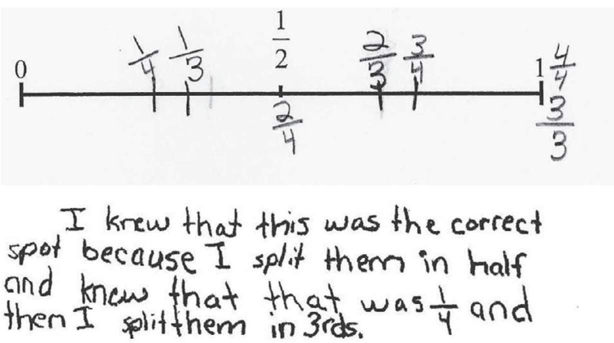


Figure 1.14 Michael's post-assessment response to the same question related to placing $\frac{1}{3}$ and $\frac{1}{4}$ on a number line from 0 to 1.



2. Figures 1.15 and 1.16 include Mark's and Kim's responses to a question about the magnitude of a fraction. Both responses include visual models generated by the students. Consider their responses and answer the following questions.

Figure 1.15 Mark's response.

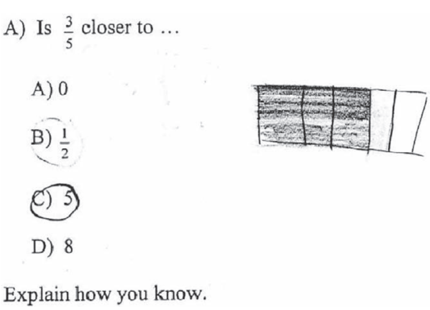
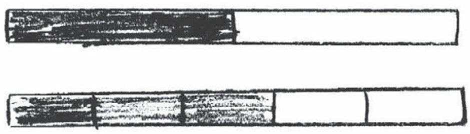


Figure 1.16 Kim's response.

A) 0
B) $\frac{1}{2}$
C) 5
D) 8



$\frac{1}{2}$ is closer to $\frac{3}{5}$ because they are only $\frac{1}{10}$ off since 5 and 8 are whole numbers they would be $4\frac{2}{5}$ and $7\frac{2}{5}$ off.

a. What is the evidence in Mark's response that leads one to believe that his ability to compare $\frac{3}{5}$ to a benchmark fraction is developing, but is still fragile and easily destabilized? Explain.
b. What is the evidence that Kim is using fractional reasoning? Explain.

3. Review Kim's solution one more time. Kim included carefully drawn and accurate area models for $\frac{1}{2}$ and $\frac{3}{5}$. To what extent did Kim's explanation require these area models?
4. Review the evidence in Willy's response found in Figure 1.17 and answer the questions that follow.

a. What is the evidence in Willy's response that he had sound fractional reasoning?
b. If Willy had the time to rewrite his response, how might his sentence be rewritten to clarify what you think Willy had in mind?
c. Do you think that Willy decided that the sum "is just going to be a little less than 1" without computing the sum? If he didn't add the fractions, what reasoning do you think Willy used to decide that the sum "is just going to be a little less than 1"?

Figure 1.17 Willy's response.

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20
B. 8
C. $\frac{1}{2}$
D. 1

D. 1

Explain your answer.

I think 1 because $\frac{7}{8}$ is almost one + $\frac{1}{12}$ is just going to be a little less than 1.

5. The idea that numbers are created from units is a major idea in this chapter. For example, we saw that 500 can be interpreted as 100+100+100+100+100. By reinterpreting this expression we can also see 500 equals:

   - 100 + 400 or 100 + (100 + 100 + 100 + 100)
   - 200 + 300 or (100 + 100) + (100 + 100 + 100)
   - 400 + 100 or (100 + 100 +100 +100) + 100
   - 100 + 400 or (100 + 100 + 100) + (100 + 100)

   a. Apply this same idea to the fraction $\frac{5}{8}$ by generating addition expressions.
   b. What properties are at work in your examples and the ones for 500 in the question?
   c. How might these different additive interpretations for $\frac{5}{8}$ be important for students?

Instructional Link: Your Turn

We suspect that some of the students in your classroom inappropriately apply whole number reasoning while solving problems involving fractions. However, this inappropriate use of whole number reasoning can be greatly reduced. If the students experience a coherent instructional program that focuses on fractions as single quantities, the students can move away from inappropriate use of whole number reasoning. To help you think about your instruction and the mathematics materials that you use, complete Table 1.3.

Table 1.3 Instructional Link—Strategies to Support Development of Reasoning With Fractions as Quantities

Do you (or does your program)	*Yes/No*
1. Encourage students to use a variety of visual models in all aspects of developing fraction understanding?	
2. Provide opportunities for students to locate fractions on number lines with more than one unit? (See Chapter 6.)	
3. Provide exercises that compare fractions to benchmarks and to each other? (See Chapter 5.)	
4. Provide opportunities for students to make estimates of sums, differences, products, and quotients? (See Chapters 9 and 10.)	
5. Have a focus on reasoning with fractions as single quantities when operating with fractions, not just a focus on procedures? (See Chapters 9 and 10.)	
6. Provide the opportunity for students, even in the early grades, to find the fractional parts of the whole where the number of parts in the whole is a factor or multiple of the denominator?	

Based on the responses to these questions, what gaps in your instruction or mathematics program did you identify? How might you address these gaps?

2
Developing Understanding of Fractions Through Visual Models

Big Ideas

- Set, area, and number line models represent fractions in different ways and present students with unique challenges and understandings.
- Instruction should provide students with rich opportunities to interact with visual models that have different perceptual features.
- The use of visual models is a means to the mathematics, not the end.
- Teachers should build instruction on student-generated visual models to help students generalize mathematical ideas.

Most mathematics instructional materials, including commercial textbooks and programs, include the use of visual models in lessons and activities. In addition, you also may have noticed the consistent mention of "visual fraction models" in the CCSSM. Many teachers were not exposed to the intentional and systematic use of visual models in their own learning of mathematics. For this reason and others, it is common for mathematics teachers to have questions about the role of visual models in effective mathematics instruction. Common questions include:

- What is the purpose of using visual models in mathematics instruction?
- My math program uses one type of visual model. Is that OK or should I expose my students to different visual models?
- Is there one best way to use visual models in instruction? For example, I am uncertain when and how my students should use the fraction strips they make at the beginning of the fraction unit.
- Why can my students successfully shade $\frac{3}{4}$ of an area model and the next day struggle to locate $\frac{3}{4}$ on a number line or find $\frac{3}{4}$ of a set of objects?
- My sixth-grade students often use visual models to compare fractions. Is that OK or should they be able to use more efficient and abstract methods for this?
- My math textbook provides visual models for students to use and interpret but do not ask students to create their own visual models. Is this OK?

This chapter begins to address these questions as well as other issues related to the effective use of visual models in fraction instruction. Subsequent chapters will deepen these ideas and discuss how the use of visual models can help students develop understanding of specific mathematics concepts related to comparing and ordering fractions, finding equivalent fractions, and operating with fractions.

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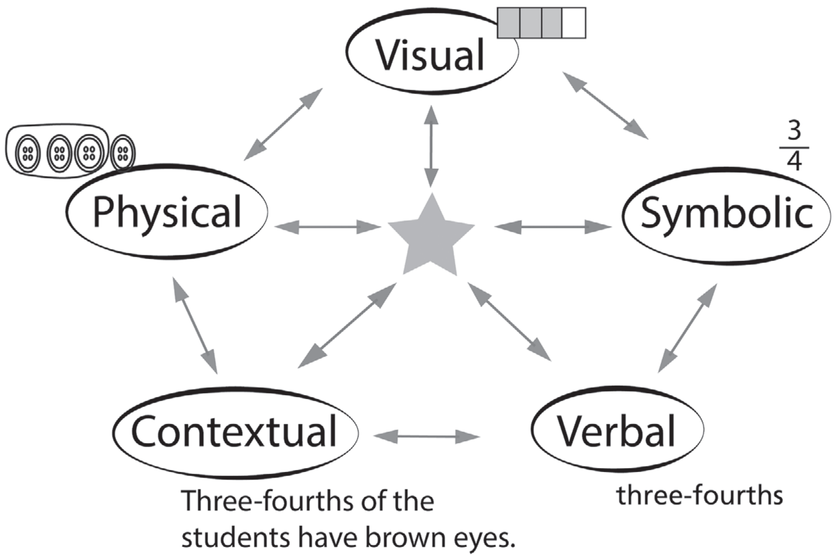
Mathematical Representations and Visual Models

Research identifies five types of mathematical representations as illustrated in Figure 2.1: visual, physical, symbolic, contextual, and verbal.

Figure 2.1 Five types of mathematical representations and the important connections among them.
Source: (Adapted from Leinwand, 2014)



The intentional use of a variety of mathematical representations is one aspect of effective mathematics instruction. Of particular importance to student learning is making connections among different representations and noticing how the different representations of a mathematical concept are alike and different (Hattie et al., 2016). For example, as students construct visual models, they connect the visual model with the symbolic representation of the fraction they are representing. In fact, representing mathematical concepts and procedures in multiple ways and focusing on the relationships among these representations can help students become better mathematical problem solvers and develop deeper understanding of mathematical concepts (Fuson et al., 2005).

The vignette that follows, as well as the message that "visual models are the means to the mathematics, not the end," sets the stage for understanding the importance of using visual models to help students build an understanding of fraction concepts. While addressing some of the teacher questions about using visual models, the vignette also illustrates the fine balance needed in using visual models to develop understandings without developing an overreliance on their use.

A Case Study—When Visual Models Are Used Like Calculators

Mr. Smith is a fourth-grade teacher who has been using the same mathematics program for the past five years. The program teaches fraction concepts through the use of only one visual model: the circle model. As a part of the instruction guided by this program, students make circle models representing halves, thirds, fourths, fifths, sixths, sevenths, through fourteenths, which are put on display and used in all aspects of the unit. Mr. Smith has always been comfortable with using only circle models for fraction instruction.

This past year, Mr. Smith participated in the Ongoing Assessment Project Study. He noticed that the OGAP questions did not always use the circle model, but instead included a variety of area models, number lines, set models, and visual models involving manipulatives such as pattern blocks and geoboards. However, because he was familiar with using the circle model, he charged ahead.

Midway through the unit, he gave the students a question that involved comparing $\frac{3}{7}$ and $\frac{7}{8}$. The students asked if they could use their circle models on display to answer the question. Mr. Smith said they could if they needed to, but he was hoping that they would not feel the need to use them.

Mr. Smith was very disappointed with what happened and was beginning to question the decision to only use circle models. With the exception of three students, all the students felt that they could *not* compare the fractions without the use of the visual models on the wall. He was hoping that his students would be able to visualize and justify $\frac{7}{8}$ as greater than $\frac{3}{7}$ using student-drawn visual models or justifications based on the benchmark $\frac{1}{2}$.

He thought that these fractions offered students a relatively easy comparison. However, instead of the visual models helping his students to internalize (generalize) the ideas behind the concepts, he realized that his students were using the premade circle models as the only way to compare fractions, in the same way that students sometimes inappropriately use calculators as the only way to make calculations.

It may be that Mr. Smith's reliance on one type of visual model limited his students' abilities to make the important conceptual leaps he intended. He was not sure. Mr. Smith realized that he needed to learn more about how to use visual models in his instruction and why using different visual models could help his students internalize and generalize important mathematical ideas.

This vignette paints a picture of a classroom in which only one type of visual model (fraction circles) was used and in which students relied on the visual models they made at the beginning of the unit as if they were reaching for a calculator to do a simple calculation. It may be possible that the students could compare these fractions without the premade visual models, but it was becoming clear to Mr. Smith that his students' use of visual models was not necessarily helping them internalize fraction concepts in the ways he intended.

According to research, Mr. Smith inadvertently made two mistakes in his use of fraction circles that may have led to his students not internalizing the concepts he intended to develop:

1. His students used the fraction circles in a "rote" way, not tied to the mathematical ideas that are embodied in the fraction circles (Clements, 1999). This led to their dependence on the circles to compare fractions.
2. He used only one visual model, while research suggests that learning is facilitated when students interact with multiple visual models that differ in perceptual features causing students to continuously rethink and ultimately generalize the concept (Dienes, cited in Post & Reys, 1979).

Visual models should be used as a way to understand and generalize mathematical ideas—visual models are a means to the mathematics, not the end (Clements, 1999; Post, 1981).

Physical Representations: Manipulatives

Physical representations or concrete models, commonly called manipulatives, are tangible objects that can be tactilely examined, sorted, counted, and grouped to represent a mathematical situation or concept. Physical representations can be everyday objects such as a collection of pennies or a jar of dried beans, or they can be manipulatives designed specifically for use in mathematics instruction such as Unifix Cubes, Cuisenaire Rods or Base-10 Blocks. Research suggests that while math manipulatives can be a valuable tool in mathematics instruction, their use alone does not automatically lead to student learning (Baroody, cited in Clements, 1999; Fennema, cited in Clements, 1999). It is the job of the teacher to help students connect the manipulative to a more generalizable and abstract mathematical understanding (Furner & Worrell, 2017). More specifically, it is the shared context that is developed around the manipulative that is most important—the ways in which students work, talk, and interact with the material towards a mathematical purpose—for "the manipulative itself cannot on its own carry the intended meanings and uses" (Ball, 1992, p. 18).

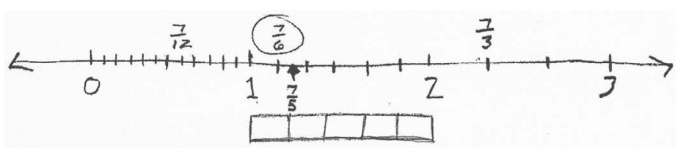
To maximize the impact of using manipulatives to build concepts, researchers suggest that teachers:

- Guard against using manipulatives in a rote manner (as Mr. Smith did with the fraction circles), and
- Make clear connections to the mathematical ideas embodied in the manipulative and do not "assume that the concepts can automatically be read off the manipulative" (Clements, 1999, p. 46).

Visual Models

Visual models, in particular student constructed visual models, are especially important in both the teaching and learning of mathematics because they help students deepen their understanding of concepts and procedures, more clearly make sense of problem situations, and support more impactful discourse (Stylianou & Silver, 2004). Student-constructed visual models (sketches, drawings, or diagrams) allow students to bring their own meaning to their developing mathematical understandings and contemplate foundational aspects of mathematical concepts. For example, when constructing visual fraction models, students may have to consider aspects such as the size of the whole and how to equally partition the model into equal shares. With manipulatives, some of these decisions are made for the student. To illustrate this point, examine the visual model Jessie constructed to order four fractions in Figure 2.2.

Figure 2.2 Jessie constructed this visual model to order $\frac{7}{3}$, $\frac{7}{5}$, $\frac{7}{6}$, and $\frac{7}{12}$ from least to greatest.



Jessie had to consider the size (i.e., length) of each whole represented in this model and equipartition the whole into 3rds, 5ths, 6ths, and 12ths. Thus, Jessie's visual model provides evidence of her developing understandings related to comparing and ordering fractions that can help inform her teacher's next instructional steps.

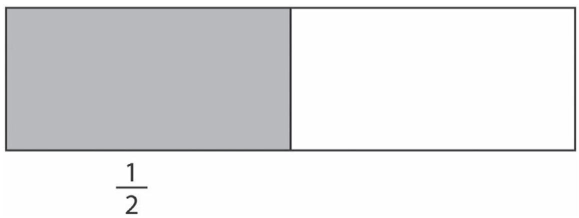
In general, instruction that includes ongoing opportunities for students to construct visual models, as well as interact with preconstructed visual models, not only supports student learning, but also can provide the teacher invaluable formative information that can inform future instructional decisions. You will interact with many examples of student constructed visual models throughout this book, where you will be asked to make sense of the student constructed model, identify developing student understandings based on the model, identify common errors or misconceptions that future instruction should address, and identify some possible next instructional steps.

There are three general categories of visual models that students interact with, use to solve problems, and use to generalize concepts related to fractions: area models, set models, and number lines. In the sections that follow, each of these types of visual models is examined in detail. Understanding the features of these visual models can help teachers better support students' use and construction of visual models.

Area Models

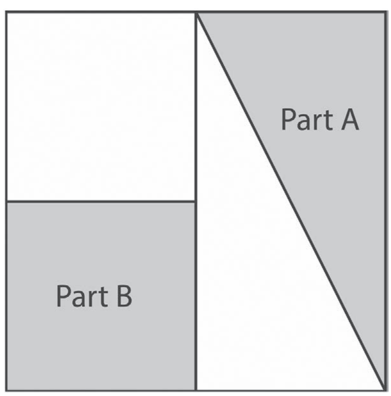
Area models that students typically interact with in mathematics programs and other instructional materials include objects or drawings such as grids, geoboards, folded paper, and pattern blocks. These visual models use the context of area or region to communicate a fraction. In the area model in Figure 2.3, the shaded region covers $\frac{1}{2}$ of the area of the entire rectangle. Thus, this is an area interpretation of the fraction $\frac{1}{2}$.

Figure 2.3 Area model representing the fraction $\frac{1}{2}$.



It is important to note that the key feature of area models is the area of the parts, not the shape of the parts. The parts need to have the equal areas, but do not need to be congruent. This idea is illustrated in Figure 2.4.

Figure 2.4 The fraction $\frac{1}{4}$ represented in equal areas of different shapes.



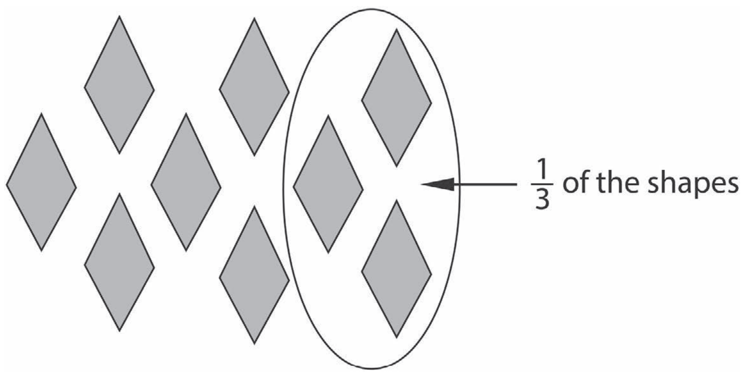
Both Parts A and B have the same area even though they look quite different and are different shapes. Each represents the fraction $\frac{1}{4}$ because they cover $\frac{1}{4}$ of the large square. Students need rich opportunities to work with area models that are partitioned into different shapes in order to develop understanding of this concept.

Set Models

Interpretation of a set model involves considering the fractional part of a set of countable objects. Sets students typically interact with in mathematics programs and instruction include collections of discrete objects such as buttons, candies, counters, marbles. An egg carton model could also be used as a set model since each cup can be counted individually.

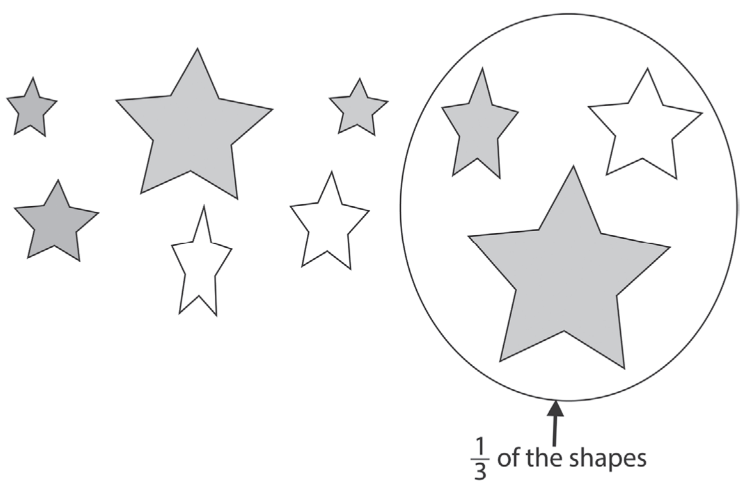
Representing a fraction in a set model is based on the definition of what is in the set, the number of items in the entire set and the number of items in the parts of a set. Figure 2.5 illustrates this feature of set models.

Figure 2.5 A set model representing the fraction $\frac{1}{3}$ as $\frac{1}{3}$ of the number of shapes.



The visual cue for the fraction $\frac{1}{3}$ in a set model is that the number of objects in the part is one-third of the number of objects in the entire set. Thus, a set model focuses on the count and not on other features of the objects in the set, such as color or size. As with the attribute of shape in an area model, this feature of a visual set model can be confusing for some students. Figure 2.6 provides an example of this idea.

Figure 2.6 Attributes such as size or color are not considerations in a visual set model.



One-third of the set of stars is circled despite the varying sizes and colors of the objects in the set and in the part. A set model is concerned solely with the number of objects. In this way thinking about fractions as "fair shares" is not accurate because someone receiving the $\frac{1}{3}$ share indicated earlier would receive the largest objects in the set. A more accurate description for a set model would be that equivalent fractions in a set have an equal count.

The way the items in the set are organized on the page can impact the difficulty of engaging with a set model. The use of physical objects such as counters allows students to move and group the items in the set, which may be easier for most students than interacting with a visual set model such as the one in Figure 2.6.

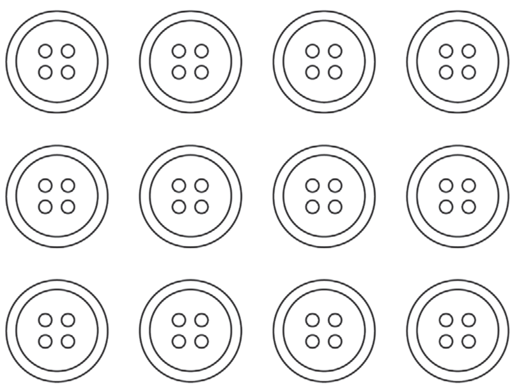
Haphazard Arrangement and Arrays

The visual set model in Figure 2.6 is an example of a set in which the objects are arranged haphazardly. This arrangement may make it more difficult for students to partition the objects into three equal groups to make thirds. The haphazard nature of the set may make counting more difficult as students need a method to make sure they count every object once without double counting. This arrangement provides no visual groupings to help one more easily see thirds. We contrast this with set models in which the objects are organized into an array.

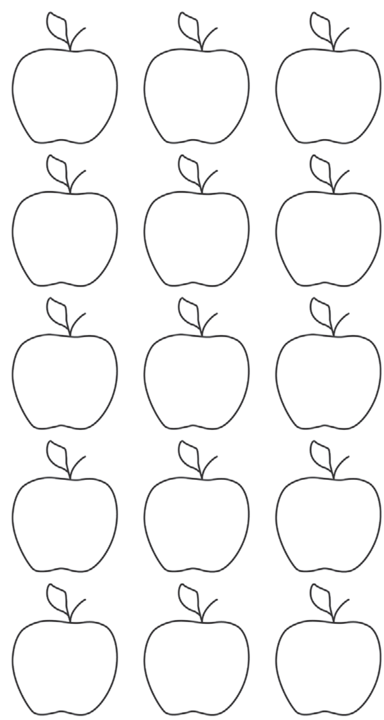
Figure 2.7 provides three different ways one can arrange a set of objects in an array. For each example, identify the ways in which the arrangement impacts the way in which a student may address the task.

Figure 2.7 Three examples of sets of objects arranged in an array.

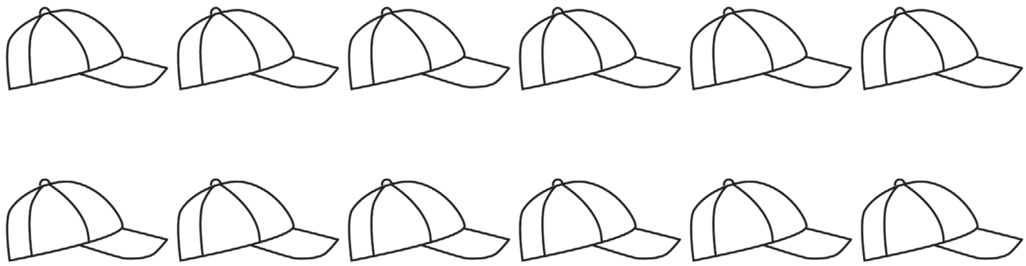
Example A: Circle $\frac{1}{4}$ of the buttons in the picture.



Example B: Circle $\frac{3}{5}$ of the apples in the picture.



Example C: Circle $\frac{2}{3}$ of the caps in the picture



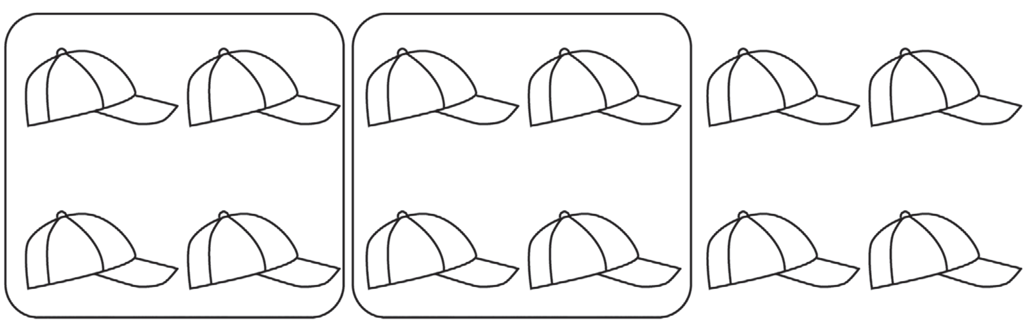
In Example A, the fourths are grouped by the columns. Each column contains one-fourth of the buttons in the set. Notice that in Example B the fifths are grouped in the rows. Each row of 3 apples is one-fifth of the apples in the set. Thus, three of the rows or 9 apples is three-fifths of the set. These two arrangements remind us that in an array, each row and each column represent a particular fraction of the whole set. Students should have plenty of opportunities identifying the fractional values of rows and columns in a variety of arrays.

Example A introduces us to another potential source of confusion for students. That is, one-fourth of the buttons is 3 buttons not 4 buttons. A common error in this type of set arrangement is for students to see each row as one-fourth because there are 4 buttons in each row.

Example C provides students a different challenge because the thirds are not represented in a row or a column. In this case, each column represents one-sixth of the caps in the set and each row represents one-half of the caps in the set. Students need a different strategy than considering the number of objects in a row or a column. This example involves partitioning the set into 3 groups each with the same number of objects. One possible solution is shown in Figure 2.8.

Circle $\frac{2}{3}$ of the caps in the picture

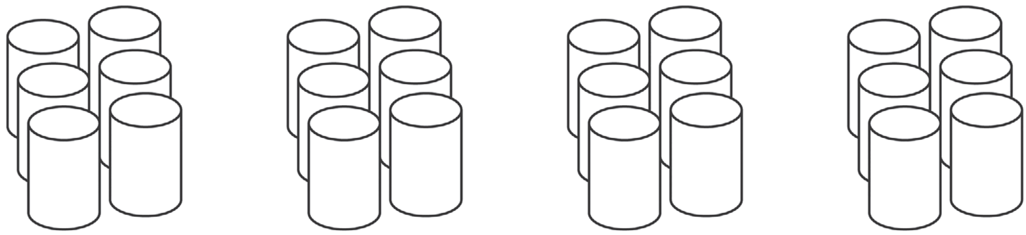
Figure 2.8 Sample solution for finding $\frac{2}{3}$ of a 2 by 6 array of objects.



Composite Sets

A composite set is comprised of subgroups or objects that are grouped together. An example of a composite set is shown in Figure 2.9.

Figure 2.9 Sample composite set.



One case of soda grouped in six-packs.

Notice there is more than one unit in this composite set. There is a case of soda, six-packs of soda and individual sodas. One could also include half of a case as another unit. This feature allows for questions such as:

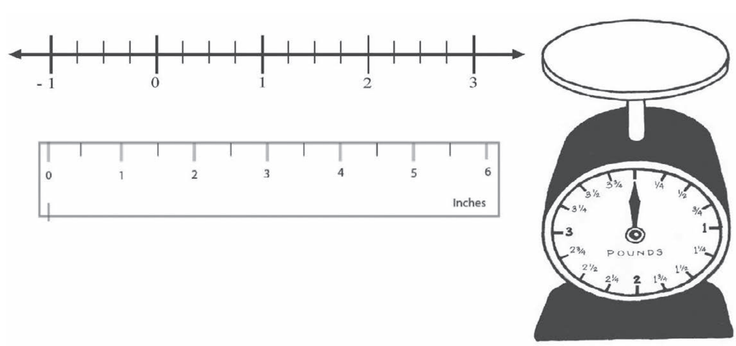
- Two sodas are what fraction of a six-pack?
- Three sodas are what fraction of a case?
- How many six-packs in half a case of soda?
- Three six-packs are what fraction of a case?

Visual and physical set models can provide students rich opportunities to develop fraction understanding in the context of collections of objects. One can change the perceptual features of a set model by varying the ways the objects in the set are arranged.

Number Lines

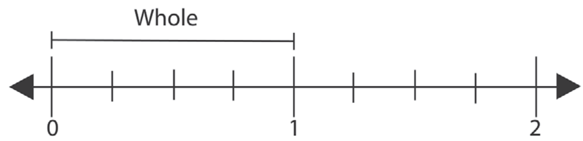
Making sense of a number line model involves reasoning about linear distance and the location of a point on a number line, ruler, or other linear measurement tool. Some examples of number lines are pictured in Figure 2.10.

Figure 2.10 Examples of number lines.



Number lines are different from area and set models for several reasons, starting with the definition of the whole or the unit. In a set model the whole is the number of objects in the set. In an area model, the whole is the area of a given region. So, what is the whole in a number line?

Figure 2.11 What is the whole on a number line?

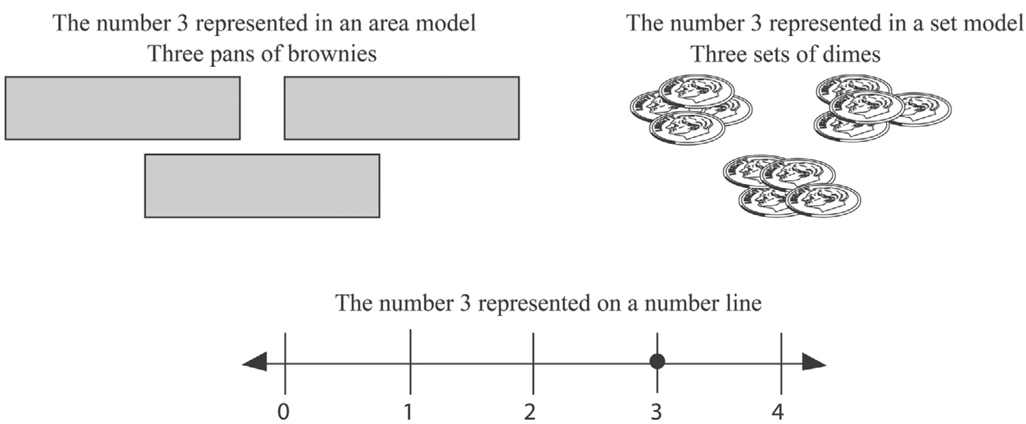


The whole on a number line is defined by the distance from 0 to 1 as shown in Figure 2.11. This concept impacts the meaning of a fraction on a number line. For example, one can think of $\frac{3}{4}$ as a line beginning at 0 with a length that is $\frac{3}{4}$ of the distance from 0 to 1, as shown in Figure 2.12. One might determine this length by partitioning the whole into fourths and drawing a line with a length equal to "three one-fourths." Using this concept, the number $1\frac{1}{2}$ is defined by a line that begins at 0 with a length that is $1\frac{1}{2}$ times the distance from 0 to 1.

Figure 2.12 Using length to define the numbers $\frac{3}{4}$ and $1\frac{1}{2}$ on number line.

Another feature that makes a number line different than an area or set model is the way it represents the whole numbers 1, 2, 3 etc. The whole numbers are continuous on a number line but separated or distinct in area or set models. See Figure 2.13.

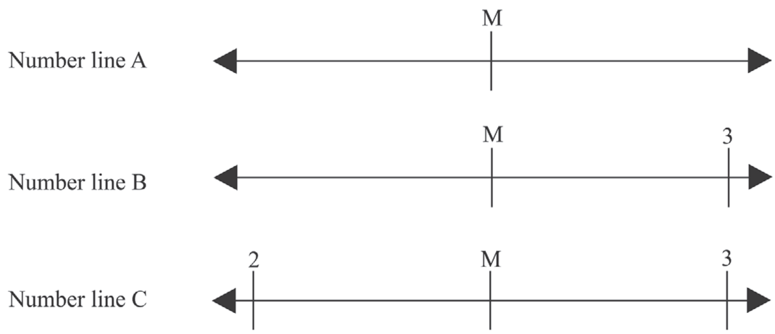
Figure 2.13 Whole numbers represented in area and set models, and on a number line.



This feature of a number line allows numbers greater and less than 1 to be represented more clearly than on the other two models Because of this, the relative magnitude of numbers is a central visual feature of a number line. For example, one can clearly see that 2 is closer to 1 than it is to 4. If we label the halves on this number line, $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, one can see that $2\frac{1}{2}$ is halfway between 2 and 3 or that $1\frac{1}{2}$ is closer to 1 than it is to 4.

Lastly, of the three visual models discussed in this chapter, the number line is the only one whose meaning requires integrating a visual component (the line) with numbers. One can only place a number on a number line or identify the value of a point on a number line if two other numbers are provided. The number lines in Figure 2.14 makes this point.

Figure 2.14 Interpreting a number on a number line requires integrating the line with numbers.



In number line A, point M could represent any number as no benchmarks are provided. In number line B we know that point M is less than 3 but we do not know how much less. The inclusion of two numbers, 2 and 3 in number line C now provides us the needed information to confidently estimate that point M represents $2\frac{1}{2}$. As mentioned earlier, notice that identifying a point or placing a number on a number line requires reasoning about magnitude and how one number or point relates to the numbers given on a number line.

Because of the continuous nature of number lines, the integration of a physical feature (the line) with numbers, and other attributes, number lines play a vital role in students' development of fraction concepts and skills. Chapter 9: Number Lines and Fractions provides a closer, more detailed examination of number lines and the teaching and learning of fraction concepts.

Number of Parts in the Whole

How the number of parts or objects in the whole relates to the magnitude of the denominator is another feature that needs to be considered when students solve problems involving visual models. Research shows that it is easier for students to find the fractional part of the whole when the number of parts in the whole is equal to the magnitude of the denominator, than when the number of parts in the whole is a multiple or factor of the magnitude of the denominator. The most difficult case is when the number of parts in the whole is a multiple of the denominator (Bezuk & Bieck, 1993).

Figures 2.15, 2.16, 2.17, and 2.18 provide examples of the relationship of the number of parts in the whole to the denominator.

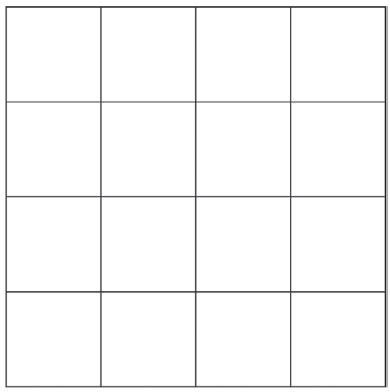
Figure 2.15 The number of parts in the whole is *equal* to the magnitude of the denominator: There are eight parts in the whole and the denominator is 8.

Shade $\frac{3}{8}$ of the figure.

Teachers found that students had the most difficult time finding the fractional part of a region when the number of parts in the whole was a multiple of the denominator, and the number of rows and columns were not equal to the denominator. The area model in Figure 2.16 is an example of this (OGAP, 2005). Note that the area has been partitioned into 16 equal parts which is a multiple of the denominator 8, and the length of each side of the square is 4, which is a factor of the denominator 8.

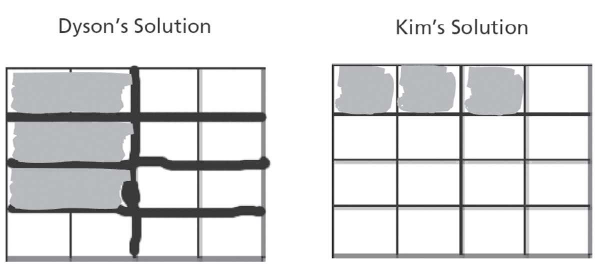
Figure 2.16 The number of parts in the whole is a *multiple* of the denominator: There are 16 parts in the whole, which is a multiple of the denominator 8.

Shade $\frac{3}{8}$ of the figure.



Finding $\frac{3}{8}$ of a figure in which the number of parts in the whole is a multiple of the denominator (Figure 2.16) requires the understanding that $\frac{3}{8} = 3\left(\frac{1}{8}\right)$. This understanding results in the whole figure being equipartitioned into eight equal parts regardless of the number parts in the whole or in each part. Compare the solutions to this problem shown in Figure 2.17

Figure 2.17 Dyson used the understanding that $\frac{3}{8} = 3\left(\frac{1}{8}\right)$ by partitioning the whole into eighths and shading $3\left(\frac{1}{8}\right)$ of the figure. Kim used the magnitude of the numerator and shaded 3 parts.



To find the fractional part of a whole when the number of parts in the whole is a factor of the denominator requires repartitioning the figure. In the case of the problem shown in Figure 2.18, this means repartitioning the rectangle into eighths.

Figure 2.18 The number of parts in the whole is a *factor* of the denominator. The whole is divided into two parts, which is a factor of 8.

Shade $\frac{3}{8}$ of the figure.

Go To For more about partitioning see Chapter 3, What Is the Whole? and Chapter 4, Equipartitioning. To explore the concept of the relationship between the number of parts in the whole to the magnitude of the denominator, answer questions 1, 2, and 3 in the Looking Back section at the end of this chapter.

Summary: Features of Set, Area, and Number Line Models

As we have seen, set, area, and number line models represent fractions in different ways, presenting students with unique challenges and understandings. We can generalize these differences by thinking about how each model represents these features of a fraction:

1. How the whole is defined.
2. How "equal parts" are defined.
3. What the fraction indicates.

Visual models differ in the challenges they present to students (Hunting, cited in Bezuk & Bieck, 1993; OGAP, 2005, 2007; Zawojewski, personal communication, November, 2005). Why visual models differ in the challenges they present and why it is important for students to encounter the three types of visual models are related, in part, to three aspects of visual models for fractions.

Table 2.1 Summarizes how the features of visual area models, sets of objects, and number lines differ.

Table 2.1 Features of the Visual Models

Visual Model	*The Whole*	*"Equal parts" Are Defined by:*	*What the Fraction Indicates*
Area model	Determined by the area of a defined region	Equal area	The area of the identified part in relation to the entire area
Set model	Determined by the number of objects the set	Equal number of objects	The number of objects in the subset of objects in relation to the total number of objects in the set
Number line	Length of the "unit": its distance from zero	Equal distance	The distance of a given point from zero in relation to length of the "unit"

Visual Models and Student Learning

The first part of this chapter dealt with examining the three general types of visual models for representing fractions and how each visual model type supports a different interpretation of a fraction. The next part of the chapter extends these ideas to the role visual models play in effective fraction instruction and in student learning. It begins with a brief look at visual fraction models and the CCSSM

CCSSM

The CCSSM and Visual Models

The Common Core State Standards in Mathematics (CCSSM) highlight the role that using visual models plays in developing understanding of fraction concepts and fluency. Students are expected to use visual models to help them understand new concepts, to solve problems, and to show their understanding of concepts. Later they are expected to use the concepts underlying the visual models to generalize concepts, build new mathematical ideas, or understand and apply procedures. For example, Standard 4.NF.1 is centered on students' use of visual models to understand why multiplying the numerator and denominator of a fraction by the same factor results in a fraction that is equivalent to the original fraction. However, it is important to note that the purpose of the standard is not to use visual models to find equivalent fractions, but to use the understanding derived from working with the visual models to recognize and generate equivalent fractions. This standard is more fully discussed in Chapter 7, Equivalence.

Providing Opportunities to Interact with Visual Models That Have Different Perceptual Features

Mathematics programs, textbooks and other instructional materials differ in their use of visual models. Some use one type of visual model such as pattern blocks or dot paper. We saw in the opening vignette that Mr. Smith's math program utilized only visual circle models. According to research, limiting instruction to one type of visual model can be problematic. The best instructional approach includes a balance of visual models that differ in perceptual features, causing students continuously to rethink the concept (and not to overgeneralize on the strength of one model) (Dienes, cited in Post & Reys, 1979).

When students are interacting with perceptual features of visual models, they are interpreting the different aspects or characteristics of the visual models. One way to think about this is to think about questions related to the different features of visual models.

Instructional Considerations Regarding Perceptual Features of Visual Models

- What is the whole in the visual model?
- In an area model, what is the shape of the whole?
- In an area model, what is the shape of the part? How does the shape of the part relate to the shape of the whole?
- In set models, how are the objects in the set arranged (e.g., arrays, haphazard)?
- In set models, are the sizes of the objects the same or different?
- Is partitioning provided or does the student have to provide the partitioning?
- What is the nature of the partitioning provided if it is provided?
- What do equal parts mean in this visual model?
- What is the relationship between the number of parts in the whole and the magnitude of the denominator?
- What would a student have to do to find the fractional part of the whole in question?

Once teachers understand the importance of using a variety of visual models with different perceptual features, they report that their instruction changes. For example, teachers in the OGAP Exploratory Study (2005) reported:

- using a greater variety of visual models in their instruction
- making explicit links between visual models
- providing more opportunities for students to use visual models to solve problems
- an increase in the use of number lines

These instructional changes appear to be reflected in student work in the OGAP (2005) Study. Some 30% $\left(\frac{39}{128}\right)$ of pre- and post-assessments for students in grade 4 were analyzed for the use of visual models to solve problems:

- In the pre-assessment, only 23% $\left(\frac{9}{39}\right)$ of the students effectively used one or more visual models to solve problems.
- In the post-assessment, 80% $\left(\frac{31}{39}\right)$ of the students effectively used one or more visual models to solve problems.

While it is important to use a variety of visual models, it is also important to recognize that some visual models can be used more effectively in some situations than in others. Review the student responses in Figures 2.19 and 2.20. In these responses, students use visual models to place $\frac{1}{3}$ and $\frac{1}{4}$ in the correct locations on a number line. Both responses show evidence of effectively using a model to solve the problem.

Figure 2.19 Wesley effectively used two number lines the same size as the original number line to locate $\frac{1}{3}$ and $\frac{1}{4}$ on the number line.

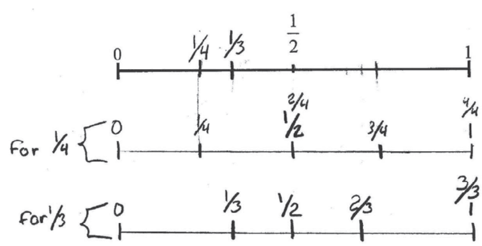
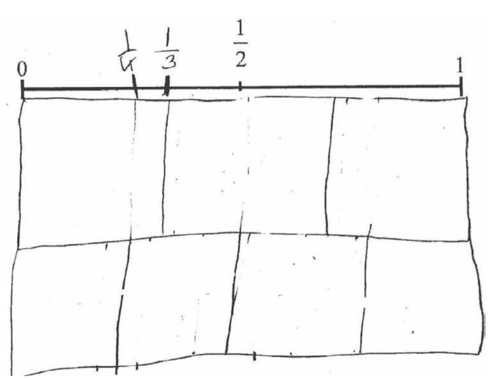
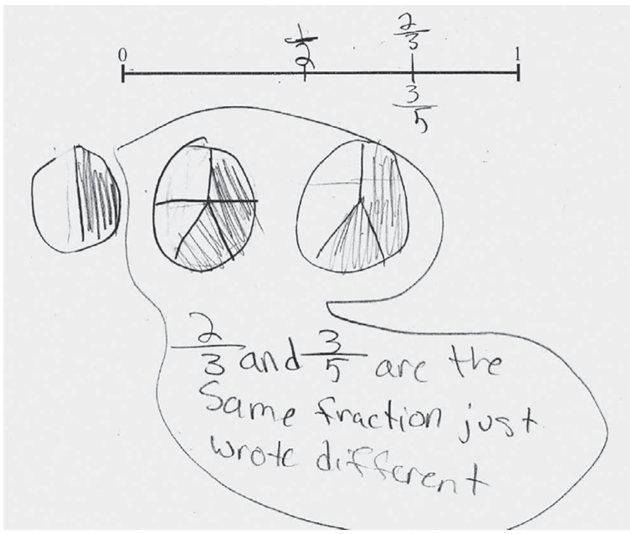


Figure 2.20 Patty effectively used the linear dimension of visual area models to locate $\frac{1}{3}$ and $\frac{1}{4}$ on the number line.



Patty's response exemplifies a strategy that many students in the OGAP (2005) studies adopted: Student-drawn area models can be effective for making comparisons or locating fractions on a number line when the linear feature of the visual model is used. However, this was only true when the lengths of the wholes were the same, and the visual models were partitioned into equal-sized parts. Using number lines and the linear feature of an area model are effective strategies for locating fractions on a number line. On the other hand, Kim's use of circles in Figure 2.21 to help locate $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{2}{3}$ was not effective, given the problem situation.

Figure 2.21 Kim's response. For this situation, the circles are less effective than the visual area models used by Patty or the number lines used by Wesley because circle models do not translate well to the linear feature of the number line.



It is important to make a distinction here: Circle models can be used effectively to compare fractions as long as students consider the size of the whole and are accurate in their partitions into equal-sized parts. However, evidence in student work has shown that circle models are not as effective in accurately locating fractions on a number line as number lines or rectangular models (OGAP, 2005).

Student-Drawn Visual Models May Have Limitations

Although it is important to use visual models to develop conceptual understandings of fractions, some student-drawn visual models may have limitations at the younger grades when a student's fine motor skills are not developed, or when older students are comparing fractions that are close in magnitude (Lamon, 1999; OGAP, 2005). In both cases, students may draw incorrect conclusions based not on lack of understanding, but on an inaccurate drawing.

Karen's response to the candy problem in Figure 2.22 provides an example of how the use of models may be limiting if the fractions are close in magnitude and the student's fine motor skills are undeveloped.

Karen incorrectly concluded that $\frac{2}{5}$ and $\frac{3}{10}$ are equal based on her visual models, which were not partitioned into equal-sized parts. This partitioning error makes it appear that the fractions are equal.

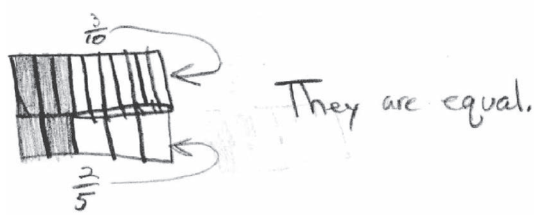
Figure 2.22 Karen's response. While the wholes in Karen's models were the same size, her inaccurate partitioning led her to the conclusion that $\frac{3}{10}$ and $\frac{2}{5}$ are equal.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or peppermint candies in the dish?

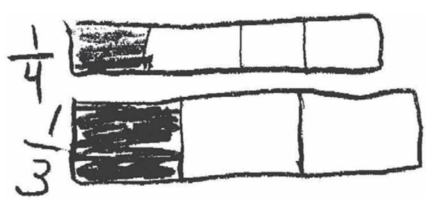


Helping Students Understand Errors When Using Visual Models

One way to help students understand errors when using visual models is to have them compare their visual models to a manipulative in which the sizes of the wholes and parts are predefined.

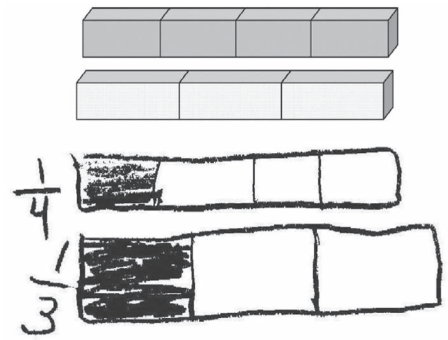
In Figure 2.23, Leslie used a visual area model to compare $\frac{1}{3}$ and $\frac{1}{4}$. Her visual model would lead to the correct conclusion that $\frac{1}{3}$ is greater than $\frac{1}{4}$; however, her modeling error (wholes not the same size) could lead to the incorrect conclusion that $\frac{2}{3}$ and $\frac{3}{4}$ are equal.

Figure 2.23 Leslie's response. Because the wholes are not the same size, Leslie may conclude that $\frac{2}{3}$ and $\frac{3}{4}$ are equal.



The inaccuracy of the visual models could be due to a misunderstanding about the importance of the wholes being the same size when comparing fractions, or it could be due to the limitations of Leslie's hand-drawn model. In either case, having Leslie compare her visual models to manipulatives may help her focus on important features of visual models (e.g., size of the whole) that she should consider when comparing fractions. See Figure 2.24.

Figure 2.24 Comparing manipulatives with predefined wholes and parts to student work.

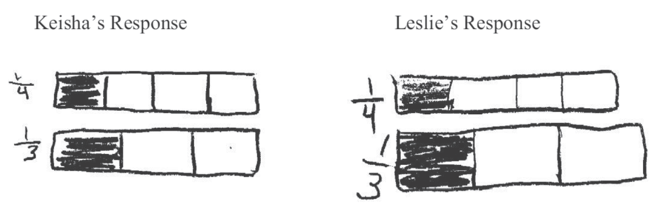


As Leslie compares the two visual models, the teacher should ask her explicit questions that focus on features of the visual models that are necessary for the effective use of the visual model in making an accurate comparison of the fractions.

Another way to help students improve their visual models is to compare different student solutions. One can place two solutions side by side as in Figure 2.25, and ask questions designed to focus student thinking on the mathematical point being made:

- How are Keisha's and Leslie's visual models alike?
- How are they different?
- Using Keisha's visual model, it appears that $\frac{3}{4} > \frac{2}{3}$. Using Leslie's visual model $\frac{3}{4}$ and $\frac{2}{3}$ appear to be equal. What feature of the visual models led to a different conclusion?

Figure 2.25 Comparing two student solutions. Both students partitioned rectangles into nearly equal parts. However, the wholes in Leslie's model are not the same size, leading to a wrong conclusion that $\frac{3}{4}$ and $\frac{2}{3}$ are equivalent. The sizes of the wholes in Keisha's model are the same, leading to the correct conclusion that $\frac{3}{4} > \frac{2}{3}$.



Visual Models Are Not the Only Way to Reason with Fractions

As important as the use of visual models is to students' learning of fractions, there are other equally important concept-based methods for reasoning about fractions. In fact, researchers have found that students use five types of reasoning when they successfully compare and order fractions that in some way involve reasoning about the relative contributions of the numerator and denominator to the magnitude of the fractions. They are:

1. Using visual models, visual and physical.
2. Using unit fraction reasoning (fractions with numerators of one, e.g., $\frac{1}{8}, \frac{1}{5}, \frac{1}{16}$).
3. Using extended unit fraction reasoning when comparing and ordering other fractions.
4. Using a reference point, such as $\frac{1}{2}$.
5. Using equivalence/common denominators (Behr & Post, 1992).

Go-To Chapter 7, Equivalence, and Chapter 8, Comparing and Ordering, for more on these reasoning strategies.

Jared and Kelyn: Closing Thoughts on Students' Use of Visual Models

The goal is not for students to use visual models to calculate answers but to use visual models to learn fraction concepts and skills deeply and flexibly. Jared and Kelyn's experiences with visual models provide us with important reminders about the important role visual models play in the rich learning we want for all students. We begin with an interview with Jared, a third-grade student.

Interviewer: I know that mathematicians use visual models, but sometimes kids in school are uncomfortable using them.
Jared: I think it's pretty comfortable because sometimes if you try to do it in your head its gets harder and if you use like blocks or diagrams or anything it will help a lot. Sometimes my favorite thing is like a number line or a T-table or something. That's what I do a lot.
Interviewer: It's nice to hear that you are comfortable to draw or get other materials or that kind of thing.
Jared: Yeah, because it helps you do the questions a lot better.
Interviewer: Well, you can see it, right? It's not just words on a page.
Jared: Yeah, because if you do it in your head you can't do it as good. Sometimes I first use blocks. Then I sort of sometimes imagine blocks. So now I sort of do it in my head.
Interviewer: Wow!
Jared: So I can imagine blocks and I can do it without real blocks and I can do it in my head now. Because I did it with blocks and got it in my head, I can do it pretty easy now.
Interviewer: Why can you do it in your head now?
Jared: Because I used blocks a lot in first and second grades, and since I did it a lot it sort of got stuck in my head.
Interviewer: What happens when a problem gets hard?
Jared: When the problems get like harder and harder, when they are really hard I sometimes need to draw or something.

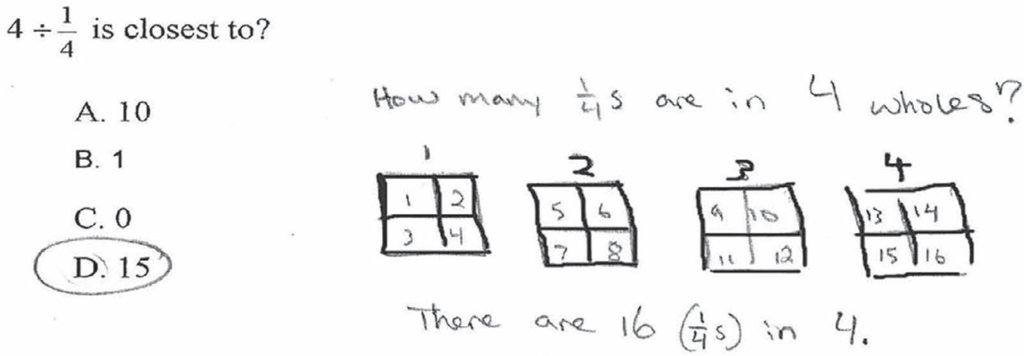
(OGAP, 2007)

> Models are mental maps mathematicians use as they solve problems or explore relationships. For example, when mathematicians are thinking about a number, they may have a number line in mind. They think about where the numbers are in relation to one another on this line, and they imagine moving back and forth along the line.
>
> (Fosnot & Dolk, 2002, p. 73)

As students are developing their understanding of concepts, they will need to construct visual models to solve problems and represent concepts if they are to construct their own mental map. Over time, students can build on their experiences with visual models to develop their own mental image of a mathematics concept. Students' acquisition of a mental model does not mean visual models are no longer an important part of learning and understanding mathematic concepts. For students, intentional and systematic use of visual models is vital, particularly as they learn new concepts or encounter more difficult problems. The interview with Jared makes this point.

One suspects that Jared's confidence in solving problems and using either "pictures in his head" or physical models are the result of experiences with a variety of visual models over time as he developed his understanding of the concepts. This suggests that fraction instruction should provide experiences in which students are encouraged to look for patterns and relationships, make and explore conjectures, and use what they learn from their visual models to generalize concepts. Kelyn's response to a division problem in Figure 2.26 also makes this case.

Figure 2.26 Kelyn's visual model suggests an accurate conceptualization of this division problem.



The evidence in Kelyn's solution leads one to believe that she understands that there are 4 one-fourths in a whole and 16 one-fourths (written in Kelyn's response as " $16\left(\frac{1}{4}s\right)$ ") in 4 wholes. Her effective use of a visual model to show the number of $\frac{1}{4}$s in 4 is important and valuable but not sufficient. Strong instruction must build on and deepen her current ideas about fraction division. Thus, the challenge for instruction is to ask questions or present situations that *capitalize* on her current understandings to lead to a generalization about dividing a whole number by a unit fraction.

Here are some examples of questions that one might ask Kelyn to help her move to a more generalized understanding of the division of fractions:

- Your visual model shows that there are 4 one-fourths in every whole. How many fourths do you think there are in 5 (or in 6, or in 10, or in 100)?
- How many thirds, fifths, or sixths are in 4 (or in 5, or in 6, or in 100)?
- What patterns do you see?
- Make and test a conjecture about the patterns that you see (giving Kelyn the chance to say "I noticed that . . .").

Jared's transition back and forth between mental models and visual models and the potential for Kelyn's teacher to capitalize on her conceptualization of the division of fractions to make a generalization about division make an important point: The use of visual models is a means to the mathematics, not the end (Clements, 1999; Post, 1981).

> Additionally, researchers indicate that one way to help students build on their use of visual models to understand and generalize fraction concepts is through the use of "multiple experiences (not the same experience multiple times) using a variety of materials."
>
> (Dienes, cited in Post, 1981)

Using visual models and regular probing and asking students to explain their thinking or demonstrate their visual models should play a key role in instruction as students are solving problems and building their understanding of unit fractions, the relative magnitude of fractions (equivalence, comparing and ordering fractions), or fraction operations.

The use of visual models (both teacher- and student-generated) should permeate instruction, not just be an incidental experience for the class, but a way of thinking and learning for students.

- Students should have opportunities to solve problems in which they interact with visual models (e.g., shade $\frac{3}{8}$ of an area model or find $\frac{3}{8}$ on a number line).
- Students should have opportunities to solve problems by generating their own visual models (e.g., Figure 2.26 Kelyn's division solution).
- Students should have opportunities to use visual models to develop understanding of concepts (e.g., use a visual model to show that $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent).
- Teachers should build instruction on student-generated visual models to help students generalize mathematical ideas. Teachers should ask students to explain their visual models and respond to probing questions that capitalize on understandings in their visual models (e.g., Kelyn's division visual models).

CCSSM

The use of a variety of visual fraction models permeates the CCSSM. In grades 1 and 2, students are introduced to the meaning of fractions. By engaging in sharing activities and problems, students partition circles and rectangles into equal shares. The focus in these grades is for students to use words, not formal fraction notation, to describe the equal shares and to begin to see that partitioning into more equal shares results in smaller pieces (see Chapter 4, Equipartitioning). In grade 3, students use area models and number lines to lay the foundation for understanding unit fractions and fractions as quantities that can be compared and written in equivalent forms. In grade 4, the use of visual fraction models extends to the use of collections of objects (sets). The use of all three visual fraction models continues through elementary and middle school as students develop their understanding and procedural fluency for equivalence and magnitude (including negative fractions) and operations with fractions (Common Core Standards Writing Team, 2013a, 2013b). The chapters on equivalence, equipartitioning, comparing and ordering fractions, number lines, and operations with fractions provide further discussion on how visual models are used in the CCSSM to develop concepts and/or to explain or justify an understanding of a concept.

Chapter Summary

This chapter presented research related to the use of visual and physical (manipulative) models. The research indicates that when used effectively, pre-drawn visual models, manipulatives, and student-generated visual models can help students develop a strong conceptual understanding of fractions. Furthermore, the research suggests that students should:

- interact with a variety of visual models that differ in perceptual features and in a variety of contexts to understand concepts
- make connections between physical and visual models, verbal descriptions, symbolic representations, and contexts

The most important goal for the use of visual models in fraction instruction is to support the generalization of concepts and skill. Visual models are a means to a greater end.

Looking Back

1. Compare questions in Figures 2.27 and 2.28. Data from the OGAP Exploratory Study (2005) showed that students had a more difficult time with the question in Figure 2.27 than with the question in Figure 2.28. Provide a possible explanation for why one question is more difficult than the other.

Figure 2.27 Shade $\frac{1}{8}$ of this figure.

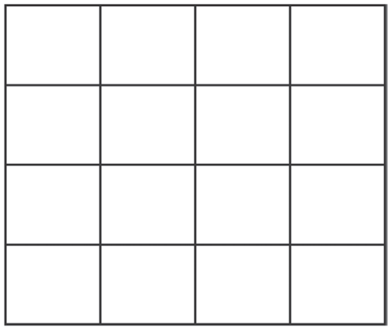
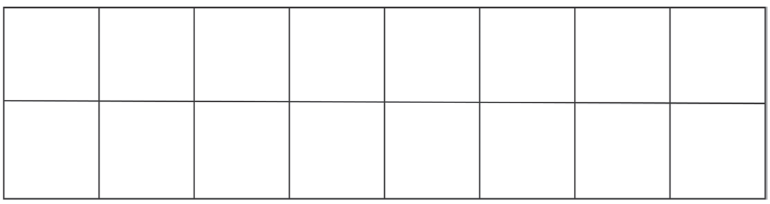


Figure 2.28 Shade $\frac{1}{8}$ of this figure.



2. Why do you think that it is more difficult for a student to determine the fractional part of a whole when the number of parts in the whole is a factor or multiple of the denominator rather than when the number of parts in the whole is equal to the denominator?
3. You have just completed the first part of your fraction unit with your third-grade students. Up to this point, your students have been finding the fractional part of an area as in Figure 2.29. Students have been very successful with questions like these. Today you are going to ask your students to find $\frac{3}{4}$ of the objects in a bag. There are four marbles and eight buttons in the bag. What aspects of the task may cause problems for your students? Explain.
4. Study the visual models in Figures 2.30–2.32. Identify the perceptual features of each that may require students to reinterpret their understanding of $\frac{3}{4}$ as they move from one visual model to another.

Figure 2.29 Shade $\frac{3}{4}$ of this square.

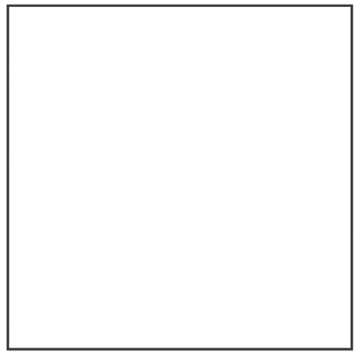


Figure 2.30 Shade $\frac{3}{4}$ of the rectangle.

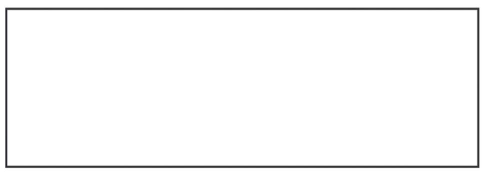
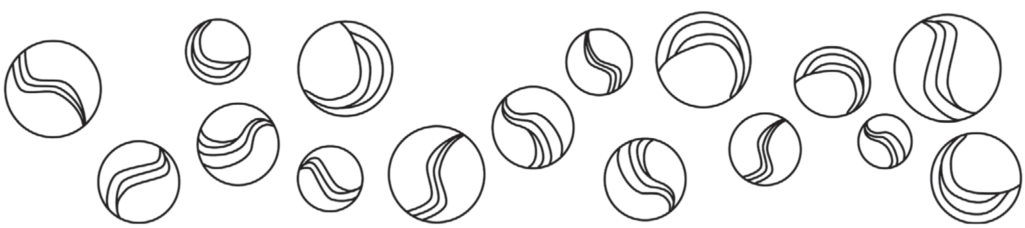


Figure 2.31 Locate $\frac{3}{4}$ on the number line.

Figure 2.32 Circle $\frac{3}{4}$ of the marbles.



Instructional Link: Your Turn

Use Table 2.2 to help you think about how your instruction or mathematics programs provide students the opportunity to use a variety of visual models to solve problems, understand concepts, or generalize ideas.

Table 2.2 Instructional Link—Using Visual Models

What visual models are used in your instructional materials for developing fraction concepts?
1. Area models 2. Sets of objects 3. Number lines 4. Manipulatives **How are the visual models used in your instructional materials?** Students are given visual models and asked questions using given visual models. ☐ never ☐ occasionally ☐ throughout Students have the opportunity to generate their own visual models to solve problems. ☐ never ☐ occasionally ☐ throughout Students have the opportunity to use manipulatives to bring meaning to concepts. ☐ never ☐ occasionally ☐ throughout Students have the opportunity to generate and use visual models to help develop concepts and/or generalize ideas. ☐ never ☐ occasionally ☐ throughout
Based on this analysis and your understanding of the importance of using visual models to build and understand fraction concepts, how might you adjust your instruction? Describe.

3
What Is the Whole?

Big Ideas

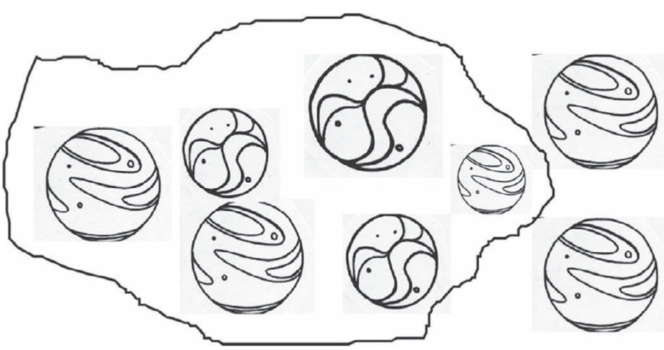
- A fraction should always be interpreted in relation to the specified or understood whole.
- Understanding fractional parts in relation to the whole is a critical foundation for other fraction concepts, such as relative magnitude, equivalence, and operations.

Defining the Whole

"The concept of the whole underlies the concept of a fraction" (Behr & Post, 1992, p. 13). In other words, a fraction should always be interpreted in relation to the specified or understood whole.

For example, the meaning of the fraction $\frac{3}{4}$ can differ depending on the nature of the whole: $\frac{3}{4}$ of a set of marbles (Figure 3.1), or $\frac{3}{4}$ of a brownie (Figure 3.2), or the distance from 0 to $\frac{3}{4}$ on a number line (Figure 3.3). In these cases the whole is specified.

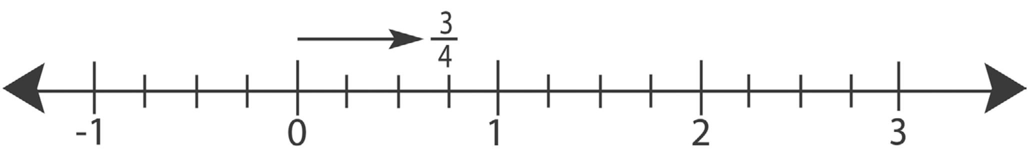
Figure 3.1 Specified whole. The whole is the set of eight marbles. Three-fourths of the set of eight marbles is six marbles.



DOI: 10.4324/9781003185475-3

Figure 3.2 Specified whole. The whole is the brownie, and the visual area model shows $\frac{3}{4}$ of the brownie remaining after one part, $\frac{1}{4}$ of the brownie, has been removed.

Figure 3.3 Specified whole. The whole is a defined length on a number line (0–1). Three-fourths represents the distance from 0 to $\frac{3}{4}$ based on the length of the defined unit.



In other cases, the whole is not specified, as in the problem in Figure 3.4. Understanding fractional parts in relation to the whole is a critical foundation for other fraction concepts, such as relative magnitude, equivalence, and operations.

Figure 3.4 Whole not specified. The number of students (the whole) in Mrs. Smith's class and in Mr. Taylor's class is not specified.

Three-fifths of Mrs. Smith's students ride the bus to school.

One-half of Mr. Taylor's students ride the bus to school.

Explain how it could be possible that Mr. Taylor has more students ride the bus to school even though Mrs. Smith has a greater fractional part of her students ride the bus to school?

CCSSM

The CCSSM and the Size of the Whole

The CCSSM explicitly states that starting at grade 3 students should "recognize that comparisons are valid only when the two fractions refer to the same size whole" (CCSSM Standard 3.NF.A.3.D). At grades 4 and 5, this understanding is extended to adding and subtracting fractions and solving related problems.

This chapter focuses on students understanding the importance of specifying the whole and the related challenges students may encounter as they compare and order fractions, find equivalent fractions, find parts of a whole, and add and subtract fractions.

The challenges students may encounter are:

- Some students have difficulty identifying the whole when there is more than one part or object in the whole (Payne, 1976).
- Some students use an "out of parts strategy" (Figure 3.8, Dominic), not an "out of equal parts strategy" (Figure 3.8, Abdi) when finding the fractional part of a whole (OGAP, 2005).
- Some students have a difficult time determining the whole when they are given just a part of the whole (Behr & Post, 1992), particularly when working with fractions that are not unit fractions (Figure 3.23) (OGAP, 2005).
- Some students make comparisons using models in which the wholes do not reflect the situation (Figure 3.21) (OGAP, 2005).
- Some students make comparisons using visual models in which the wholes are not equal in size (Figure 3.22) (OGAP, 2005).

Each of these challenges is illustrated with examples of student work in the following sections.

Identifying the Whole When There Is More Than One Part

Some students have difficulty identifying the whole when there is more than one part or object in the whole (Payne, 1976). This sometimes results in students using an "out of parts strategy," not an "out of equal parts strategy," when finding the fractional part of a whole (OGAP, 2005).

Tom's response (Figure 3.5), although showing one understanding of $\frac{1}{2}$, may provide evidence that he is unsure of the whole by interpreting each heart as a whole. Sonia, on the other hand, treats the set of hearts as the whole (Figure 3.6).

Figure 3.5 Tom's response. Tom circled one-half of each heart.

Circle $\frac{1}{2}$ (one-half) of the set of hearts.

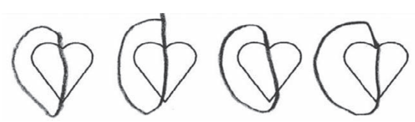


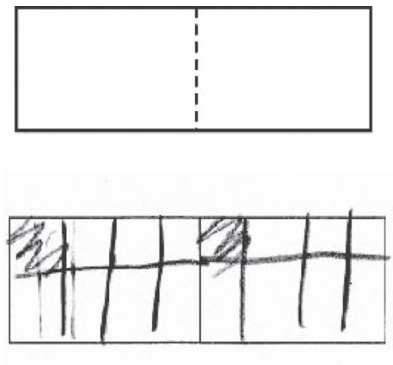
Figure 3.6 Sonia's response. Sonia correctly circled one-half of the set of hearts.

Circle $\frac{1}{2}$ (one-half) of the set of hearts.

Another way that this issue is evidenced is when students treat an area partitioned into two parts as two wholes, as in Karen's response in Figure 3.7. Although the total area shaded in Karen's response is $\frac{1}{8}$ of the figure, Karen's use of unnecessary partitions may indicate confusion regarding the whole.

Figure 3.7 Karen's response. Karen partitioned each half-rectangle into eighths and shaded $\frac{1}{8}$ of each half using a "one out of eight" strategy.

Shade $\frac{1}{8}$ of Figure 3.7.

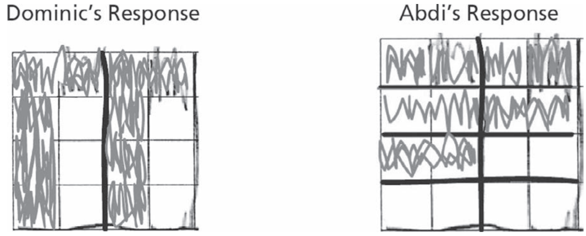


In Figure 3.8, Dominic shaded five out of eight parts twice, whereas Abdi equipartitioned the whole into eight equal parts (8 one-eighths) and then shaded 5 one-eighths.

Figure 3.8 Dominic shaded five out of eight parts twice. Abdi divided the 16 parts into eight equal parts.

Shade $\frac{5}{8}$ of the figure.

Dominic's Response　　　　Abdi's Response



It appears that Abdi is seeing the whole as sixteen parts, while Dominic is seeing two wholes of eight parts. Some teachers argue that both answers are correct, and they are right. However, Abdi's strategy is more efficient and generalizable and suggests an understanding of how $\frac{5}{8}$ is built from the unit fraction, $\frac{1}{8}$.

Consider each student solving a problem in which they have to find $\frac{5}{8}$ of \$168. If Dominic used the same strategy, he may have to draw a figure or set of objects with 168 parts and then shade/circle five out of eight parts 21 times. Once that is done, Dominic would have to count the parts in all the shaded regions.

On the other hand, if Abdi used his out-of-equal-parts strategy, he would first divide \$168 into eight equal parts to find that each $\frac{1}{8}$ of \$168 is equal to \$21.00. Extending his unit fraction knowledge that $\frac{5}{8} = 5\left(\frac{1}{8}\right)$, he would understand that $\frac{5}{8}$ of \$168 is $5 \times$ \$21.00, or \$105.00.

One important idea from this student work and the accompanying discussion is that in order for students to develop more sophisticated fractional strategies, instruction should focus on strategies to help students understand the whole and to use their developing unit fraction understanding.

One strategy that may help students to see the whole is through the use of a familiar context in which it is not sensible to divide each object/part in the whole. This strategy is more likely to promote thinking about the whole, not each object/part in the whole. For example:

- How many dinosaurs are in $\frac{1}{2}$ of a set of 18 dinosaurs?
- Circle $\frac{1}{2}$ of a set of 20 coins.

Another strategy that helps to focus students on the whole is suggested by Lamon (1999). She suggests that students may have an easier time identifying the whole and subsequently will make fewer partitions if they have an opportunity first to visualize the whole from a distance. Some teachers project diagrams onto a classroom screen, as shown in Figures 3.9 to 3.14. Students discuss how they visualized the fractional parts of the whole.

Because the goal of projecting a set of objects is to help students visualize the whole and identify different fractional parts of the whole, select a set of objects that lends itself to exploring a range of fractional parts. For example, a set of 24 objects allows one to explore halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths.

This type of visualization can also help students see equivalent fractions such as $\frac{1}{3} = \frac{2}{6}$ (Figure 3.14).

Figure 3.9 A set of 24 apples displayed at a distance.

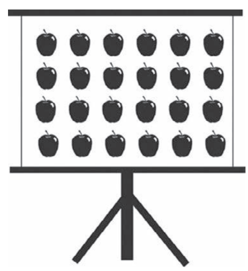


Figure 3.10 Some students may visualize $\frac{1}{2}$ of the set of apples like this.

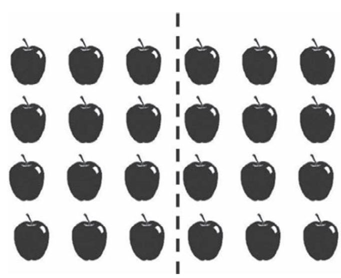


Figure 3.11 Other students may visualize $\frac{1}{2}$ of the set of apples like this.

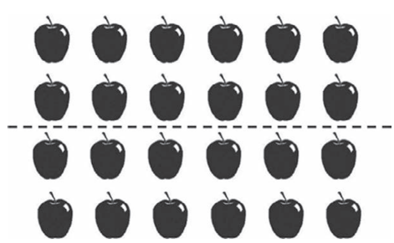


Figure 3.12 A student might visualize thirds like this.

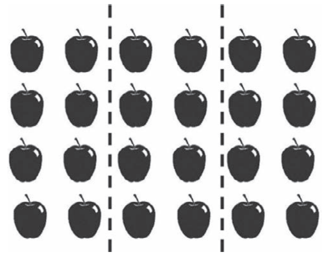
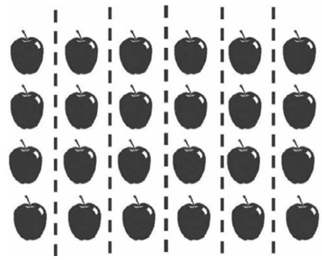
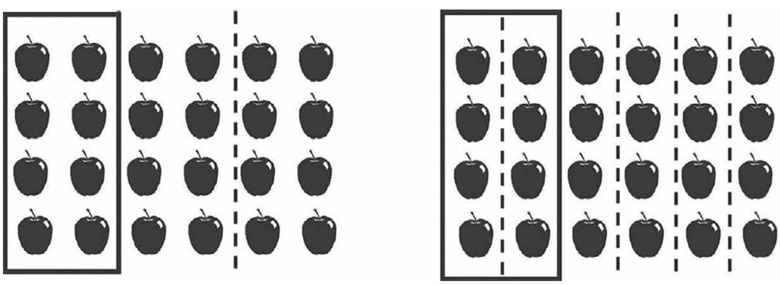


Figure 3.13 A student might visualize sixths like this.



Go-To See Chapter 7, Equivalence, for more on how to use models to develop understanding of equivalence.

Figure 3.14 One-third of the set of 24 apples is equivalent to $\frac{2}{6}$ of the set of 24 apples.



Another way to help students focus on the whole is to provide problems such as fractions of a square (Figure 3.15), in which students have to reinterpret the parts, which are not the same shape or area.

Figure 3.15 Fractions of a square.

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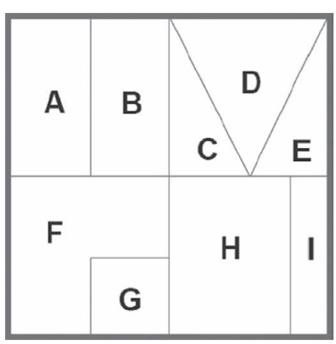
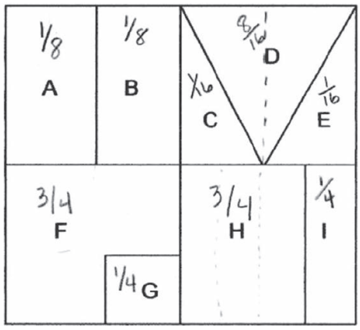


The large outer square represents one whole unit. It has been partitioned into pieces. Each piece is identified with a letter. What fractional part of the whole is each piece?

Matt's response (Figure 3.16) illustrates difficulties that students may encounter when they lose sight of the whole as they are solving the problem. Matt correctly wrote the fraction $\frac{1}{8}$ in part A.

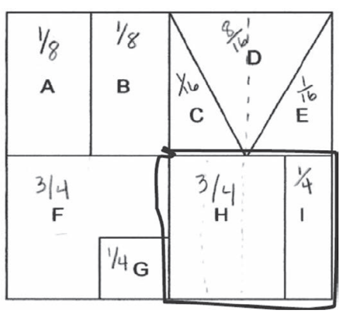
Figure 3.16 Matt's response to fractions of a square.



Matt incorrectly wrote the fraction $\frac{1}{4}$ in part I. Part I contains $\frac{1}{4}$ of the area of the square in the lower right corner that is comprised of parts H and I, but not $\frac{1}{4}$ of the large square.

How can we understand Matt's selection of fractions? When Matt was looking at sections A and B, he used the large square as the unit. Each of parts A and B contains $\frac{1}{8}$ of the area of that unit (the large square). When Matt was looking at sections H and I, however, he incorrectly focused on only a portion of the whole figure in Figure 3.17. It appears that Matt made the same error when he considered sections F and G.

Figure 3.17 The part of the whole square that Matt considered when determining the fractional parts H and I is circled.



For a closer look at Matt's response to this problem, and to consider possible next instructional steps for Matt, answer question 3 in Looking Back.

Considering the Size of the Whole When Comparing Fractions

An important concept necessary for solving problems involving fraction comparisons is to understand how the size of the whole impacts the fractions being considered. Some students make comparisons using models that do not reflect the whole in the given context (OGAP, 2005).

The vignette that follows illustrates the importance of considering the whole (specified or understood) when comparing fractions.

Candy Bar

Mr. Brown is a third grade teacher. At the start of his fraction unit, he always does the following activity. He comes to class with a paper bag filled with candy bars. He then tells his students, "I have candy bars in this bag. Who would like $\frac{1}{2}$ of a candy bar and who would like a whole candy bar?"

Every year that Mr. Brown has done this activity, almost all the students want a whole candy bar. The few students who don't like candy often ask for $\frac{1}{2}$ of a candy bar. Mr. Brown then hands out mini candy bars to the students who want a whole candy bar and $\frac{1}{2}$ of a large candy bar to those who only wanted $\frac{1}{2}$. Naturally, none of the students are happy!

Mr. Brown's activity makes clear that the size of the whole is critical when determining a fraction of a whole. In this case, the size of $\frac{1}{2}$ of the candy bar is dependent on the size of the whole candy bar.

The problem in Figure 3.18 is an example in which the size of the whole (i.e., the number of students in each classroom) is not specified, but the context implies that they are not the same.

Figure 3.18 Students who ride the bus to school.

Three-fifths of Mrs. Smith's students ride the bus to school.

One-half of Mr. Taylor's students ride the bus to school.

Explain how it could be possible that Mr. Taylor has more students ride the bus to school even though Mrs. Smith has a greater fractional part of her students ride the bus to school?

The evidence in Toni's and Samantha's responses (Figures 3.19 and 3.20) for this problem suggests that they understand that the two wholes being considered (the number of students in Mrs. Smith's class and the number of students in Mr. Taylor's class) are different sizes. To solve the problem, they each specified wholes that would prove their case.

Figure 3.19 Toni's response. Toni used a visual set model to show that $\frac{3}{5}$ of 20 students (12 students) is less than $\frac{1}{2}$ of 26 students (13 students).

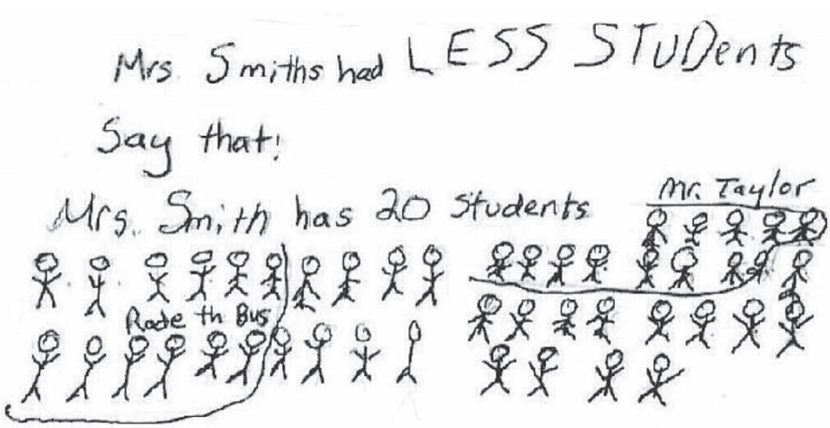
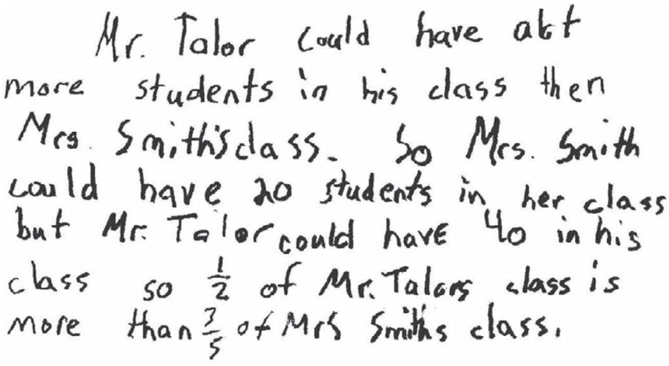
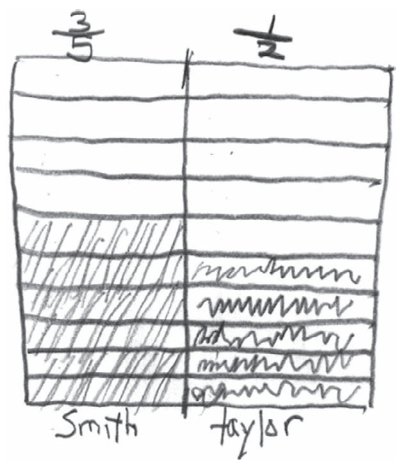


Figure 3.20 Samantha's response. Samantha did not construct a visual model, rather she described a situation in which Mr. Taylor had more students ride the bus to school.



Toni's and Samantha's solutions both reflected the situation. Contrast their solutions with Jayden's and Bill's solutions in Figures 3.21 and 3.22. Jayden solved the same problem as Samantha and Toni. However, Jayden's model that includes the same size and same number of parts for each class does not accurately reflect the situation.

Figure 3.21 Jayden's response. Jayden used visual area models to compare students who ride the bus to school in both classrooms. However, the models and number of parts in the whole are equal while the context implies that they are not equal.



Bill's response (Figure 3.22) to a problem in which the understood wholes are the same is an example of a common problem found when students use models to compare fractions. In his solution, he uses different-sized wholes when solving a problem in which the wholes should be the same size.

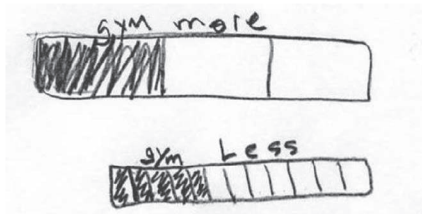
To analyze student work and consider instructional implications relative to the explicit or understood whole, complete questions 4 and 5 in Looking Back.

Figure 3.22 Bill's response. Although Bill's visual models represent $\frac{5}{12}$ and $\frac{1}{3}$, they are different sizes, leading him to incorrectly conclude that $\frac{1}{3}$ of the gym is greater than $\frac{5}{12}$ of the same gym.

$\frac{5}{12}$ of the gym was used for a kickball game.

$\frac{1}{3}$ of the same gym was used for a football game.

Which game used more of the gym?



Given the Part, What Is the Whole?

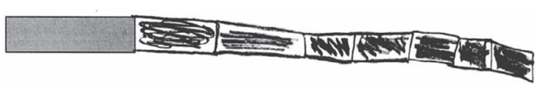
Some students also have a difficult time determining the whole when they are given just a part of the whole (Behr & Post, 1992), particularly when dealing with fractions that are NOT unit fractions (OGAP, 2005). Examples of student work that address this research are shown next.

Bob's responses in Figure 3.23 provide an example of a solution in which a student was successful finding the whole when given a part with a unit fraction, but not when given a non-unit fraction.

Figure 3.23 Bob's response. Bob appears to have applied the same strategy to both problems by making the number of parts in the whole equal to the magnitude of the denominator.

(A) This is $\frac{1}{5}$ of a candy bar. Draw the whole candy bar.

(B) This is $\frac{7}{8}$ of another candy bar. Draw the whole candy bar.



Although Bob's diagram for part A of the problem in Figure 3.23 is "not pretty," it does show the relationship between $\frac{1}{5}$ and the whole. However, it appears that Bob applied the same strategy to both problems, treating part B as if the problem pictured $\frac{1}{8}$ of a candy bar, not $\frac{7}{8}$ of the candy bar. It is unclear, therefore, if Bob even understands the concept for $\frac{1}{5}$. In each case he may have drawn the number of parts so that the total number of parts is equal to the magnitude of the denominator.

Contrast Bob's response to Beth's response in Figures 3.24 and 3.25. Beth's responses provide evidence that, in this situation, she is able to find a whole when given a part.

Unlike Bob's response to part B of this problem, Beth partitioned (divided) the given part into seven equal-sized pieces and then added one more piece (equal to the size of one of the seven pieces) to make a whole.

Figure 3.24 Beth's response. Beth's visual model shows the relationship between $\frac{1}{5}$ of the candy bar and the whole candy bar.

(A) This is $\frac{1}{5}$ of a candy bar. Draw the whole candy bar.

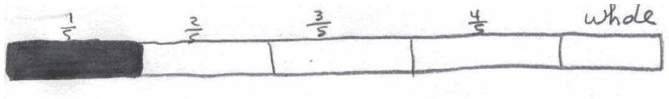
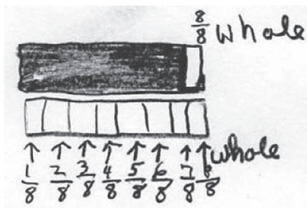


Figure 3.25 Beth's response. Beth successfully found the whole when given a non-unit fraction $\left(\frac{7}{8}\right)$.

(B) This is $\frac{7}{8}$ of another candy bar. Draw the whole candy bar.



To consider the next instructional steps for Bob as he deepens his understanding of identifying the whole when given a part, answer question 2 in Looking Back.

Chapter Summary

This chapter focused on the importance of the whole in developing fraction concepts. Students should solve a range of problems in which they have to carefully consider the meaning of the fraction in terms of the whole.

Students should encounter problems in which they:

- find fractional parts of wholes with multiple parts
- compare fractions involving different-sized wholes
- compare fractions involving the same size wholes
- find a whole when given a part

Looking Back

1. Explain how the lesson learned from the candy bar vignette was applied by the students in the following vignette.

   A group of fourth-grade students compared $\frac{5}{8}$ and $\frac{2}{3}$. As the students were presenting and discussing their solutions, one student said that it didn't really matter which was bigger because $\frac{2}{3}$ is only $\frac{1}{24}$ bigger than $\frac{5}{8}$, and that wasn't very big. Another student immediately piped up and said that it depends upon the size of the whole. If the whole is really big, than $\frac{1}{24}$ could be really big (OGAP, 2005).

2. It was suggested that Bob (Figure 3.23) may have used the same strategy to solve parts A and B of the candy bar problem. In both cases, Bob added the number of pieces that resulted in a candy bar with the total number of pieces equal to the denominator of the fraction given. Although this method resulted in a correct response to part A, the

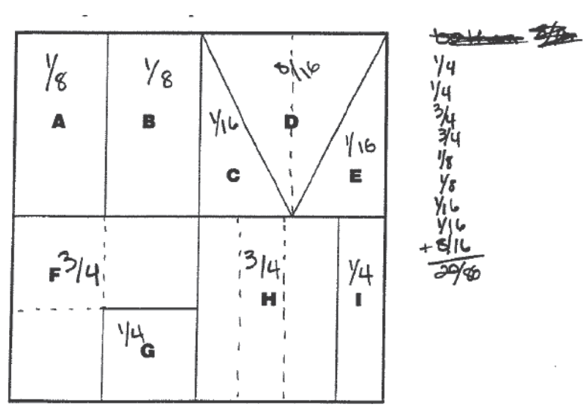
question remains, did Bob use inappropriate whole number reasoning to solve both questions?

What questions might you ask to determine if Bob is using inappropriate whole number reasoning and to help Bob deepen his understanding of finding the whole when given a fractional part?

3. Earlier in the chapter we examined part of Matt's work on the Fractions of a Square problem. Shown in Figure 3.26 is all of Matt's written work, in which Matt indicated that the sum of all the parts is $\frac{20}{80}$. Matt's teacher asked him to describe the part of his solution., where he explained, "Twenty-eightieths. 20 + 80 = 100, that's the whole!"

Figure 3.26 Matt's response. Fractions of a square.

The large outer square represents one whole unit. It has been partitioned into pieces. Each piece is identified with a letter. What fractional piece of the whole is each piece? Write that fraction on the piece.



Use the evidence in the student work (Figure 3.26) to answer the following questions.

a. What understandings of fractional parts of an area model are evidenced in Matt's response? Describe the evidence.
b. What errors are evidenced in Matt's response? Describe the evidence.
c. What potential questions might you ask Matt that would help him focus on identifying the whole?
d. What potential questions might you ask to help Matt rethink his conclusion that 20 + 80 = 100, and that is the whole? Provide a rationale for each question.

4. Read Tiara's and Maggie's responses to the Candies in a Dish problem in Figures 3.27 and 3.28. Although both students successfully answered the question, they each used a different strategy. Explain how an understanding of the whole is reflected in each of their solutions.

5. Review Jayden's work in Figure 3.29 and answer the following questions.

a. Jayden's solution does not reflect the situation. What is the evidence?
b. What are some questions that you could ask that may help Jayden rethink her solution?

Figure 3.27 Tiara's response.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies?

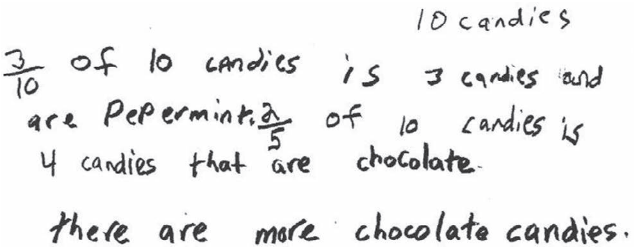


Figure 3.28 Maggie's response.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

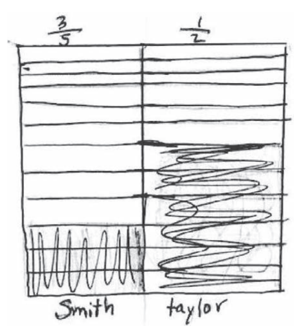
Are there more chocolate candies or more peppermint candies?

Figure 3.29 Jayden's response.

$\frac{3}{5}$ of Mrs. Smith's students ride the bus to school.

$\frac{1}{2}$ of Mr. Taylor's students ride the bus to school.

Explain how it could be possible that Mr. Taylor has more students ride the bus to school even though Mrs. Smith has a greater fractional part of her students ride the bus to school?



Instructional Link: Your Turn

Use the questions in Table 3.1 to help you think about how your mathematics program provides students with opportunities to consider the *whole* and to develop understandings about the impact that the whole has on determining a fractional part of a whole.

Describe any adjustments you need to make in your unit and lesson plans to ensure that the research from this chapter is addressed in your instruction.

Table 3.1 Instructional Link—Strategies to Support the Development of Solving Problems Involving Understanding of the Whole in a Range of Situations

Do you (or does your program) provide opportunities for students to:	*Yes/No*
1. Solve problems involving fractional parts of a whole?	
2. Solve problems that involve finding the fractional part of a whole that is partitioned into more than one part?	
3. Find the whole when given a part?	
4. Solve problems that involve comparing fractions that relate to the same size wholes?	
5. Solve problems that involve comparing fractions that relate to different size wholes?	
6. Construct models that reflect the size of the whole in various fraction problems and contexts?	

4
Equipartitioning

Big Idea

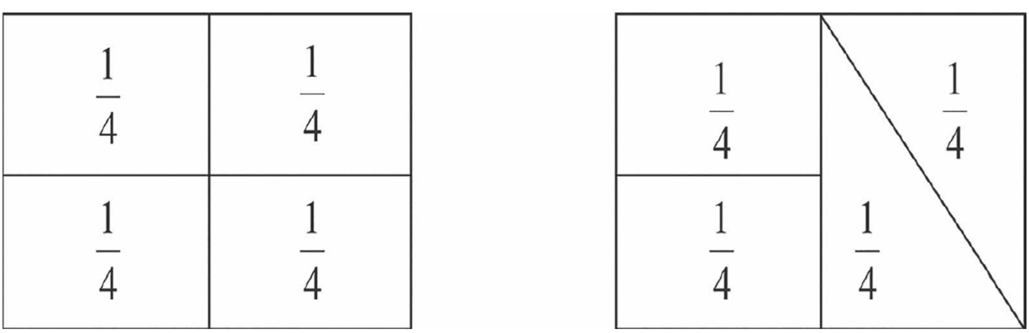
Equipartitioning is central to understanding and generalizing concepts related to fractions, such as:

- fair shares
- unit fractions
- ordering and comparing fractions
- equivalence
- density of fractions
- operating with fractions

What Is Equipartitioning?

Equipartitioning is the act of dividing into equal-sized groups or equal-sized parts. "Fractions have their roots in equipartitioning" (Confrey et al., 2011). The action of equipartitioning is key to using visual models to develop and generalize fraction concepts. Equipartitioning any visual model will separate the models into sections that do not overlap each other. When each section represents the same fractional part of the whole, the sections will all be the same size. To understand these concepts, we will explore examples using an area model (Figure 4.1), a set of objects (Figure 4.2), and a number line (Figure 4.3).

Figure 4.1 Each of these rectangles is partitioned into sections that do not overlap. Within each rectangle, each section has the same area representing $\frac{1}{4}$ of the whole region.



DOI: 10.4324/9781003185475-4

Figure 4.2 This set of eight dinosaurs has been partitioned (divided) into fourths (four groups of an equal count of dinosaurs). Each group of two dinosaurs is $\frac{1}{4}$ of the set of eight dinosaurs.

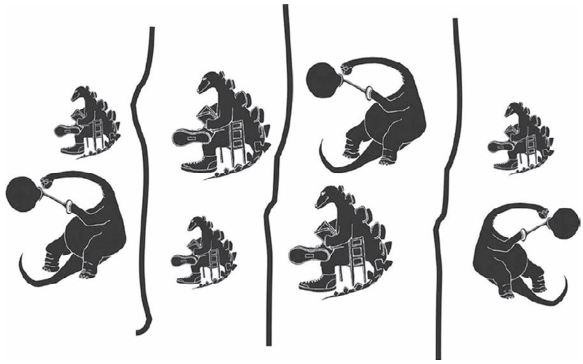
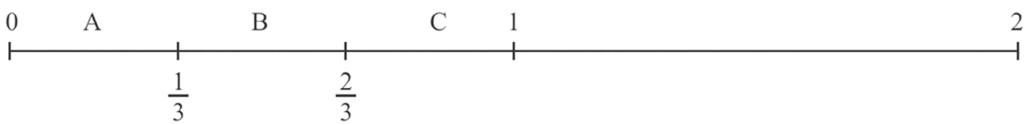


Figure 4.3 In this number line, the whole is taken to be the section of the number line from 0 to 1. This whole has been partitioned into three parts (A, B, and C), each of which is the same length.



The Importance of Equipartitioning in Developing Fraction Concepts

Equipartitioning is a "fundamental mechanism for building up fraction concepts" (Lamon, 1999, p. 77) and is key to understanding and generalizing concepts related to fractions, such as:

- identifying fair shares
- identifying fractional parts of a region
- identifying fractional parts of sets of objects
- comparing and ordering fractions
- locating fractions on number lines
- understanding the density of rational numbers
- evaluating whether two fractions are equivalent or finding equivalent fractions
- operating with fractions
- measuring

(Lamon, 1999).

Some researchers indicate that "early experiences with physically partitioning objects or sets of objects may be as important to a child's development of fraction concepts as counting is to their development of whole number concepts" (Behr & Post, 1992, p. 14). The goal, however, according to the research, is for students to use early experiences with physically partitioning wholes to understand the impact of partitioning as they solve problems and generalize fraction concepts. Later, "just imagining the impact of partitioning will suffice and ultimately be desirable" (Behr & Post, 1992).

Students can develop an understanding of the properties of fractions by equipartitioning manipulatives or student-created visual models. At a higher level of understanding, students are able to recognize how equipartitioning applies in solving a problem even if the model and its partition are not explicit in the students' responses.

CCSSM

The CCSSM and Equipartitioning

Students in grades 1 through 3 use equipartitioning to develop foundational understanding of fraction concepts in the CCSSM.

At grades 1 and 2, students engage in sharing activities and problems that involve partitioning rectangles and circles into two, three, and four equal shares. At grade 1, students use the words halves, fourths, and thirds and phrases half of, fourth of, and quarter of to describe the equal shares. Importantly, students use their observations about the impact of equipartitioning to notice that partitioning into more equal shares results in smaller pieces.

At grade 2, students build on the descriptions of the shares used at grade 1 and describe the whole as two-halves, three-thirds, and four-fourths. They recognize that in terms of area, equal shares of the same whole do not have to have the same shape but do have to be the same size.

At grade 3, students use unit fractions to name equally partitioned parts that make up a whole. Through work with visual area models and number lines, students learn to build fractions from unit fractions by "interpreting $\frac{a}{b}$ as *a* copies of $\frac{1}{b}$" (Small, 2014, p. 17). This supports development of understanding of fractions as numbers, equivalent fractions, and comparing and ordering fractions.

Equipartitioning of visual models (including sets of objects, area models, and number lines) to build understanding of fraction concepts, to solve fraction problems, and to justify solutions are explicitly stated in the CCSSM through seventh grade as students work with a range of fraction and rational number concepts.

Given the requirements in the CCSSM for equipartitioning visual models, it is important to understand some challenges that arise as students partition visual area models into equal parts.

Understanding Equal Parts

Researchers indicate that some students, when using or interpreting an area model to represent a fraction, do not consider the sizes (areas) of the different parts that result from the partitioning. Instead, the students consider just the number of parts (Bezuk & Bieck, 1993).

The vignette that follows highlights another misunderstanding students have when they partition regions: that partitions must result in pieces that are both the same size and the same shape. Students have difficulty recognizing fractional parts as equal in size if the pieces are not congruent (same size and shape) (Bezuk & Bieck, 1993).

Is $\frac{1}{4}$ of the Square Shaded?

When Mrs. Armstrong started a unit on fractions with her third-grade students, she introduced the first lesson with the problem shown in Figure 4.4. She learned from the OGAP (2005) study that only 30% of third-grade students ($n = 127$) correctly responded to this problem. She knew her students well because she taught them in second grade the previous year. With that second-grade class, she had paid careful attention to her lessons on fractions, giving her students many experiences in which they partitioned regions into equal-sized parts. She was confident that her students would be more successful than the students in the OGAP study.

Only 7 of her 19 students answered the question correctly. She was shocked. Many of her students shared a misinterpretation of the fractional part of an area model even after she had devoted significant instructional time to reinforcing the concept of equal-sized parts when the students were in second grade.

Correct responses to her problem looked much like Maria's, shown in Figure 4.5. These responses pleased Mrs. Armstrong and were what she had expected from all her students. It was a surprise to her that the majority of her students' solutions were incorrect. William's response, shown in Figure 4.6, was typical of these. He and many others considered only the number of parts and not the size of the parts.

Figure 4.4 Problem: Is $\frac{1}{4}$ of the square shaded?

Figure 4.5 Maria's response. Maria appropriately considered the size of the pieces.

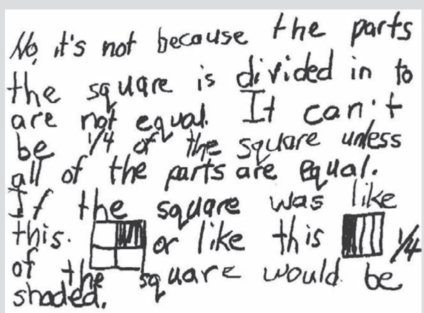
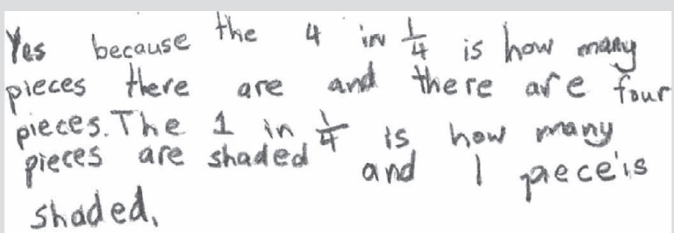
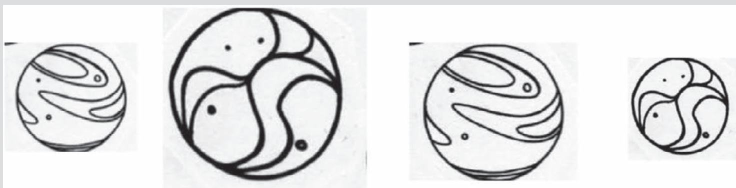


Figure 4.6 William's response. William inappropriately considered only the number of pieces.



Mrs. Armstrong decided to probe a little deeper so that she could understand what her students were thinking. She asked William about his answer. William took four marbles out of his desk to help him explain his thinking. He said that even though the marbles were not the same size, one of these marbles was $\frac{1}{4}$ of his set of four marbles, and the shaded portion of the square is also one piece out of four pieces, so it is $\frac{1}{4}$ of the square (see Figure 4.7).

Figure 4.7 William used these four marbles to explain his thinking. William thought that finding $\frac{1}{4}$ of an area is the same as finding $\frac{1}{4}$ of this set of marbles.



Mrs. Armstrong was encouraged that her question to William had prompted a clear explanation of his thinking. His response suggested that some of her students incorrectly used a feature of the fractional part of a set of objects with an area model.

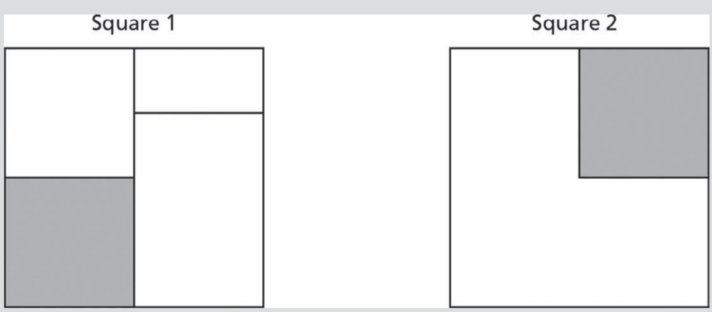
This interaction with William raised Mrs. Armstrong's curiosity about Maria's understanding. The two area visual models that Maria included in her response clearly showed $\frac{1}{4}$ of the squares had been shaded. Moreover, the squares were fit into a statement that is correct: "If the square was like (either of the partitioned and shaded squares), $\frac{1}{4}$ of the square would be shaded." That statement is clearly written.

But now Mrs. Armstrong looked back at Maria's previous statement that "It can't be $\frac{1}{4}$ of the square unless all of the parts are equal." It appeared to Mrs. Armstrong that Maria might believe that the partitions had to result in parts of the same size and the same shape (that is, all of the parts needed to be congruent) in order to contain a part representing the fraction $\frac{1}{4}$.

Mrs. Armstrong then asked Maria if both of the squares shown in Figure 4.8 had $\frac{1}{4}$ of the square shaded.

Even though $\frac{1}{4}$ of each of the figures is shaded, Maria said, "No, $\frac{1}{4}$ is not shaded because square 1 is not divided into four equal parts, and square 2 is divided into two unequal parts, not four parts." Mrs. Armstrong was surprised to find that Maria did not believe that $\frac{1}{4}$ of each of the regions was shaded.

Figure 4.8 Is $\frac{1}{4}$ of each square shaded?

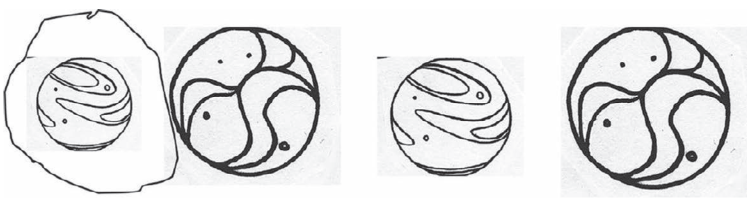


However, Maria's and William's levels of understanding helped her to reflect on her instruction. While Mrs. Armstrong emphasized the importance of partitioning regions into equal-sized parts last year, the students had never experienced questions such as those in Figure 4.4 and 4.8.

She realized that last year the only visual models that students saw or drew were ones in which all the pieces were the same size and the same shape. This year she decided to be more intentional about providing her students with experiences that challenge their understandings of the meaning of equal-sized parts.

This vignette raises some important issues as students are introduced to and begin to use equipartitioning to solve problems involving finding the fractional part of a whole. Like William, students may be inappropriately applying their understanding of partitioning sets to partitioning an area model. For example, Mrs. Armstrong realized that William appears to understand that the objects in a set did not have to be the same size (different-sized marbles) but had to have the same number of marbles in each group (see Figure 4.9).

Figure 4.9 One-fourth of the marbles is circled, even though the marbles are not the same size.



Mrs. Armstrong realized that some students, like Maria, may be taking a literal application of equal-sized parts to mean that the parts always need to be the same size and the same shape. As teachers reinforce the concept of equal-sized parts, they need to deepen students' understanding by providing examples and counterexamples as well as developing other instructional strategies. There are many strategies for teachers to choose from that reinforce the idea that fractional parts in an area model need to be the same area, but not the same shape, and that help students move away from thinking about a fractional part of an area in the same way they think about fractional parts of sets of objects. One such activity asks students to come up with as many different ways as they can to show $\frac{1}{2}$ of a geoboard or $\frac{1}{2}$ of a square, as shown in Figure 4.10.

An important aspect of this activity is having students share their solutions as well as probing their understanding with questions such as the following:

- What fractional part of each shape is shaded? (Don't assume that all students see these as representing $\frac{1}{2}$ of each shape.)
- What did you notice about the shading on all your sketches? (For example, the shading covers the same area, but is not necessarily the same shape.)
- How did you decide what to shade? (For example, counting the total number of boxes and shading a number equal to half of the boxes.)

The examples cited here directly relate to solving problems on finding the fractional part of a region (e.g., shade $\frac{1}{2}$ of a square). This concept is important as students experience real-world situations that often involve finding the fractional part of an irregular area. For example, 20 acres divided equally among four people means that all four people get five acres. It does not necessarily mean that the shape of each five-acre parcel is the same (see Figure 4.11).

Figure 4.10 Even though the shading is not the same shape, $\frac{1}{2}$ of each of these figures is shaded.

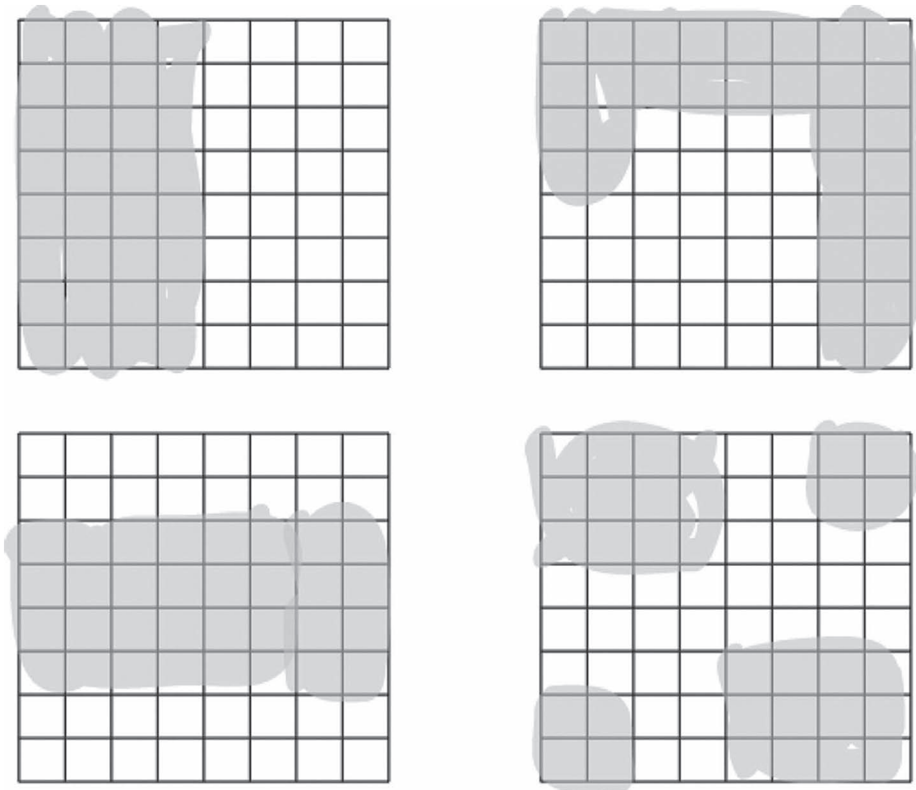
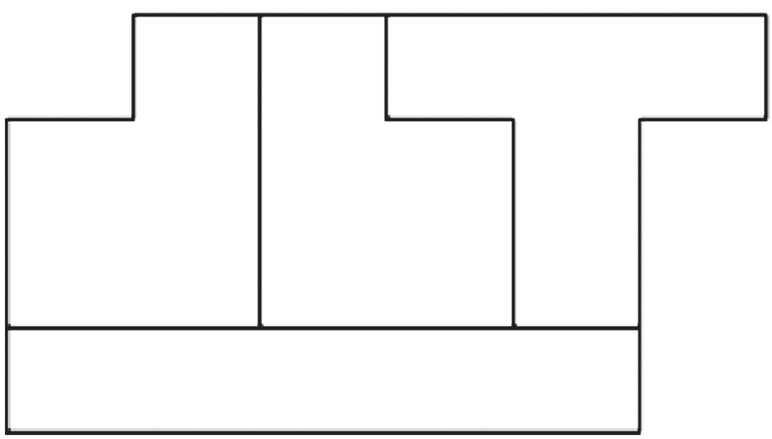


Figure 4.11 Each person gets $\frac{1}{4}$ of the 20 acres (or 5 acres). Even though the pieces are not the same shape, they have the same area.



Using Equipartitioning and the Impact of Equipartitioning in Solving Fraction Problems and Developing Understanding of Fraction Concepts

In the previous section we focused on an important concept about area visual models: two parts can represent the same fraction even though the parts have different shapes. What matters is that the two parts must have the same size (area). In this section we will focus on the use of equipartitioning to:

- find fair shares
- compare and order fractions
- develop understanding of equivalent fractions
- develop an understanding of the density of rational numbers
- operate with fractions

To effectively use visual models to solve problems involving these mathematical topics, it becomes important to partition regions into the same size and shape. As you read through the student responses on the pages that follow, think about the following questions.

1. What attention are students paying to equipartitioning regions, lines, or sets of objects into parts that have equal sizes?
2. How are students using their understanding of equipartitioning into equal-sized parts to solve the problems?
3. How are students using their understanding of the impact of equipartitioning to solve the problems?

Equipartitioning and Fair Shares

Students in the early grades use equipartitioning to divide objects into fair shares, as in Katie's response in Figure 4.12. When first determining fair shares, most students need to physically partition the region, as in Katie's response. Over time and with enough experience, it is sufficient and desirable for students to visualize the impact of the equipartitioning. This can be seen in Thomas's response in Figure 4.13.

Figure 4.12 Katie's response. Katie physically partitioned the pizzas into (approximately) equal-sized parts. Katie did not, however, respond with a fraction to indicate how much pizza each child got.

Four children share 2 pizzas equally.

What fraction of a pizza does each child get?

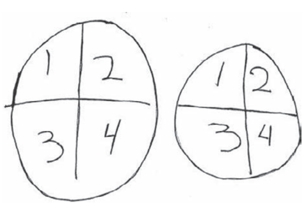
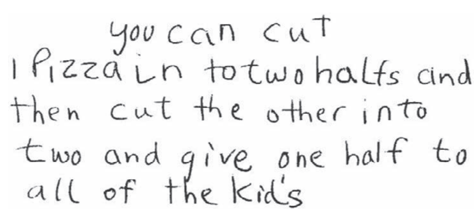


Figure 4.13 Thomas's response. Thomas used an understanding of the impact of partitioning by explaining how you can cut each pizza in half without having to draw each pizza. Using this strategy, he was able to determine that each student gets $\frac{1}{2}$ of a pizza.

Four children share two pizzas equally.

What fraction of a pizza does each child get?



To explore the concept of developing students' abilities to use equipartitioning to solve problems involving fair shares, answer question 2 in Looking Back.

Equipartitioning to Compare Fractions

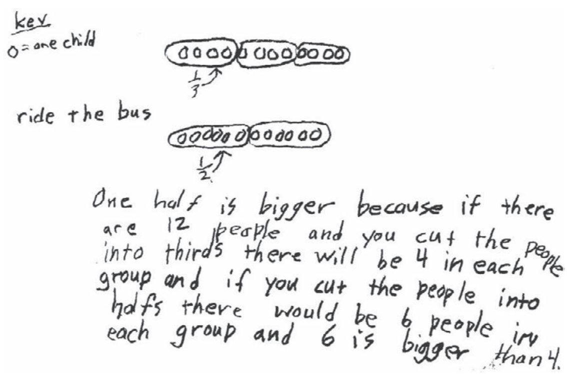
Students also use equipartitioning strategies to compare the magnitude of fractions. In Tom's response in Figure 4.14, he used a set of 12 objects to represent the class. Using two copies of this set of circles, he partitioned one copy into thirds, and he partitioned the other copy into halves. His two sets of objects show $\frac{1}{3}$ of the set containing four children and $\frac{1}{2}$ of the set containing six children. His answer is that $\frac{1}{2}$ (of the class) is bigger, and his explanation is based on counting the pieces in his partitions and saying that "6 is bigger than 4."

Figure 4.14 Tom's response. Tom understood that the number of students in the class was the same. He chose a set of 12 objects to represent the students in the class. He then physically partitioned the 12 objects into halves and thirds.

$\frac{1}{3}$ of the students in Joe's class walk to school.

$\frac{1}{2}$ of the students in Joe's class ride the bus.

Do more students walk to school or ride the bus?



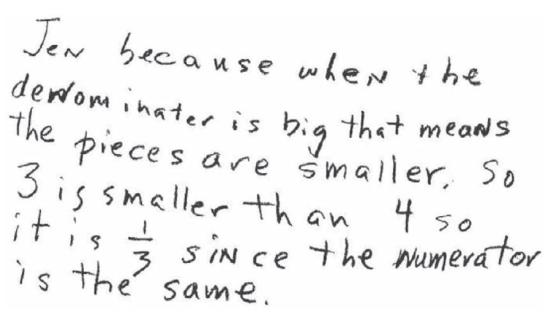
Later, students will use their understanding of the impact of equipartitioning on the size of the equal parts in the whole to compare and order fractions, as Mike did in his response in Figure 4.15.

Figure 4.15 Mike's response. Mike used his knowledge of the impact of partitioning.

Linda hiked one-fourth of the way up Mount Mansfield.

Jen hiked one-third of the way up Mount Mansfield.

Who hiked farther?



See Chapter 8, Comparing and Ordering, for a more in-depth discussion on comparing fractions.

Equipartitioning and Equivalent Fractions

The use of equipartitioning is at the foundation of using visual models to understand the procedure for determining equivalent fractions. See Chapter 7, Equivalence, for more information on this topic.

Equipartitioning and Density of Rational Numbers

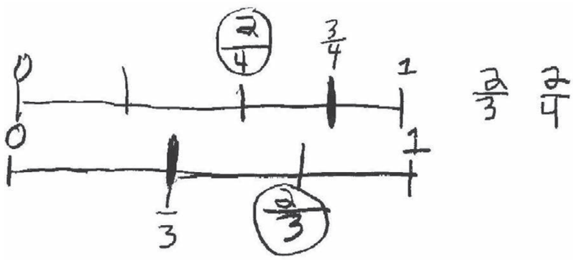
Students have a difficult time understanding the density of rational numbers. That is, between any two rational numbers there are an infinite number of rational numbers (Orton et al., 1995). Experience with equipartitioning and repartitioning can help students visualize the concept of density. Review Chris's response in Figure 4.16. You will notice that Chris used number lines to identify two fractions $\left(\frac{2}{3} \text{ and } \frac{2}{4}\right)$ that are located between $\frac{1}{3}$ and $\frac{3}{4}$. In part B, Chris outlined a strategy for finding other fractions between $\frac{1}{3}$ and $\frac{3}{4}$. His response indicates that you can find more fractions between $\frac{1}{3}$ and $\frac{3}{4}$ "if you partition it more."

Chris's response displays a comfort with equipartitioning and with equipartitioning as a way to find fractions between two given fractions.

See Chapter 10, The Density of Fractions, for a more in-depth discussion on developing student understanding of the density of fractions.

Figure 4.16 Chris's response. Chris physically partitioned two number lines in order to identify two fractions between $\frac{1}{3}$ and $\frac{3}{4}$, and he showed his understanding of partitioning in B by indicating that "if you partition it more" you can find more fractions.

A) Name two fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.



B) Do you think that there are other fractions besides the two that you named between $\frac{1}{3}$ and $\frac{3}{4}$? Explain why or why not.

There are more in between 1/3 and 3/4ths because
if you partition it more you get 8ths and 6ths at
and that would still — count as a fraction inbetween
1/3 and 3/4ths.

Equipartitioning and Operating with Fractions

Max (Figure 4.17) used equipartitioning strategies to explain his solution when subtracting $\frac{2}{5}$ from $\frac{4}{5}$. Kasey (Figure 4.18), on the other hand, was visualizing the relative magnitude of $\frac{1}{12}$ and $\frac{1}{8}$ when estimating their sum.

Figure 4.17 Max's response. Max physically partitioned a number line from 0 to 1. While one does not know for sure, Max may have used the number line to subtract the two fractions or to determine that the difference $\frac{2}{5}$, was closest to 0.

$\frac{4}{5} - \frac{2}{5}$ is closest to:

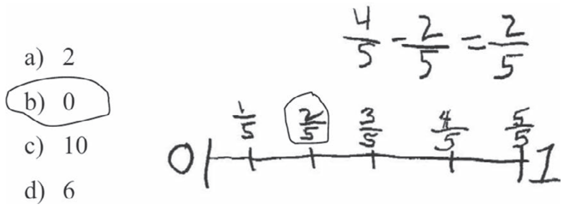
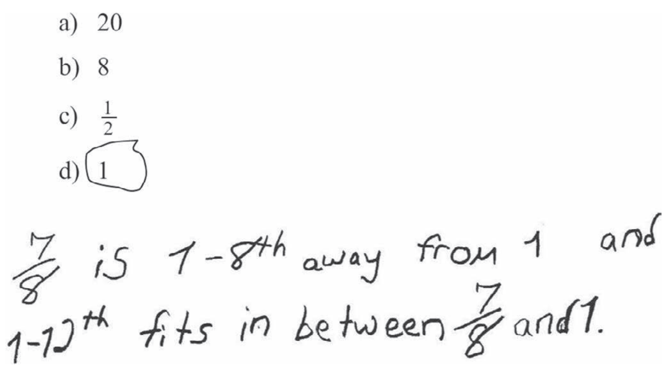


Figure 4.18 Kasey's response. Kasey used number sense that may have resulted from an understanding of the impact of partitioning.

The sum of $\frac{1}{12} + \frac{7}{8}$ is closest to:



See Chapter 11, Addition and Subtraction of Fractions, and Chapter 12, Multiplication and Division of Fractions, for more in-depth discussions on developing student understanding of operating with fractions.

Stages of Partitioning

Because physically equipartitioning visual models is a foundational activity for students as they develop an understanding of fractions and generalize ideas, understanding how students develop partitioning skills is important. This section focuses on the stages in which students develop their equipartitioning skills. Understanding these stages helps to provide guidance about the possible difficulties that students might have when using partitioning to solve problems involving fractions with different denominators. The following vignette provides a context for why understanding the stages of partitioning is important for teachers.

Mrs. Murray, a fourth-grade teacher, gave her students a pre-assessment prior to beginning a unit on fractions. She hoped to gain an insight into her students' understanding of foundational fraction concepts. She planned to use this information to inform the design and implementation of her upcoming fraction unit.

She analyzed her students' responses to a question that asked them to place $\frac{1}{3}$ and $\frac{1}{4}$ on a 0 to 1 number line. In general, Mrs. Murray found that her students were more successful in locating and justifying the location of $\frac{1}{4}$ on the number line and less successful locating $\frac{1}{3}$. Mrs. Murray was surprised by this and wondered why her students would display a different level of understanding when placing these two fractions on the number line.

Mrs. Murray's findings are not surprising and are related to the development of equipartitioning strategies. Researchers suggest that students progress through stages of equipartitioning that include:

(a) sharing
(b) algorithmic halving
(c) evenness
(d) oddness and
(e) composition

(Pothier & Sawada, 1983)

These stages are described more fully in the following sections.

Sharing

Most students first explore equipartitioning through sharing activities. According to research, students who successfully use a sharing strategy are able to partition a whole into two equal parts (Pothier & Sawada, 1983). Rhonda's work (Figure 4.19) exemplifies this sharing strategy.

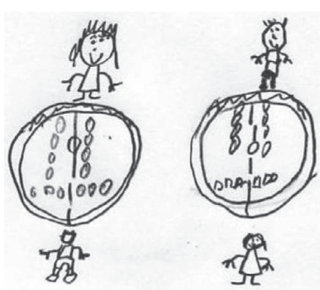
Algorithmic Halving

Students usually move easily from sharing to algorithmic halving, which is the process of continuing the halving process to obtain fourths, eighths, sixteenths, and so on (Pothier & Sawada, 1983). Fraction strips are used in Figure 4.20 as examples of the impact of algorithmic halving. Each fractional piece, starting with the whole strip, is halved to create the next smaller piece.

Figure 4.19 Rhonda's response. Rhonda partitioned each pizza into two halves using a sharing strategy.

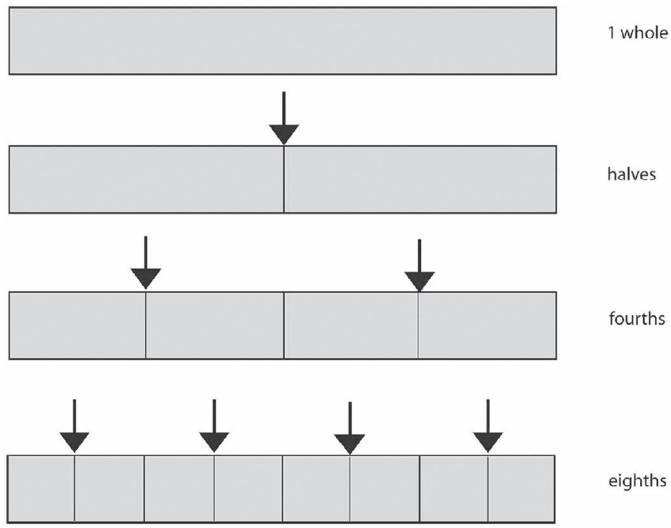
Four children are sharing two pizzas equally.

What fraction of a pizza does each child get?



Partitioning visual models into equal parts that are powers of two (i.e., fractions with denominators of 2, 4, 8, 16, 32, etc.) is easier than equipartitioning that involves odd numbers or even numbers that have odd number factors (Pothier & Sawada, 1983). This research suggests that students should be introduced to equipartitioning with fractions whose denominators are powers of two $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \text{etc.}\right)$.

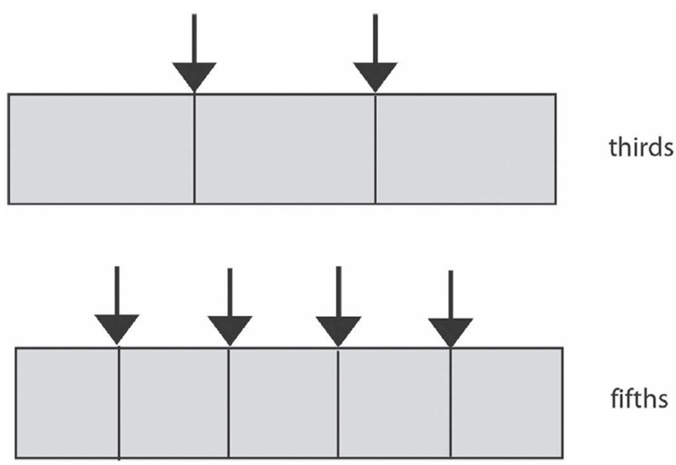
Figure 4.20 Fractions strips partitioned using algorithmic halving. First the strip is partitioned into two halves. Then each half is partitioned into halves, making fourths. Then each fourth is halved, making eighths.



Evenness and Oddness

Moving from equipartitioning that involves even numbers that are powers of two to other numbers offers students a number of challenges. Studies have shown that students have a more difficult time partitioning a whole into equal parts that are odd numbers (3, 5, 7, etc.) (Figure 4.21) and even numbers with odd number factors ($6 = 2 \times 3$, $10 = 2 \times 5$, $12 = 2 \times 2 \times 3$, etc.) than partitioning using algorithmic halving strategies (Pothier & Sawada, 1983).

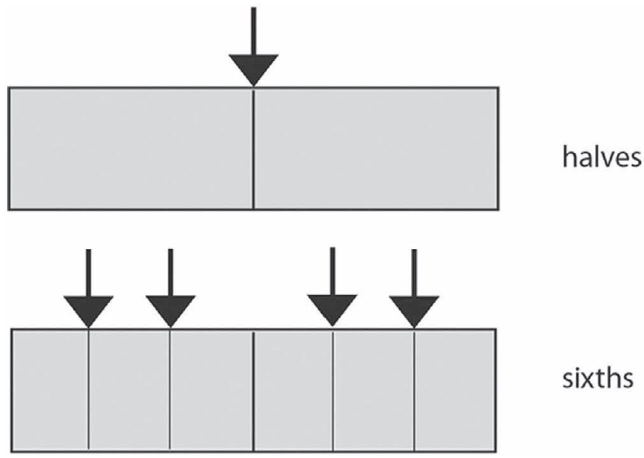
Figure 4.21 Oddness stage—partitioning into an odd number of equal parts involves thinking about the relative size of each part in relation to the whole.



Partitioning into even numbers with odd number factors (6, 10, 12, etc.) might involve a two-step process: first halving, and then partitioning into an odd number of parts (Figure 4.22). We use $6 = 2 \times 3$ to illustrate this point. One way students partition wholes into even numbers with odd number factors, such as 6, is to halve the whole first and then partition each half into thirds.

Other students partition first into thirds and then halve each third. Still others will estimate and partition directly into sixths.

Figure 4.22 Evenness stage—to partition into sixths one may first partition the whole in half then partition each half into thirds.



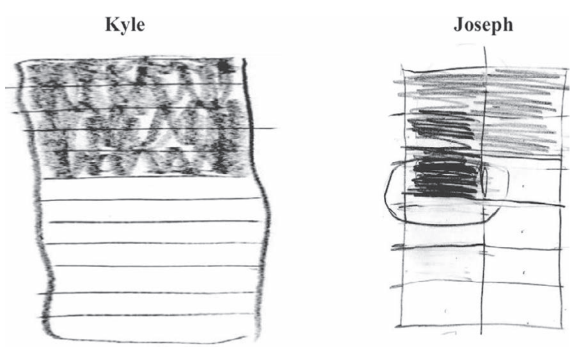
Composition

As students become flexible with the equipartitioning concepts previously described, and as students' multiplicative reasoning develops, they often use multiplicative strategies to partition wholes into equal-sized parts.

Kyle and Joseph each partitioned the regions in Figure 4.23 into twelfths. Joseph's model shows evidence of a multiplicative strategy to represent five-twelfths of the region.

To explore the concept of stages of equipartitioning, answer question 2 in Looking Back.

Figure 4.23 Kyle's and Joseph's responses. Joseph's strategy shows evidence of using a multiplicative strategy by partitioning the figure into six rows of two columns.



Instruction and Stages of Equipartitioning

Some teachers have asked if they should explicitly teach each strategy (e.g., "today we are going to do algorithmic halving"). Our answer is no. Rather than providing steps to teach equipartitioning, the stages suggest that teachers make intentional choices about which fractions they use to teach, reinforce, and strengthen concepts that can be built on understanding the impact of equipartitioning.

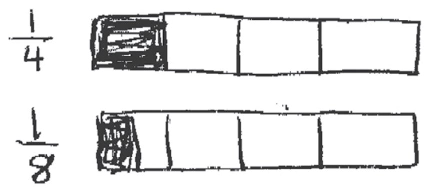
To do this, a teacher should provide students with experiences in which they equipartition a variety of visual models (regions, sets of objects, and number lines). Students should have experiences equipartitioning the visual models into a variety of fractional parts, starting first with even numbers that are powers of two (halves, fourths, eighths, sixteenths, etc.). To strengthen their equipartitioning skills, a teacher should have students share their strategies so that all students are exposed to a variety of ways of thinking. Over time and as they are ready, students will develop other strategies.

More importantly, students should start developing several generalizations from their experience with equipartitioning that will help them find fractional parts of the whole, compare and order fractions, develop concepts related to equivalence, and operate with fractions.

Visualizing and Generalizing the Impact of Equipartitioning

Teachers often ask, "How can I get my students to generalize concepts or use an understanding of the impact of equipartitioning?" One answer is to build from visual models that students use to solve problems. Another answer is to create an environment in which students are encouraged to look for patterns and relationships, ask questions, and make conjectures as Tom did in Figure 4.24.

Figure 4.24 Tom's response. Tom drew these visual area models to show $\frac{1}{4}$ and $\frac{1}{8}$.



Tom made the following observation: "I noticed that the larger the denominator, the smaller the part that I shaded so the smaller the fraction." He went on, "I wonder if it always works that if I make the denominator larger, I will get a smaller fraction?"

Building on this observation, Tom made the following conjecture: When making a drawing to show a fractional part of a whole, the larger the denominator, the smaller the piece that will be shaded.

His teacher capitalized on his observation by engaging the class in testing Tom's conjecture. The class generated a list of the kinds of examples that can be used to test Tom's conjecture (e.g., unit fractions, fractions with the same numerator but different denominators, fractions with the same denominators but different numerators) and then tested it with the different examples. After considerable exploration, they made more observations that led to modifying the conjecture.

Modified conjecture: When making a drawing to show a fractional part of a whole, the larger the denominator, the smaller the piece that is shaded and the smaller the fraction if the magnitude of the numerators is the same.

The teacher strengthened this understanding by providing the students with problems that involved comparing and ordering fractions. Read Mike's response in Figure 4.25. This type of response exemplifies responses after students start moving from depending on constructing a visual model when solving a problem to visualizing the impact of equipartitioning.

Figure 4.25 Mike's response. Mike used his knowledge of the impact of partitioning.

Linda hiked $\frac{1}{4}$ of the way up Mt. Mansfield.

Jen hiked $\frac{1}{3}$ of the way up Mt. Mansfield.

Who hiked farther?

Jen because when the denominater is big that means the pieces are smaller. So 3 is smaller than 4 so it is $\frac{1}{3}$ since the numerator is the same.

Ultimately, students should internalize their understanding of the impact of equipartitioning visual models and use unit fraction and benchmark reasoning to compare fractions as is evidenced in Ted's response in Figure 4.26. Ted extended his unit fraction reasoning to determine that $\frac{7}{6}$ is $\frac{1}{6}$ away from 1. He then determined the distance each of the other given fractions are from one and compared his findings to $\frac{1}{6}$.

Figure 4.26 Ted's response. Ted's response shows evidence of having internalized the impact of equipartitioning.

Which fraction is closest to 1? Show your work or thinking.

$\frac{7}{3}$ $\frac{7}{5}$ $\frac{7}{6}$ $\frac{7}{12}$

7/6 is 1/6 away from 1
7/12 is 5/12 away from one, 5/12 is larger then 1/6
7/5 is 2/5 larger then 1, 2/5 is larger then 1/6
7/3 = 2 1/3

Go-To See Chapter 7, Equivalence; Chapter 8, Comparing and Ordering; Chapter 10, The Density of Fractions; Chapter 11, Addition and Subtraction of Fractions; and Chapter 12, Multiplication and Division of Fractions, for additional opportunities to see how equipartitioning is applied across fraction concepts.

Chapter Summary

This chapter focused on how equipartitioning visual models helps students develop skills and understandings and how students use these understandings to generalize fraction concepts. Equipartitioning visual models is a means to understanding the mathematics. In the end, we want students to solve problems and generalize concepts based on what they have internalized through equipartitioning experiences.

Looking Back

1. Study Mandy's and Mark's responses in Figures 4.27 and 4.28 and then answer questions 1a, 1b, and 1c that follow.

Figure 4.27 Mandy's response. Mandy's strategy for placing $\frac{1}{3}$ and $\frac{1}{4}$ on a number line from 0 to 1.

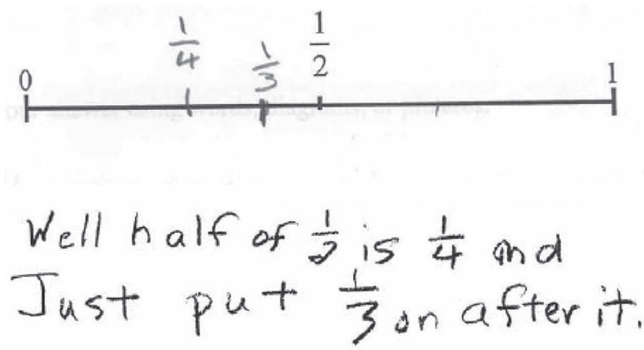
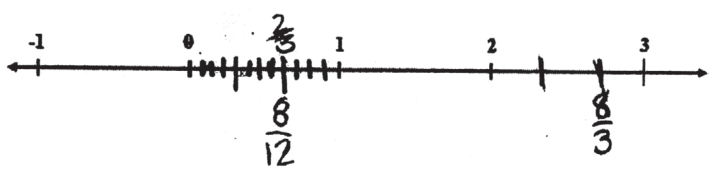


Figure 4.28 Mark's strategy for locating $\frac{2}{3}$, $\frac{8}{12}$, and $\frac{8}{3}$ on the number line.



a. What strategy does Mandy use to place $\frac{1}{4}$ on the number line? Does she use the same strategy to place $\frac{1}{3}$ on the number line? Explain, using evidence from Mandy's response.
b. What strategy does Mark use to place $\frac{2}{3}$, $\frac{8}{12}$, and $\frac{8}{3}$ on the number line? Explain, using evidence from Mark's work.
c. Because this is the only evidence that you have about each student's level of equipartitioning, what else might you want to know to determine the next instructional steps?

2. John (Figure 4.29) and Kim (Figure 4.30) answered different problems that involve dividing into "fair shares." Study their responses, and then answer the following questions.

Figure 4.29 John's response.

Twelve students are sharing four pizzas equally.
How much will each student get?

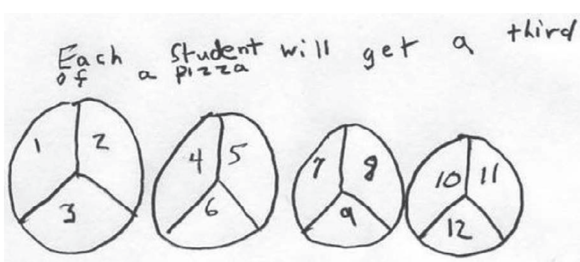
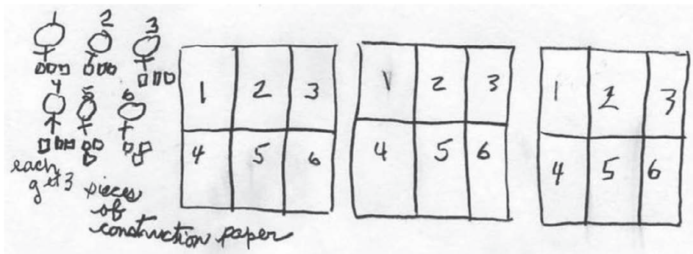


Figure 4.30 Kim's response.

Six students equally share three pieces of construction paper.

How much construction paper does each child get?



a. John and Kim both used equipartitioning in their solutions. How are their strategies different? Explain.
b. What activity or questions might help Kim partition each piece of paper into halves instead of sixths and to recognize that each student receives $\frac{1}{2}$ of a piece of construction paper?

3. Tom and Tiara were both asked questions about the ordering of fractions. Both of them chose to use set visual models in their responses. They each selected a specific number of objects in their sets, even though the numbers were not specified by the problem. Read Tom's response (in Figure 4.14) and Tiara's response (from Chapter 3, shown again in Figure 4.31), and answer the following questions.

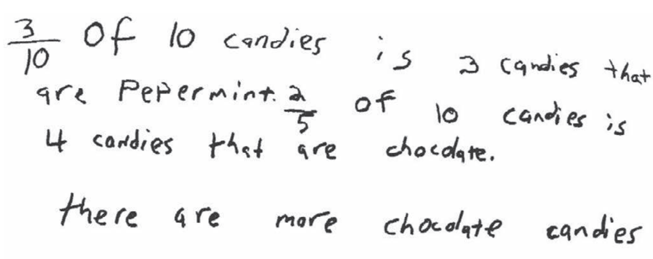
Figure 4.31 Tiara's response.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate or more peppermint candies?



a. Do the numbers that each selected lead to correct solutions to the problems? Explain.
b. Are there other numbers that Tom and Tiara could have chosen? Explain.
c. The solutions written by Tom and Tiara point out a unique feature of the set model as someone attempts to equipartition the set into parts that are all the same size. What is that unique feature? Describe.

Instructional Link: Your Turn

Use the guiding questions in Table 4.1 to help you think about how your mathematics programs provide students with opportunities to experience equipartitioning and to use their understanding of the impact of equipartitioning to generalize concepts and develop understandings of concepts.

Table 4.1 Instructional Link—Strategies to Support the Use of Partitioning to Develop and Generalize Concepts

Do you (or does your program):	*Yes/No*
1. Provide opportunities for students to physically partition a variety of regions, sets of objects, and number lines?	
2. Pay attention to the stages of partitioning? For example, do students solve problems involving halving strategies before partitioning into an odd number?	
3. Encourage students to use their understanding of the impact of partitioning to solve problems?	
4. Use partitioning to help develop ideas or generalize concepts?	

Describe any adjustments that you need to make to your instruction to ensure that students have opportunities to use equipartitioning to develop an understanding of concepts, as well as to generalize concepts.

5 Understanding the OGAP Fraction Framework and Progression

Big Ideas

- The *OGAP Fraction Progression is* a learning progression based on mathematics education research and is designed as a tool for teachers to gather evidence of student thinking to inform instruction and monitor student learning.
- Accumulating evidence by researchers indicates that knowledge and use of learning progressions positively impacts both teachers' knowledge and instruction and students' motivation and achievement.

The first four chapters of this book focused on important underlying concepts regarding fractions (i.e., fractions as numbers, using visual models, understanding fractions in relation to the whole, and equipartitioning). The focus of this chapter shifts to the use of the *OGAP Fraction Framework and Progression* for formative assessment and instructional decision-making. It starts with a vignette in which a group of third-grade teachers are meeting to analyze some student work.

> A group of third-grade teachers have all completed some exercises with their students on using area models to represent unit fractions. They wondered the degree to which their students would use this knowledge to order a set of unit fractions. The teachers decided they would all administer the same formative assessment question (Figure 5.1) and discuss the student work at their next PLC. Why do you think this is a good formative assessment question given their goal?

Figure 5.1 Formative assessment question.

Place the following fractions in order from the smallest to the largest. Show your work.

$$\frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{8}$$

> When the teachers gathered, they were very excited about the work their students completed. They decided to sort the work into two piles—"got it" and "did not get it." That is, correct and incorrect answers. They noticed that students were using a range of different strategies, and students were making many different types of errors, but they did not come away with any insight into how to help students who "weren't getting it" other than to provide small group instruction or reteaching. They also weren't sure what to do with the students who could solve the problem correctly. Without a systematic way to analyze the

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evidence across the set of student work, the teachers had little information that could be used to inform their instruction.

The *OGAP Fraction Framework and Progression*, developed from mathematics education research on how students learn fraction concepts, was created expressly to help teachers systematically analyze and understand the evidence in student work (both written and classroom discussions), inform instructional decisions, provide actionable feedback to students, and help design and select tasks. This chapter provides an overview of both the framework and the progression. As you progress through each of the chapters that follow you will engage in case studies designed to illustrate how teachers can systematically analyze student work based on the mathematics education research and apply this knowledge to instruction.

There are two major elements in the framework which are discussed in this chapter:

1. *Fraction Problem Structures*
2. The *OGAP Fraction Progression*, which shows evidence of student work along a continuum of student understanding

The two parts of the framework are interrelated. That is, the structure of a problem impacts the nature of the evidence of student understanding that is elicited by that problem. This chapter draws on sample problems and student responses to explore both the structures of fraction problems and the *OGAP Fraction Progression*, including underlying issues and errors.

To access the full *OGAP Fraction Framework* go to www.routledge.com/9781138816442

The *OGAP Fraction Progression*

The *OGAP Fraction Progression* is designed to help teachers gather descriptive evidence of thinking related to students' developing understanding of fraction concepts and skills, as well as to identify the underlying common errors, preconceptions or misconceptions that may interfere with students learning new concepts or solving related problems. The *OGAP Fraction Progression* is based on mathematics education research on how students learn fraction concepts organized along a continuum of development, from using non-fractional reasoning and strategies to effectively applying fraction knowledge and skills at an application level. Importantly, there is accumulating evidence that knowledge and instructional use of learning progressions, together with the mathematics education research that underpins progressions, positively affects instructional decision-making, student motivation, and achievement in mathematics (Carpenter et al., 1989; Clarke, 2004; Clarke et al., 2001; Clements et al., 2011; Fennema et al., 1996; Supovitz et al., 2018, 2021; Wilson, 2009).

Study the *OGAP Fraction Progression* that follows. What do you notice about the progression?

When you reviewed the *OGAP Fraction Progression*, you probably noticed that there are levels that reflect different kinds of evidence that might be found in student work, ranging from *Non-Fractional* to *Application* as students learn new fraction concepts and solve related problems. You probably noticed examples of evidence found at each level. These levels are at a grain size that is usable by teachers to gather actionable evidence across students' development of fraction concepts and skills. This important idea will be explored in case studies in each of the subsequent chapters.

Also notice the double-arrow on the left side of the progression. This arrow communicates an important idea about learning progressions: as students are introduced to new concepts or interact with different problem structures for the same concept, their solutions may move up and down the progression levels. That is, movement on a progression is not always linear. The graphic in Figure 5.2 illustrates this important point.

However, by middle school, students' fraction fluency and understanding should be stabilized at the *Application* level so that they can fully engage in middle-school mathematics without interference from weak fraction understanding and fluency.

OGAP Fraction Progression

For OGAP Fraction Progression with additional student work examples go to http://ogapamathllc.com

Movement on a progression is not always linear

Use of visual models, equipartitioning, unit fractions, equivalence and properties of operations

Application

Middle school topics and concepts in which rational number understanding and procedures are applied:

Area, Volume, Surface Area
Percents
Expressions and Equations

Proportionals
Rates
Scaling

Probability
Functions
Similarity

Transformations
Measures of Central Tendency
Other

Fractional Strategies

Accurately locates fractions on a number line of any length, compares and orders fractions using a range of strategies, finds equivalent fractions, and operates efficiently when solving mathematical and contextual problems.

Uses reasoning about relative magnitude
Uses benchmark reasoning
Uses unit fraction reasoning
Uses efficient algorithm
Uses properties of operations
Demonstrate understanding of a concept
Equipartitions given visual model

Unit Fraction Reasoning

Which fraction is closest to 1?
$7/3 \quad 7/5 \quad 7/6 \quad 7/12$

7/6 is 1/6 away from 1
7/12 is 5/12 from 1, 5/12 is larger than 1/6
7/5 is 2/5 from 1, 2/5 is larger than 1/6
$7/3 = 2\,1/3$

Distributive Property

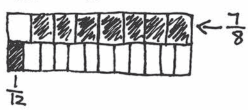
Bob ran 4 times this week.
Each run was $3\,3/4$ miles long.
How far did Bob run this week?

$4 \times 3\frac{3}{4} = 12\frac{12}{4} = 15$

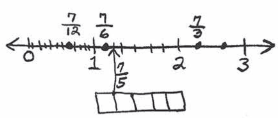
Transitional Strategies

Generates an effective visual model.

The sum of $1/12$ and $7/8$ is closest to:
a) 20
b) 8
c) $1/2$
d) 1



Which fraction is closest to 1?
$7/3 \quad 7/5 \quad 7/6 \quad 7/12$



Evidence of fractional reasoning, but the strategy is not efficient.

Bob ran 4 times this week.
Each run was $3\,3/4$ miles long.
How far did Bob run this week?

$$\begin{array}{r} 3\frac{3}{4} \\ 3\frac{3}{4} \\ 3\frac{3}{4} \\ 3\frac{3}{4} \\ \hline 12\,\frac{12}{4} - 15 \end{array}$$

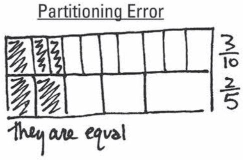
The distance from Billy's house to work is $2\,1/5$ miles. His car broke down $3/5$ of a mile from work. How far is Billy from his house?

$2\frac{1}{5} - \frac{3}{5}$
$2\frac{1}{5} - \frac{1}{5} = 2\text{ miles}$
$2 - \frac{1}{5} = 1\frac{4}{5}\text{ miles}$
$1\frac{4}{5} - \frac{1}{5} = 1\frac{3}{5}\text{ miles}$

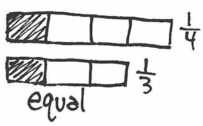
Early Fractional

Uses a fractional or transitional strategy or an operation appropriate for the situation, but the solution includes an error.

Partitioning Error



Size of whole error



The distance from Billy's house to work is $2\,1/5$ miles. His car broke down $3/5$ a mile from work. How far is Billy from his house?

$$\begin{array}{r} \cancel{2}^{1}\,\frac{5}{5} \\ -\ \frac{3}{5} \\ \hline \frac{2}{5} \end{array}$$

Non - Fractional

Applies rules without evidence of understanding, uses inappropriate whole number reasoning, or uses incorrect operation given the problem context.

Which fraction is closest to 1?

$\frac{7}{3} \quad \frac{7}{5} \quad \frac{7}{6} \quad \frac{7}{12}$

Big Headed   not Big headed

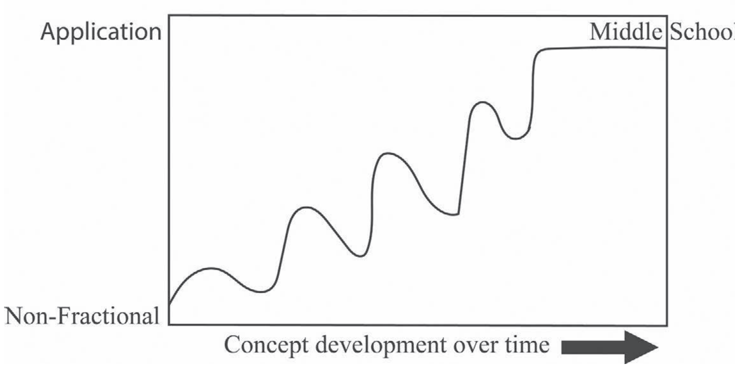
The sum of $1/12$ and $7/8$ is closest to:
a) 20
b) 8
c) $1/2$
d) 1

$\frac{1}{12} + \frac{7}{8} = \frac{2}{24} + \frac{21}{24} = \frac{23}{24}$ is closest to 20

Underlying Issues/Errors

Applies rules without evidence of understanding, inappropriate whole number reasoning, misinterprets a visual model, incorrect operation, concept or property error, partitioning error, remainder error

Figure 5.2 Hypothesized movement on the progression as concepts are introduced and developed across grades.



Levels on the *OGAP Fraction Progression*

Looking more closely at the levels on the progression, one notices five levels: *Application, Fractional Strategies, Transitional Strategies, Early Fractional*, and *Non-Fractional*. Naturally, all the evidence that can be exemplified at each level cannot be included on the progression. Thus, the examples have been selected to provide a sample of common solution strategies. You will find more examples of student solutions and their relationships to the progression as you read through Chapters 6 through 12.

To become familiar with the problems and solutions used as examples on the progression, the authors suggest you solve each of the following problems found on the *OGAP Fraction Progression*. As you solve the problems anticipate how students might solve the problems or errors they might make.

a. Which fraction is closest to 1? $\left(\frac{7}{3}, \frac{7}{5}, \frac{7}{6}, \frac{7}{12}\right)$ Show your work.
b. Bob ran 4 times this week. Each run was $3\frac{1}{4}$ miles long. How far did Bob run this week? Show your work.
c. The sum of $\frac{1}{12} + \frac{7}{8}$ is closest to: a) 20; b) 8; c) $\frac{1}{2}$; d) 1 Show your work.
d. The distance from Billy's house to work is $2\frac{1}{5}$ miles. His car broke down $\frac{3}{5}$ of a mile from work. How far is Billy from his house? Show your work.

As you read about the different levels on the progression in the pages that follow, study the examples from the images that are provided of each level being discussed. After you have made sense of these examples for yourself, read the explanation of how the examples represent evidence at that level.

Application

The ultimate goal is for students to be able to apply their understanding of fraction concepts and strategies. Study the *Application* level of the progression shown in Figure 5.3. What do you notice about the topics and concepts that are listed in this level?

Figure 5.3 *Application* level of the progression.

Application

Middle school topics and concepts in which rational number understanding and procedures are applied:

Area, Volume, Surface Area	Proportionals	Probability	Transformations
Percents	Rates	Functions	Measures of Central Tendency
Expressions and Equations	Scaling	Similarity	Other

You probably noticed that there are a wide variety of topics in middle school where students must apply their understanding of rational number concepts and procedures. Moreover, these topics appear across all mathematical domains (number and operations, measurement, algebra, ratio and proportion, geometry, data analysis, and probability). This explains, in part, why middle school teachers, as well as high school teachers, often express frustration with the lack of students' fluency with fraction concepts. In fact, "algebra teachers ranked poor understanding of fractions as one of the two most important weaknesses in students' preparation for their course" (Siegler et al., 2010, p. 6).

Tania's response in Figure 5.4 illustrates the effective application of fraction understanding and fluency when solving a problem involving proportions.

Figure 5.4 Tania's response shows evidence of applying fraction knowledge in a proportion problem at the *Application* level on the progression.

Paul's dog eats 20 pounds of food in 30 days. How long will it take Paul's dog to eat a 45-pound bag of dog food? Show your work.

$\times 2\frac{1}{4}$

$\frac{20 \text{ lbs}}{30 \text{ days}} = \frac{45 \text{ lbs}}{x}$ $\quad 30 \text{ doys} \times 2\frac{1}{4} = 67\frac{1}{2} \text{ days}$

$\times 2\frac{1}{4}$

Notice that Tania was able to use her understanding of fraction multiplication to solve the proportion correctly.

Fractional Strategies

Study the text and examples at the *Fractional* level on the progression in Figure 5.5. What do you notice about the difference between the *Fractional Strategy* level and the *Application* level?

You probably noticed that at the *Fractional Strategy* level, students understand fractions as quantities at an abstract level and use reasoning and efficient strategies when solving contextual and non-contextual problems. At the *Application* level, they can use these abstract strategies to solve problems in a range of mathematical domains.

Figure 5.5 The *Fractional* level of the progression.

Fractional Strategies

Accurately locates fractions on a number line of any length, compares and orders fractions using a range of strategies, finds equivalent fractions, and operates efficiently when solving mathematical and contextual problems.

Uses reasoning about relative magnitude
Uses benchmark reasoning
Uses unit fraction reasoning
Uses efficient algorithm
Uses properties of operations
Demonstrate understanding of a concept
Equipartitions given visual model

Unit Fraction Reasoning

Which fraction is closest to 1?
7/3 7/5 7/6 7/12

7/6 is 1/6 away from 1
7/12 is 5/12 from 1, 5/12 is larger than 1/6
7/5 is 2/5 from 1, 2/5 is larger than 1/6
7/3 = 2 1/3

Distributive Property

Bob ran 4 times this week.
Each run was 3 3/4 miles long.
How far did Bob run this week?

$4 \times 3\frac{3}{4} = 12\frac{12}{4} = 15$

Consider the two solutions shown on the progression for this level. The solution on the left, in which a student used unit fraction and benchmark reasoning to determine which fraction is closest to 1, is an example of using reasoning about relative magnitude without a reliance on constructing a visual model. The problem could be solved using common denominators, but that would involve a lot of calculation and is not the most efficient method. The problem can be much more efficiently solved using reasoning based on unit fraction understanding (e.g., $\frac{7}{6}$ is $\frac{1}{6}$ more than 1; $\frac{7}{5}$ is $\frac{2}{5} > 1$; $\frac{1}{5} > \frac{1}{6}$; therefore, $\frac{7}{6}$ is closer to 1), or benchmark reasoning (e.g., $\frac{7}{12}$ is $\frac{5}{12} < 1$). The other solution shown at this level illustrates the use of the distributive property to multiply a mixed number by a whole number.

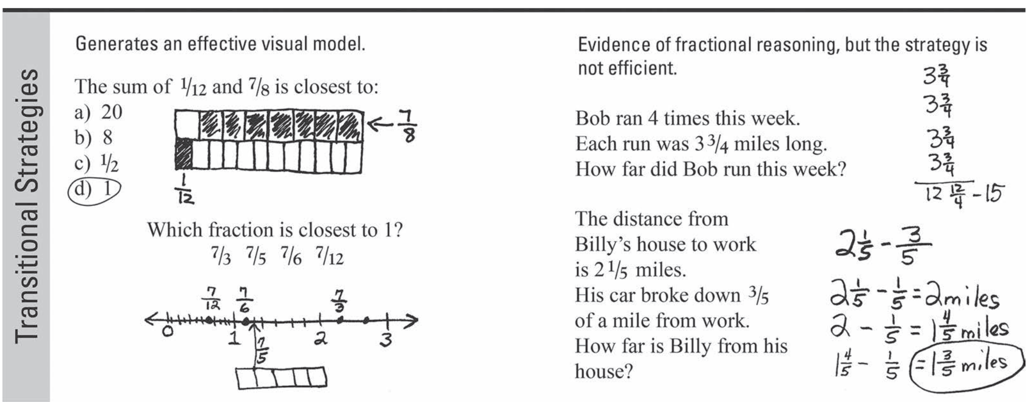
$$4\times 3\frac{3}{4} = (4\times 3) + \left(4\times\frac{3}{4}\right) = 12 + \frac{12}{4} = 12 + 3 = 15$$

Notice the list of additional strategies on the left side of this level of the progression that are evidence of fractional strategies.

Transitional Strategies

Now study the examples at the *Transitional Strategy* level in Figure 5.6. What do you notice about the differences between solutions at the *Transitional* level and the *Fractional* level?

Figure 5.6 *Transitional Strategy* level of the progression



Visual models were effectively used in the two examples on the left side of Figure 5.6. Both solutions show evidence of using equipartitioning to show understanding of the relative magnitude of each of the fractions when solving the problems. As discussed in Chapter 2, using visual models supports the development of conceptual understanding and fluency of fractions. You can imagine a class discussion using these solutions that help the students develop the kind of unit fraction reasoning evidenced in the *Fractional* level example about determining which fraction is closest to 1. The *Transitional Strategy* level, therefore, is an important move toward fluency.

The examples on the right side of the *Transitional Strategy* level are evidence of using a fractional strategy, but the reasoning or strategy is not efficient. In the solution to the problem "Bob ran 4 times this week. Each run was $3\frac{3}{4}$ miles long. How far did Bob run this week?" in Figure 5.6 there is evidence that repeated addition of fractions was used to multiply, whereas the solution to the same problem at the *Fractional Strategy* level utilizes a multiplicative strategy.

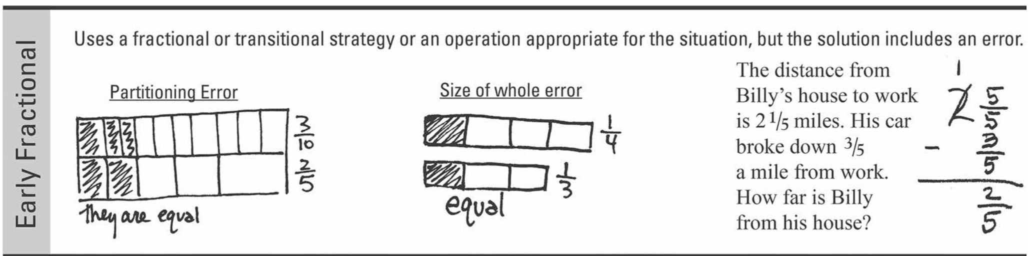
In the problem involving subtraction of $\frac{3}{5}$ to determine Billy's drive to work, the solution has evidence of repeated subtraction of $\frac{1}{5}$. While this shows a good understanding that can be built on ($\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$), the strategy is not efficient.

While both repeated addition and repeated subtraction strategies work to solve these problems and show an understanding of the operation, they are not efficient. However, as with strategies that use visual models, these strategies can provide an opportunity to transition students to the use of more efficient strategies. In subsequent chapters, case studies provide examples on how to use solutions such as these to help move students toward fluency.

Early Fractional Strategies

Study the examples at the *Early Fractional* level in Figure 5.7. What do you notice about the difference between solutions at the *Early Fractional* level and the *Transitional Strategy* level?

Figure 5.7 *Early Fractional* level of the progression.

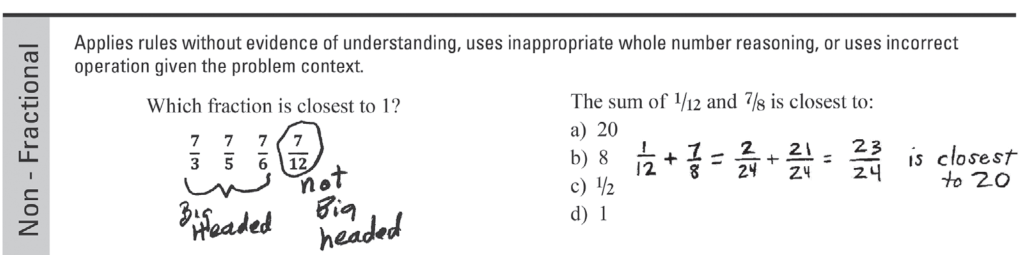


There is evidence at this level of an attempt to use a fractional or transitional strategy, but the solution contains a major flaw or conceptual error. For example, there may be evidence of partitioning visual models, but not equipartitioning, or the student may not have considered the size of the whole when using visual models to compare fractions. In other situations, there may be a conceptual error in part of the problem. Study the solution to the Billy's car problem on the right. The student understood subtraction could be used to find the missing distance between Billy's location and his house, but the solution contains a major error (which may be either conceptional or procedural) in regrouping $2\frac{1}{5}$ as $1\frac{5}{5}$.

Non-Fractional Strategies

Study Figure 5.8. What do you notice about the solutions at this level?

Figure 5.8 Non-fractional level of the progression



At this level, there is evidence of any of the following: rules being applied without understanding, inappropriate use of whole number reasoning, or the application of the incorrect operation given the problem context. The example on the left of the progression at this level is an example of applying a rule without understanding, specifically, improper fractions being "big headed" leaving $\frac{7}{12}$ closest to 1 because it is "not big headed."

The problem on the right side of the progression at the *Non-Fractional* level is probably familiar from Chapter 1, which introduced a major conceptual error of inappropriately applying whole number reasoning when solving fraction problems. Unlike the solution to the same problem at the *Transitional* level, this solution does not show evidence of understanding that the sum, $\frac{23}{24}$, is closest to 1. The fact that the student thought it was closest to 20 shows a lack of understanding of fractions as numbers, even though the procedure of finding common denominators and adding the fractions was correctly applied.

Important Ideas about the OGAP Fraction Progression

There are three important ideas about the *OGAP Fraction Progression* (and other OGAP progressions) that should always be kept in mind.

Movement Along the Progression Is Not Linear

The *OGAP Fraction Progression* provides a guide to students' development as they progress toward fraction fluency and understanding over time. Understanding the evidence in student solutions (both written and verbal) and understanding where that evidence falls on the progression provides important instructional guidance. The arrow on the right side of the progression (shown in Figure 5.9) provides guidance on fundamental fraction concepts that can be focused on to help student strategies progress from *Early* and *Non-Fractional* levels to the *Transitional* level, as well as from the *Transitional* level to the *Fractional Strategy* level. For example, when students first learn a fraction concept or there is evidence of *Early Fractional* or *Non-Fractional* strategies or reasoning, instruction should be grounded in using visual models, equipartitioning, and unit fraction understanding. Then understandings derived from visual models (e.g., unit fraction reasoning when comparing fractions) and application of equivalence and properties of operations can help move student solutions from the *Transitional Strategy* level to the *Fractional Strategy* level. In addition, as you will read in Chapters 11 and 12, strategic use of problem contexts can play a key role in helping students make sense of operations with fractions and develop procedural fluency.

Figure 5.9 Instructional strategies used to move students from one level to the next on the progression.

Using these concepts to help students progress should not be a matter of direct instruction on how to make models, on equipartitioning, or on unit fractions. Rather, it is through the interaction of these ideas with targeted instructional strategies (e.g., connecting mathematical ideas, classroom discourse, and purposeful questioning) that students engage in making sense of these concepts and relationships. Chapter 6 provides a framework for using the *OGAP Fraction Progression* to inform instruction. Chapters 7–12 contain case studies that provide some samples of how targeted instructional strategies along with the evidence collected in student work can be purposely used to help students make sense of concepts and to build fluency.

Collection of Underlying Issues and Errors Is Important

At the bottom of the progression there is a list of potential underlying issues or errors that may interfere with students learning of new concepts and/or solving problems. This information, coupled with the location of a strategy used along the progression, provides teachers with actionable evidence to inform instruction and support students' continual learning.

The OGAP Fraction Progression Is Not Evaluative

You'll notice that there are no numbers associated with the levels on the progression. The OGAP learning progressions are designed to help teachers gather *descriptive* evidence about student learning to inform instruction and student learning, not to assign a number or grade. This descriptive evidence includes both the level of the students' strategy on the progression and any underlying issues or errors evidenced in the student work.

More information on how to use the *OGAP Fraction Progression* as an instructional tool is provided in Chapter 6 and in the case studies in Chapters 7 to 12.

Problem Structures and Engineering Problems

The first part of this chapter focused on using research on the teaching and learning of fraction concepts to inform instruction and student learning. However, another important part of this research is related to how the structures of fraction problems interact with students' strategies and learning of new concepts and how those structures can be used to engineer and strategically select problems for students based on the evidence in their work. The arrow on the left side of the progression indicates that movement on the progression is not always linear, and this in part has to do with the impact of problem structures on student solutions.

Figure 5.10 shows the fraction problem structures on the *OGAP Fraction Framework*. Some of these structures are probably more familiar to you than others.

This page of the framework displays the structures of fraction problems that should be considered when selecting or designing questions for instruction or assessment purposes. Some of these structures, as you probably found, are straightforward and familiar, such as the topics being addressed (e.g., comparing fractions, operating with fractions) and the types of fractions in a problem (e.g., unit fractions, mixed numbers, negative fractions). Others may be less familiar, like the levels of partitioning discussed in Chapter 4 or classes of fractions discussed in Chapter 8.

Understanding the structures in problems that students solve is important because structures can influence student thinking and solution strategies. Looking at the example problems and student responses on the progression can provide you a sense of the potential for problems to elicit a range of strategies as well as a range of misunderstandings based on different structures. OGAP formative assessment questions are engineered to elicit developing understandings, common errors, and misconceptions by careful design. For example, in the question "Which fraction is closest to 1? $\left(\frac{7}{3}, \frac{7}{5}, \frac{7}{6}, \frac{7}{12}\right)$" the numerators are the same. This question can be solved using common denominators but is designed to elicit unit fraction understanding, as shown by the student solution in the *Fractional Level* on the progression (Figure 5.5). The question can also elicit the use of visual models, inappropriate whole number reasoning, and other conceptual errors (as shown in examples on the other levels on the progression). As you progress through subsequent chapters, study the sample problems in each chapter to consider how the structure of the problem influences student solutions.

Chapter Summary

This chapter focused on the *OGAP Fraction Framework and Progression*.

- The *OGAP Fraction Progression* is an example of a learning progression founded on mathematics education research, and it is written at a grain size that is usable across a range of fraction concepts and by teachers and students in a classroom.
- The *OGAP Fraction Progression* was specifically designed to inform instruction and monitor student learning from a formative assessment perspective.

Figure 5.10 Structures of fraction problems.

Fraction Problem Structures

Fraction Topics
- Partitioning
- Comparing and ordering
- Equivalence
- Number lines
- Operations
- Density of fractions

Fractions Types
- Unit fractions
- Non-unit fractions
- Proper fractions
- Improper fractions
- Mixed numbers
- Negative fractions

Classes of Fractions
- Different numerators, same denominators
- Same numerators, differemt denominators
- Different numerators, different denomnatos

Fraction Reasoning Strategies
- Number sense
- Unit fraction
- Extended unit fraction
- Use of visual models
- Benchmark
- Equivalence
- Properties of operation

Uses of Visual models
- To solve problems
- To understand concepts
- To generate concepts

Types of Visual Models
- Area
- Set
- Number line

Types of Number Lines
- 0 -1
- Negative to positive
- Two or more units
- Unpartitioned
- Partitioned

Levels of Partitioning
- Algorithmic halving
- Oddness
- Evenness
- Composition

Nature of Wholes
- Same size
- Different size
- Given part, find whole

Number of Parts in Whole
- Equal to denominator
- Multiples of denominator
- Factors of the denominators

Operations
- Estimation
- Algorithms
- Impact of operations
- Types of division
  - Partitive
  - Quotative

Design of Problems
- Requires interaction with a visual model
- An exact numerical answer IS or IS NOT required
- Contextual or non-contextual
- Support a claim
- Concept explanation
- Link an equation to a problem situation or visual model
- Extended multiple choice
- Impact of operations
- Multi-step

- *Fraction Problem Structures* are important to consider when deciding what evidence to collect and to make sure that students are exposed to a wide range of structures when learning about fractions.

Looking Back

1. The following addition estimation problem, which was discussed earlier in the book, was designed to elicit evidence of student understanding of unit fractions and of the magnitude of fractions.

   The sum of $\frac{1}{12} + \frac{7}{8}$ is closest to:

   a. 20
   b. 8
   c. $\frac{1}{2}$
   d. 1

   Show your work.

   Solve the problem and then analyze the evidence in each of the solutions that follow. Based on the evidence:

   a) Where on the *OGAP Fraction Progression* is each solution? What is the evidence?
   b) Are there any underlying issues or errors?
   c) Is the answer correct?
   d) For each piece of student work shown, identify understandings that can be built upon and a strategy you would use to help the student build understanding toward the next level on the progression.

Figure 5.11 Noah's response.

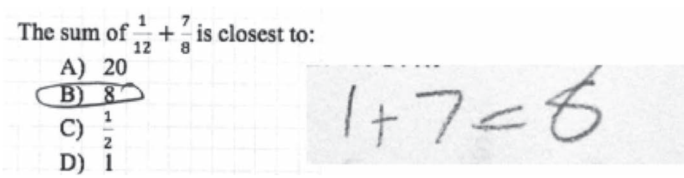


Figure 5.12 Jayden's response.

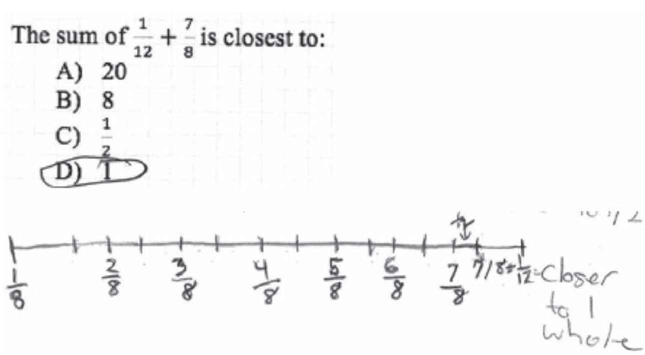


Figure 5.13 Emma's response.

The sum of $\frac{1}{12} + \frac{7}{8}$ is closest to:

A) 20
B) 8
C) $\frac{1}{2}$
D) 1

Show your work.

$\frac{7}{8} + \frac{1}{8} = 1$

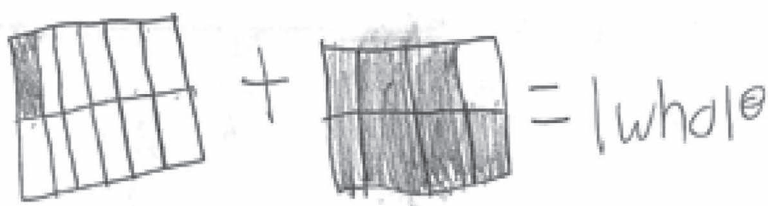
It couldn't be $\frac{1}{2}$ because $\frac{7}{8}$ is bigger than $\frac{1}{2}$, but if you add $\frac{1}{12}$ it wouldnt be 1 but that is the the closest chace

Figure 5.14 Ethan's response.

The sum of $\frac{1}{12} + \frac{7}{8}$ is closest to:

A) 20
B) 8
C) $\frac{1}{2}$
D) 1

Show your work.



2. Briefly explain the instructional importance of each of the following features of the *OGAP Fraction Progression*.

   a. Movement along the progression is not linear.
   b. The progression provides instructional guidance.
   c. The progression is not evaluative.
   d. Collection of underlying evidence is important.

3. Study the questions in a–c used throughout this chapter. For each question review the responses to the question found on the *OGAP Fraction Progression* and other examples in the chapter and identify structures that may have influenced student solutions.

   a. Bob ran 4 times this week. Each run was $3\frac{1}{4}$ miles long. How far did Bob run this week? Show your work.
   b. The sum of $\frac{1}{12}+\frac{7}{8}$ is closest to: a) 20; b) 8; c) $\frac{1}{2}$; d) 1. Show your work.
   c. The distance from Billy's house to work is $2\frac{1}{5}$ miles. His car broke down $\frac{3}{5}$ of a mile from work. How far is Billy from his house? Show your work.

Table 5.1 Instructional Link: Your Turn

1.	To what degree does your math program or instructions focus on regularly gathering formative information about student learning to inform instruction?
2.	To what degree does your math program or instruction vary problem structures.
3.	Based on this analysis, identify specific ways you can enhance your math instruction by utilizing ideas in this chapter.

6
Using the *OGAP Fraction Progression* to Inform Instruction and Support Student Learning

- The OGAP formative assessment system is cyclical, intentional, and systematic.
- The analysis of student work should focus on strategies student use when solving problems, errors they make, and correctness of solutions.
- There are a range of instructional responses to evidence in student work.

Chapter 5 focused on understanding the *OGAP Fraction Framework and Progression*. This chapter introduces you to ways to use the *OGAP Framework and Progression* to systematically analyze and record the evidence in student work and make instructional decisions. The ideas in this chapter are dependent on understanding the concepts in Chapter 5; thus, it is strongly recommended that you read Chapter 5 prior to reading this chapter.

Assessment experts propose that formative assessment involves the relentless attention to evidence in student thinking (Popham, 2012) and that the evidence must be "elicited, interpreted, and used by teachers and learners" (Wiliam, 2011, p. 43). These ideas are at the heart of the OGAP formative assessment system introduced in this chapter.

The OGAP formative assessment process involves gathering evidence of student thinking throughout the teaching/learning process, including at the beginning of a lesson, during a lesson as students are working and discussing, and at the end of a lesson. This evidence is then analyzed using the *OGAP Fraction Progression* to inform instructional decisions.

The strategies for informed instructional decision-making introduced in this chapter were developed and refined through interactions with thousands of teachers and their students over the last two decades. Instructional decisions can be informed by knowledge of specific mathematics content, such as fractions, knowledge of the mathematics education research reflected in the *OGAP Fraction Progression*, and knowledge of instructional strategies that elicit evidence of students' developing understanding.

Let's return to the example from Chapter 5, where third-grade teachers were considering the degree to which their students applied their understanding of unit fractions in area models to order unit fractions. Over the summer, the teachers attended an OGAP training and were introduced to the *OGAP Fraction Progression*. The next year, when they gave the same problem (Figure 6.1) to their students, they approached it from a formative assessment perspective utilizing ideas and strategies they learned in the OGAP fraction training.

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Figure 6.1 Formative question.

Place the following fractions in order from the smallest to the largest. Show your work.

$$\frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{8}$$

Instead of categorizing the student work by correct and incorrect solutions as they did previously, the teachers now sorted the student solutions based on the levels of the strategies described on the *OGAP Fraction Progression* and used this analysis to inform their planning and instruction. This process is referred to throughout the book as the *OGAP Sort*.

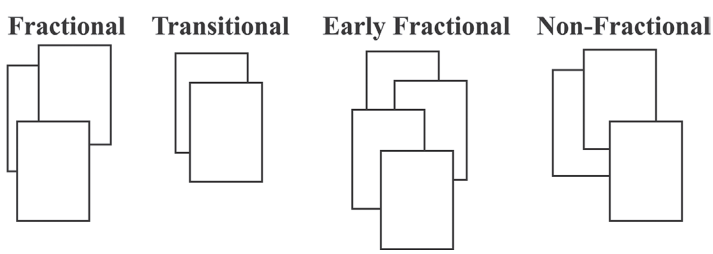
The *OGAP Sort* involves analyzing the student solutions for three different types of information: (see Figure 6.2).

These are:

1. Detailed strategy and its level on the progression (e.g., *Fractional Strategy* using the distributive property)
2. Underlying issues or errors in the solution
3. The accuracy of the solution.

When looking at student work, it is common to focus primarily on the correctness of the answer or accuracy of the solution. Although accuracy is important, by itself it can be an incomplete assessment of student understanding and provides little guidance for next instructional steps. For this reason, the *OGAP Sort* begins with first understanding each solution in relation to the levels on the *OGAP Fraction Progression*. Next, teachers identify any underlying issues or errors

Figure 6.2 Sorting student work into *Fractional, Transitional, Early Fractional*, and *Non-Fractional* solutions.



that may interfere with students understanding a concept or solving the problem. It is only after making sense of these two aspects of a student solution does the teacher consider its accuracy. In this way, teachers can gain a deeper understanding of student thinking while collecting usable evidence to inform the next instructional steps.

After analyzing the solutions based on the levels of the *OGAP Fraction Progression*, the teachers recorded the information on an *OGAP Evidence Collection Sheet*. In the example shown in Figure 6.3, one of the teachers, Ms. Smith, returned to the student work, made notes about the strategies used to order the fractions, and noted any underlying issues or errors. Ms. Smith also circled the incorrect solutions. This recording sheet parallels the *OGAP Fraction Progression* and provided Ms. Smith with a picture of the evidence in the student solutions in relation to the progression, how it is related to errors that may be interfering with learning, and the accuracy of the answer. Together, these pieces of information help inform instruction for the whole class, for small groups, and for individual students.

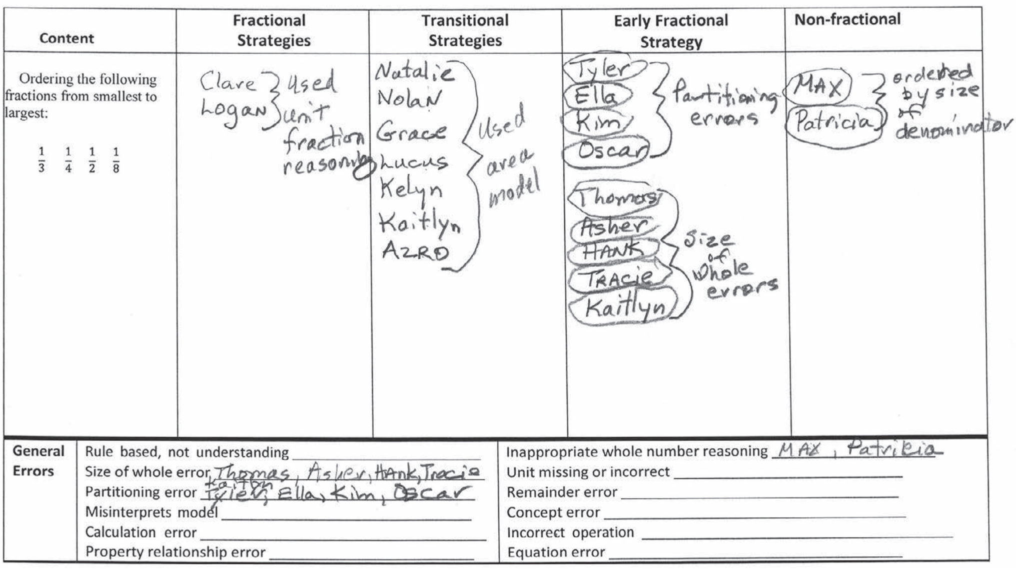
Remember that the question the teachers were considering was the degree to which their students applied their understanding of unit fractions in area models to order unit fractions.

By using the *OGAP Sort*, the analysis can move beyond whether students were or were not successful in ordering these unit fractions, to look for information and patterns in the work that informs instructional decisions. Addressing the three questions that follow can help structure discussion and reflection on this valuable information.

1. What developing understandings can be built upon?
2. What issues or concerns are evidenced in student work?
3. What are the potential next instructional steps for the whole class, for small groups, and for individuals?

Based on the information in the *OGAP Evidence Collection Sheet* in Figure 6.3 and the teacher's goals, how would you answer each of these questions?

Figure 6.3 Completed *OGAP Fraction Evidence Collection Sheet*.



Content	Fractional Strategies	Transitional Strategies	Early Fractional Strategy	Non-fractional
Ordering the following fractions from smallest to largest: $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{2}$ $\frac{1}{8}$	Clare, Logan – used unit fraction reasoning	Natalie, Nolan, Grace, Lucus, Kelyn, Kaitlyn, AZRO – used area model	Tyler, Ella, Kim, Oscar – partitioning errors; Thomas, Asher, HANK, TRACIE, Kaitlyn – size of whole errors	MAX, Patricia – ordered by size of denominator

General Errors		
	Rule based, not understanding	Inappropriate whole number reasoning MAX, Patricia
	Size of whole error Thomas, Asher, Hank, Tracie, Kaitlyn	Unit missing or incorrect
	Partitioning error Tyler, Ella, Kim, Oscar	Remainder error
	Misinterprets model	Concept error
	Calculation error	Incorrect operation
	Property relationship error	Equation error

Here is a sample of how Ms. Smith responded to these questions based on the information recorded on the *OGAP Evidence Collection Sheet* (Figure 6.3).

1. What developing understandings can be built upon?

   Clare and Logan used unit fraction reasoning to order unit fractions. Seven of the students successfully used area models to order the unit fractions. Nine of the students attempted to use area models.

2. What issues or concerns are evidenced in student work?

   Partitioning and size of whole errors and some evidence of inappropriate whole number reasoning.

3. What are the potential next instructional steps for the whole class, for small groups, and for individuals?

   While there are lots of possibilities for next steps, I think I am going to use some student work and lead a classroom discussion that helps all students move to a new understanding.

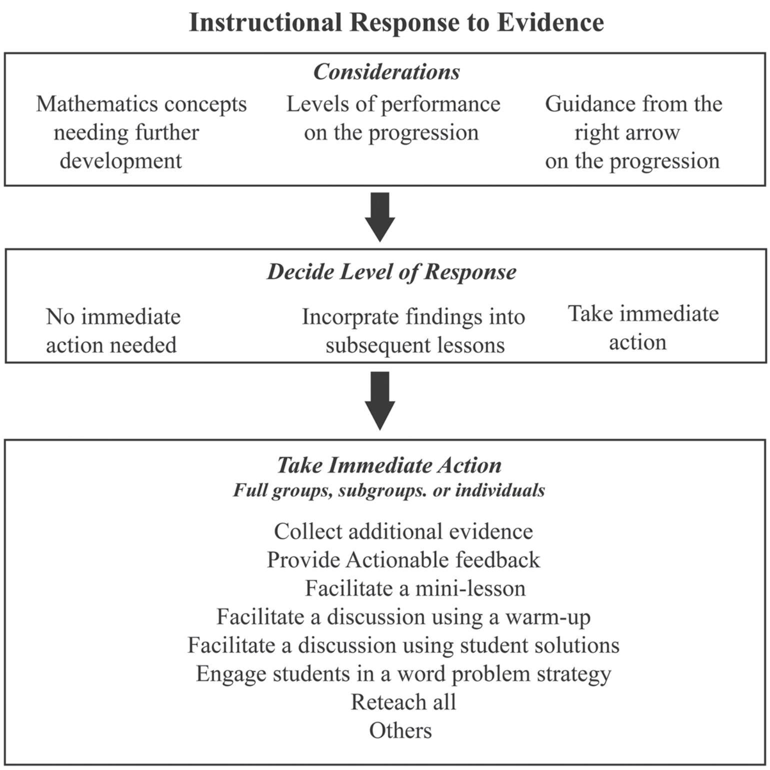
Notice the level of actionable information in these responses and how valuable these findings can be for the third-grade teachers as they determine ways to better support their students' understanding of ordering unit fractions.

It is worth repeating that the goal of *OGAP Fraction Sort* is helping all students move to more sophisticated strategies and mathematics, not just focusing on the incorrect solutions.

Instructional Response to Evidence

In working with teachers in OGAP trainings, facilitators find that many teachers' first instructional response to the evidence is to put students into small groups. Although this is one potentially helpful instructional move, it is vital that teachers understand the range of potential instructional responses to evidence collected through the *OGAP Sort*. To do this, OGAP facilitators have found it helpful to consider three levels of decision-making. Study Figure 6.4. Notice that the first level is to think about the evidence to be considered (the mathematics, the level of performance on the progression, and guidance from the right arrow on the *OGAP Fraction Progression*), then decide on the level of response, and finally, if action is taken select an appropriate instructional response.

Figure 6.4 Instructional response to evidence (Petit et al., 2020, p. 93).



Given the information on the *OGAP Fraction Evidence Collection Sheet* in Figure 6.3, Ms. Smith's analysis of the evidence and the ideas outlined in Figure 6.4, she decided to take immediate action by facilitating a classroom discussion using student solutions as modeled in the following.

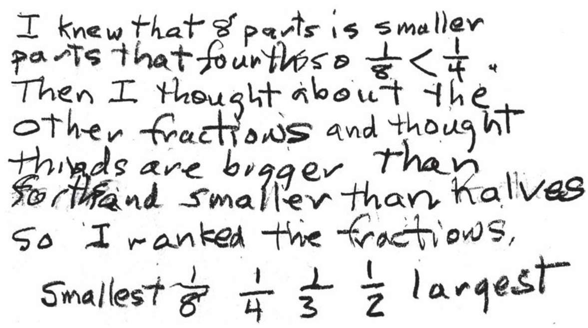
Study the three student solutions shown in Figure 6.5. Why do you think she chose these three solutions to guide the class discussion?

Figure 6.5 Three formative assessment question responses.

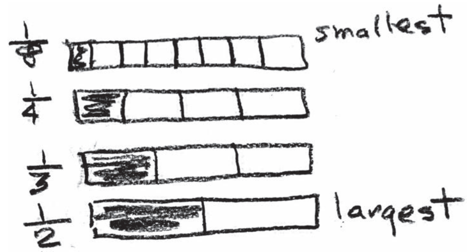
Place the following fractions in order from the smallest to the largest. Show your work.

$$\frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{8}$$

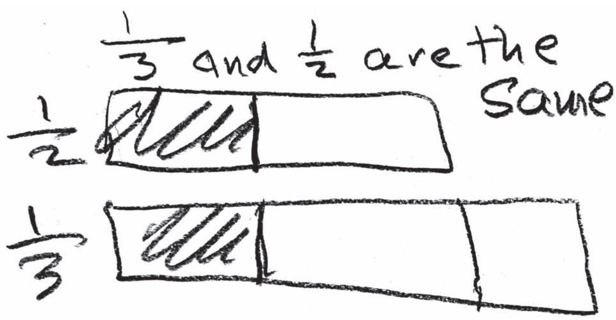
Logan's Response



Grace's Response



Jada's Response



Ms. Smith chose these solutions to help students use their understanding of partitioning an area model to develop a more abstract understanding of the impact of partitioning. She used the following questions and a "think, pair, share" strategy to engage students in discussion.

a. What do you notice about Grace's solution?
b. How is Grace's solution the same or different from your own solution?
c. A student said that $\frac{1}{2}$ is the smallest because 2 is smaller than 3, 4, or 8. Is this reasoning correct? How does Grace's response show this?
d. Look at Jada's response. Do you agree or disagree with Jada? How could Jada revise her model to get a more accurate answer?
e. Now show Logan's solution. How is Logan's solution the same or different from Grace's solution?

After looking at these solutions, the teacher decided to pose some additional questions to focus more intentionally on unit fraction reasoning:

a. Name a unit fraction smaller than $\frac{1}{8}$. How do you know it is smaller than $\frac{1}{8}$?
b. Name a unit fraction that is greater than $\frac{1}{6}$ and smaller than $\frac{1}{2}$. How do you know?

At the end of the discussion Ms. Smith gave the following exit question (Figure 6.6) to gather information about the students' developing understandings based on the classroom discussion.

Figure 6.6 Ms. Smith's exit question.

Put the following fractions in order from the smallest to the largest. Show your work.

$$\frac{1}{6} \quad \frac{1}{2} \quad \frac{1}{8} \quad \frac{1}{5}$$

This description of Ms. Smith's instructional response to evidence collected through the *OGAP Sort* illustrates the cyclical nature of the formative assessment process. After you have reviewed the work and made instructional decisions, you should again consider the structures in the problems students will solve during the next lesson and in the next formative assessment question. Once more, analysis of student responses from the next formative assessment question from one lesson can inform your planning for the next lesson.

Each subsequent chapter of this book includes a case study indicated by this icon. Case studies focus on analyzing evidence using the *OGAP Sort* and making instructional decisions. Some chapters have the case study embedded in the chapter and others are found in the Looking Back section of the chapter. Each case study is an example of one way to respond to the evidence in student work, but there are many ways teachers can respond to the same evidence. As you read through the case studies, consider other instructional approaches that could be used that respond to the evidence.

Table 6.1 provides sample rationales for selecting specific instructional strategies. Notice that the table is divided into two sections: the level of response and potential rationale for the response. As you work through the case studies in other chapters you may want to reference this table.

Table 6.1 Instructional Response Rationales (Petit et al., 2020, pp. 105–106)

Level of Response	*Potential Rationale*
No immediate action necessary	The class is ready for the next mathematics concept.
Incorporate findings into subsequent lessons	The instructional materials support further development of the concepts that students are struggling with or there is a common error that students are making that can be incorporated into subsequent lessons (e.g., students are not labeling answers).
Take immediate action	Additional instruction is necessary before proceeding with new concepts (see the following).

(*Continued*)

Table 6.1 (Continued)

Instructional Strategy	*Potential Rationale*
Take Immediate Action Plan Instruction and Implement for Full Class or Subset of the Students *Strategies that can take up to 10–15 minutes of instructional time	
Instructional Strategy	*Rationale*
Collect additional evidence	For a variety of reasons, you need to collect additional evidence of student understanding to understand students' thinking or strategies.
Provide actionable feedback	Regardless of which instructional strategy is used, students should be provided with feedback. It can be whole class feedback or individual feedback. Students should be provided time to address the feedback.
Implement a mini-lesson*	There is an instructional issue or a specific concept that merits full group discussion.
Facilitate a discussion using student solutions*	Select and sequence student solutions to focus on a specific aspect of a problem, mathematics content, errors students are making, or to extend thinking to a new concept or a higher level on the *OGAP Progression*.
Facilitate a discussion using a warm-up problem*	Can be used in a variety of situations (e.g., to gather more evidence at the beginning of a lesson; to bridge one concept to a new concept).
Engage students in making sense of word problems*	There are a variety of ways to engage students in understanding word problems.
Reteach to all	Few students have progressed based on the evidence; the instructional materials did not further student understanding nor help them develop strategies. This does not mean repeating the same lesson. Rather it means focusing on the same goal using a different instructional approach and different instructional materials.

Chapter Summary

- The OGAP formative assessment system is an intentional and systematic approach to gathering evidence and making instructional decisions.
- The *OGAP Sort* is a strategy to gather evidence of student understanding and fluency in relationship to the *OGAP Fraction Progression*.
- There are a range of instructional responses to evidence in student work.

Looking Back

1. Use the *OGAP Fraction Progression* to analyze student work from students in your own classroom. First, design or select a fraction question based on the mathematical goal of your lesson. Administer the question at the end of your lesson. Finally, analyze your students' responses using the *OGAP Sort* and record the information on the *OGAP Evidence Collection Sheet* in Figure 6.7.

Figure 6.7 OGAP Evidence Collection Sheet.

Item #	Content	Fractional	Transitional	Early Fractional	Non-Fractional

General Errors/Issues	
Rule based, not understanding __________________	Inappropriate whole Number reasoning __________________
Size of whole error __________________	Unit missing or incorrect __________________
Partitioning Errors __________________	Remainder error __________________
Misinterprets models __________________	Concept error __________________
Calculation error __________________	Incorrect operation __________________
Property/relationship error __________________	Equation error __________________

Use the evidence to address the following questions.

a. Where on the *OGAP Fraction Progression* is each student's response? What is the evidence?
b. What underlying issues or errors do you notice that may interfere with each student learning new concepts or solving related problems?
c. Based on the evidence in the work, what does it appear that each student understands, upon which future instruction can built?
d. Based on the evidence, what else do you want to know about the student's understanding?
e. What might be effective next instructional steps for each of these students?

Instructional Link: Your Turn

Use the questions that follow to analyze the ways your math instruction and program supports the use of a formative assessment.

1. To what degree does your math instruction or program focus on intentionally and systematically gathering formative assessment information to inform your instruction?
2. In what ways does your math program or instruction use instructional strategies like those found in Figure 6.4 to respond to evidence?
3. Based on this analysis, identify specific ways you can enhance your math instruction by utilizing ideas from this chapter.

7
Equivalence

Big Ideas

- Saying that two fractions are equivalent is saying that the two fractions are different names (symbols) for the same number.
- There are an infinite number of different names for a given fraction.
- Understanding equivalence and having an efficient procedure to find equivalent fractions is critical as students encounter problems involving comparing, ordering, and operating with fractions.
- Decimals are a way of expressing fractional amounts in our base-ten number system with a denominator that is a power of 10.
- Understanding decimal fractions involves understanding the base-ten number system, fractions as numbers, and fraction equivalence.

Understanding equivalence of fractions is crucial to a student's ability to compare and order fractions and add and subtract fractions. However, researchers say that "students who do not understand what a fraction means will have a hard time finding another fraction equivalent to it" (Bezuk & Bieck, 1993, p. 129). In addition, some students "have a continuing interference from their knowledge of whole numbers" (Post et al., 1986, p. 4).

Samir, Tania, and Corey (Figures 7.1, 7.2, and 7.3) show a range of understanding about equivalence. Samir is experiencing "interference" from whole number reasoning, Tania uses a model to effectively compare the two fractions, and Corey finds equivalent fractions.

Tania's solution includes an effective use of area models representing $\frac{3}{8}$ and $\frac{1}{4}$. Her solution presents evidence that she is ready to engage in discussions about equivalent fractions and their

Figure 7.1 Samir's response. Samir incorrectly compared the magnitude of the denominators and numerators, not the magnitude of the fractions.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies?

I think there are more
peppermint than chocolate becquse
10 is higher than 5 and 3 is also
higher than 2 so I thought my
answer was peppermint.

DOI: 10.4324/9781003185475-7

Figure 7.2 Tania's response. Tania used a model to compare $\frac{3}{8}$ and $\frac{1}{4}$.

There are some candies in a dish.

$\frac{3}{8}$ of the candies are chocolate.

$\frac{1}{4}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies?

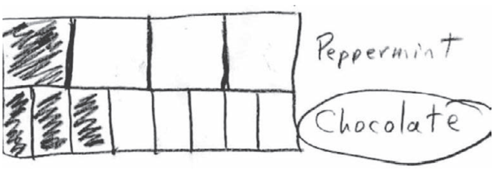


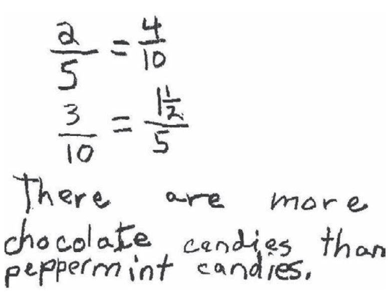
Figure 7.3 Corey's response. Corey used an understanding of equivalent fractions $\left(\frac{2}{5} = \frac{4}{10} \text{ and } \frac{3}{10} = \frac{1\frac{1}{2}}{5}\right)$ to compare the fractions $\frac{2}{5}$ and $\frac{3}{10}$.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies?



use in solving problems. Corey used an understanding he has developed about equivalent fractions to decide whether the dish contains more chocolate or more peppermint candies.

This chapter focuses on the concept of equivalence and on how students develop an understanding of fraction equivalence.

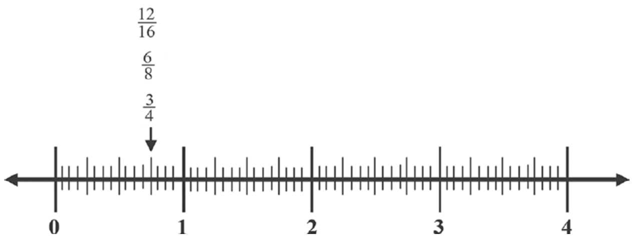
Understanding the Concept

Two concepts are central to the equivalence of fractions:

1. Saying that two fractions are equivalent is saying that the two fractions are both names (symbols) for the same number.
2. The number of different names for a given fraction is infinite. For example, the fraction $\frac{1}{2}$ can be expressed by the names (symbols) $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}, \ldots$, and so on.

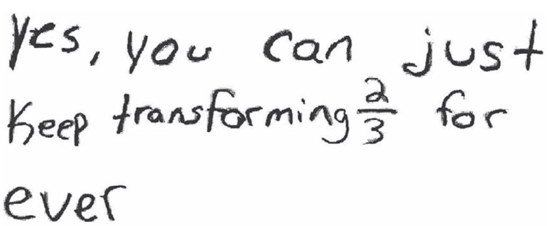
One way to understand these concepts is to think about the concept in terms of a number line. Because all equivalent fractions have the same value, they are located at the same point on a number line. The number line in Figure 7.4 is partitioned into 16ths and can be used to illustrate that $\frac{3}{4} = \frac{6}{8} = \frac{12}{16}$. If one continued to partition the number line into 32nds, 64ths, 128ths, and so on, one could picture an infinite number of possible names for $\frac{3}{4}$, all stacked on top of the $\frac{12}{16}$. That is, if one partitions the number line an infinite number of times, there will be an infinite number of equivalent fractions for that point.

Figure 7.4 A number line partitioned into 16ths, illustrating that $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{12}{16}$ are all located at the same point on the number line. Thus, $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{12}{16}$ are different names for the same number.



Mathew was asked to identify two fractions that are located between $\frac{1}{3}$ and $\frac{3}{4}$. He responded by listing $\frac{2}{3}$ and $\frac{4}{6}$. The two fractions Mathew identified are in fact equivalent fractions representing the same number. When asked whether there are any other fractions between $\frac{1}{3}$ and $\frac{3}{4}$, Mathew responded, "Yes, you can just keep transforming $\frac{2}{3}$" (Figure 7.5). While he did not show evidence of understanding the density of rational numbers (the focus of Chapter 10), he did provide evidence of understanding that there is an infinite set of fractions equivalent to $\frac{2}{3}$.

Figure 7.5 Mathew's response. Mathew's response shows an ability to find an infinite number of equivalent fractions equal to $\frac{2}{3}$.



A Framework for the Development of Equivalence Concepts

"Conceptual understanding of equivalent fractions involves more than remembering a fact or applying a procedure" (Wong & Evans, 2007, p. 826); that is, understanding equivalence, as well as procedures for finding equivalent fractions, is important for the development of other concepts and should be built in a way that brings meaning to both. Researchers suggest developing the connections between the concept and procedure through interaction with visual models and manipulatives. Using visual models and manipulatives helps to reveal patterns and relationships built on an awareness of the connections between the size and number of equal parts in a whole (Behr & Post, 1992; Behr et al., 1984; Payne, 1976; Wong & Evans, 2007).

For example, Michelle (Figure 7.6) is ready for questions that capitalize on her visual models to help build (deepen) her understanding of equivalence. In Michelle's visual model, it is easy to see the relationship between the size and number of parts in the whole. Michelle may be ready to describe $\frac{2}{10} = \frac{1}{5}$ by saying that "fifths are twice as large as tenths" or "there are two-tenths in every fifth."

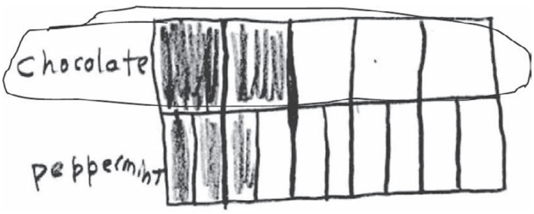
Figure 7.6 Michelle's response.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies?

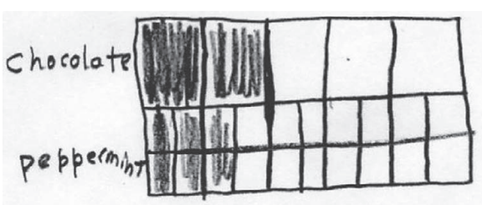


Questions for Michelle, such as "How many 10ths are equal to $\frac{2}{5}$?" and "How many fifths are equal to $\frac{8}{10}$?" will help her to see the relationships between tenths and fifths. These questions are like standard questions in some drill exercises, perhaps in the form $\frac{2}{5} = \frac{n}{10}$ or $\frac{8}{10} = \frac{n}{5}$ but are built on Michelle's developing understanding of equivalence in her models.

Taking this one step further, imagine that Michelle partitioned each 10th in half, resulting in her peppermint model being partitioned into 20ths (Figure 7.7). Michelle could then consider how many 20ths are equal to $\frac{2}{5}$. The model in Figure 7.7 shows that $\frac{2}{5} = \frac{8}{20}$.

Michelle can begin to build a set of equivalent fractions $\left(\frac{2}{5} = \frac{4}{10} = \frac{8}{20}\right)$. This can be followed by additional partitioning to identify other fractions equivalent to $\frac{2}{5}$ (and $\frac{3}{5}$, $\frac{4}{5}$, and $\frac{5}{5}$). In this way, Michelle can look for patterns in her models that eventually lead to an efficient procedure for finding equivalent fractions based on understandings that grew out of her models. According to Van de Walle (2004), the "goal is to help students see that by multiplying (or dividing) the top and bottom numbers by the same number, they will always get an equivalent fraction" (p. 260).

Figure 7.7 Michelle's peppermint model partitioned into 20ths.



The importance of building this understanding through patterns and relationships in models and not just teaching the procedure directly cannot be overemphasized. Teachers have told us that because students are multiplying (or dividing) when they apply a procedure, they have a difficult time believing that equivalent fractions represent the same number (OGAP, 2007).

Additionally, some teachers indicated that they never understood the relationship among multiples, factors, partitioning of models, and equivalence that underpins finding common denominators and simplifying fractions. In their own words, they were applying an algorithm without understanding (OGAP, 2007).

Given these findings, it is no surprise that the CCSSM focuses strongly on using visual area models, fraction strips, and number lines to develop understanding of equivalence at grades 3 and 4.

CCSSM

The CCSSM and Equivalence

At grade 3, students use number lines to observe that many fractions can be located at the same point on a number line (see Figure 7.4). They learn that fractions at the same location on a number line are equal (equivalent) to each other. They also use visual area models and fraction strips to understand equivalent fractions. They begin to understand that fractions are equivalent if they are the same size or at the same location on a number line, and they are expected to recognize simple equivalent fractions $\left(\text{e.g.,} \frac{1}{2} = \frac{2}{4}; \frac{1}{3} = \frac{2}{6}\right)$.

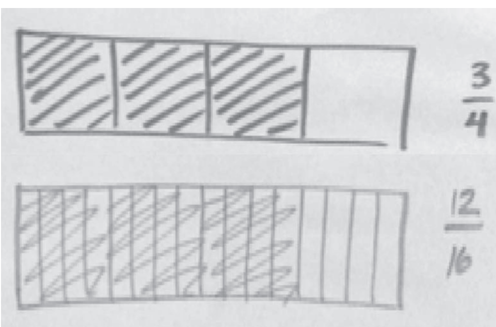
At grade 4, developing understanding and applying the concept of equivalence to compare fractions using visual models is a major focus. Students are expected to use visual area models and number lines to develop an understanding of why $\frac{a}{b}$ is equivalent to $\frac{n \times a}{n \times b}$, observing that even though the number and size of the parts in a whole are different, the fractions are equivalent. Later, fourth grade students use this understanding to compare fractions with different denominators using common denominators, common numerators, or by comparing fractions to a benchmark.

At grade 5, students are expected to use the understanding of equivalence to add and subtract fractions (including mixed numbers) with different denominators, applying their understanding that $\frac{a}{b}$ is equivalent to $\frac{n \times a}{n \times b}$ without relying on visual models or number lines (Common Core Standards Writing Team, 2013a).

From Visual Models to Efficient Procedures

Evan's visual area model in Figure 7.8 shows that $\frac{3}{4} = \frac{12}{16}$. Using visual models like Evan's and using number lines are important ways for students to start visualizing and understanding that fractions are equivalent if they represent the same area or are at the same location on a number line.

Figure 7.8 Evan's response showing that $\frac{3}{4}$ is equivalent to $\frac{12}{16}$.



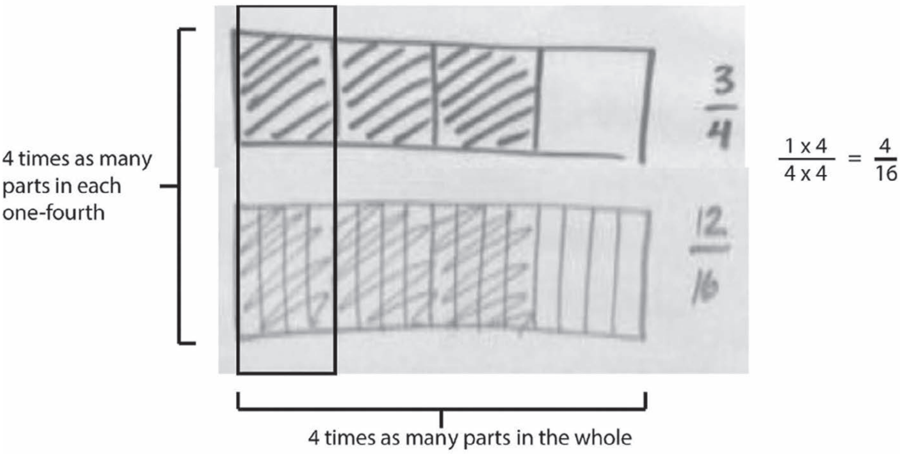
However, a transition from relying on visual models to using an efficient procedure for finding equivalent fractions requires students to understand the multiplicative relationships in their visual models. Figure 7.9 illustrates how Evan's visual model can be used to help understand why $\frac{3}{4} = \frac{12}{16}$, and this can be used to begin to help him understand why $\frac{a}{b}$ is equivalent to $\frac{n \times a}{n \times b}$.

Using visual models in this way helps students see the multiplicative relationships that underpin the procedure for finding equivalent fractions. To internalize and generalize this understanding, students need to interact with a variety of visual models and examine the multiplicative relationships to truly understand why $\frac{a}{b}$ is equivalent to $\frac{n \times a}{n \times b}$.

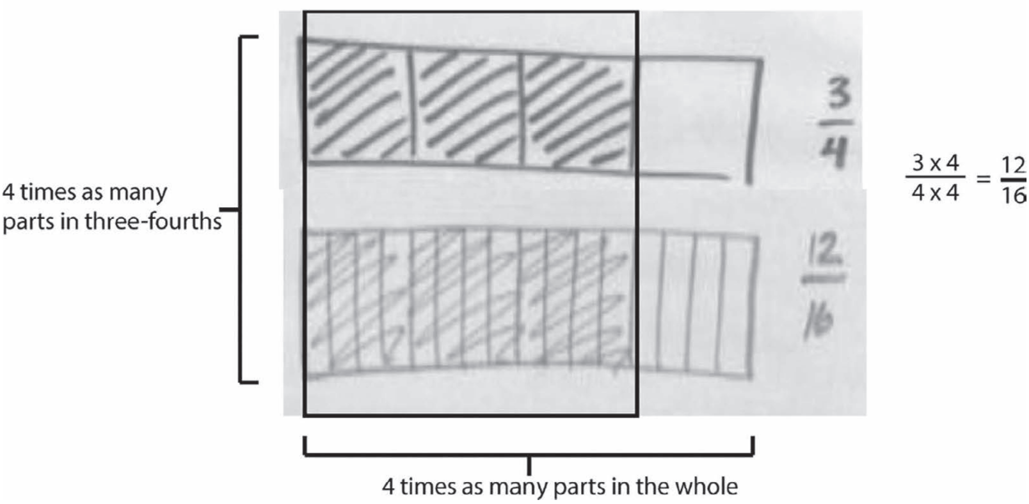
Another way to understand why $\frac{a}{b}$ is equivalent to $\frac{n \times a}{n \times b}$ is using the identity property for multiplication: A number does not change its value when multiplied or divided by 1 (e.g., $4 \times 1 = 4$; $4 \div 1 = 4$). In Evan's visual model, we noticed that both the numerator and the denominator were

Figure 7.9 Understanding the multiplicative relationships that form the foundation for why $\frac{a}{b}$ is equivalent to $\frac{n \times a}{n \times b}$.

FIRST: Using Evan's visual model notice that when going from fourths to sixteenths there are 4 times as many parts in each fourth and four times as many parts in the whole. Also notice that each sixteenth is a fourth the size as fourths.



NEXT: Notice that this relationship remains the same in his representation of $\frac{3}{4} = \frac{12}{16}$. That is, there are 4 times as many parts in three-fourths and 4 times as many parts in the whole.



multiplied by 4. Multiplying both the numerator and denominator by 4 is the same as multiplying by 1 because $\frac{4}{4} = 1$.

Therefore, multiplying $\frac{3}{4} \times \frac{4}{4}$ (or 1) does not change the value of $\frac{3}{4}$; it just changes the name to $\frac{12}{16}$. You will notice in Figure 7.4 that $\frac{3}{4}$ and $\frac{12}{16}$ are at the same location on the number line. Following the same line of thinking, dividing by 1 does not change the value of $\frac{12}{16}$, and one can find an equivalent fraction by dividing both the numerator and the denominator by 4. That is, $\frac{12 \div 4}{16 \div 4} = \frac{3}{4}$.

Over time, and with enough examples, students should see this multiplicative pattern consistent with all equivalent fractions that leads to the strategy of multiplying (or dividing) both the numerator and the denominator by the same number.

To help students believe that equivalent fractions are different names for the same number, point out the visual cues for equivalence in the various visual models. In area models and fraction strips, equivalent fractions are represented by the same region, as in Evan's and Michelle's responses. On number lines, equivalent fractions are located at the same place (Figure 7.4), and, as you will see in the discussion related to sets of objects in Figures 7.10 to 7.12, equivalent fractions result in the same count of the objects in the fractional parts.

Figures 7.10 to 7.12 illustrate $\frac{1}{4}=\frac{2}{8}=\frac{4}{16}$ using a set of 32 apples partitioned into fourths (Figure 7.10), eighths (Figure 7.11), and sixteenths (Figure 7.12). Notice that the number of parts changes, not the number of apples that are circled; that is, $\frac{1}{4}$ of 32 apples is 8 apples, $\frac{2}{8}$ of 32 apples is 8 apples; and $\frac{4}{16}$ of 32 apples is 8 apples, because $\frac{1}{4}=\frac{2}{8}=\frac{4}{16}$. This is another way, in addition to

Figure 7.10 Circled here are $\frac{1}{4}$ of the 32 apples. One-fourth of 32 apples is 8 apples.

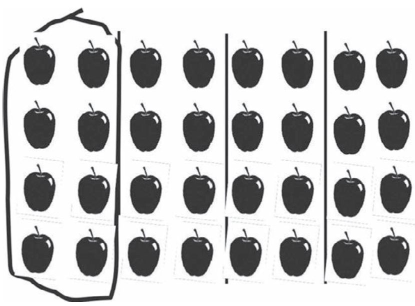
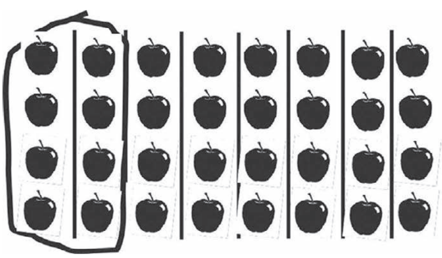
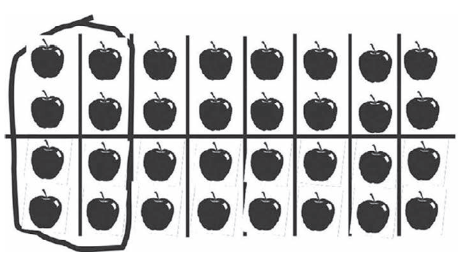


Figure 7.11 One column contains $\frac{1}{8}$ of the apples. Circled here are $2\left(\frac{1}{8}\right)=\frac{2}{8}$ of the apples. Two-eighths of 32 apples is 8 apples.



$$\frac{1}{4} \text{ x } \boxed{\frac{2}{2}} = \frac{2}{8}$$

Figure 7.12 One part contains $\frac{1}{16}$ of the apples. Circled here are $4\left(\frac{1}{16}\right)=\frac{4}{16}$ of the apples. Four-sixteenths of 32 apples is 8 apples.



$$\frac{1}{4} \text{ x } \boxed{\frac{4}{4}} = \frac{4}{16}$$

number lines and area models, to show that the value of equivalent fractions is the same and to illustrate again the mathematics essential to an understanding of equivalent fractions and the importance of equipartitioning (discussed in Chapter 4).

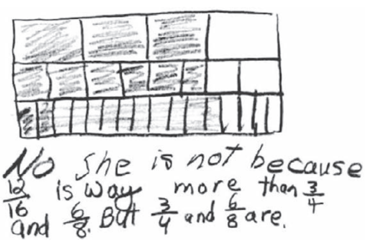
Researchers have found that instruction that helps students to move flexibly between representations (spoken and written words [two-fifths], pictorial representations, manipulatives, contexts, and symbols) and within representations (e.g., $\frac{3}{4} = \frac{6}{8}$) will help students move toward equivalence reasoning that becomes free of the need to model (Post et al., 1985).

To illustrate what it means to move flexibly between representations, let's return to Michelle's model comparing $\frac{2}{5}$ and $\frac{3}{10}$ (Figure 7.6). If Michelle is to add $\frac{2}{5} + \frac{3}{10}$ (using symbols), she would need to apply her understanding derived from the model that $\frac{2}{5} = \frac{4}{10}$. This would allow her to add four-tenths + three-tenths = seven-tenths (represented in words). Four-tenths + three-tenths = seven-tenths is adding up units in the same way that 3 hours + 2 hours = 5 hours (Gross & Gross, 1999). Converting both fractions to tenths allows Michelle to add fractions represented by the same-sized pieces in her model, just like adding hours to hours. In this way, intentional connections are made between the words, models, and symbols that bring meaning to each.

Because of the expectation at grade 5 in the CCSSM that students use equivalent fractions (common denominators) to add and subtract fractions, students need a firm understanding of equivalence, as well as an efficient procedure to find equivalent fractions. Students who continue to rely solely on models to solve problems will be at a disadvantage (Figure 7.13).

Figure 7.13 Kyle's response. Kyle's response shows the limitation of visual modeling as a strategy for determining equivalence. In this case, partitioning accurately to sixteenths is a limiting factor.

Kim said that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$ and to $\frac{12}{16}$. Is Kim correct?



Kyle's response (Figure 7.13) shows the potential limitation of a visual model based solution. Although all three fractions are equivalent, Kyle's inability to accurately partition 16ths (understandably so) led him to incorrectly conclude that $\frac{12}{16}$ is not equivalent to $\frac{3}{4}$.

Conversely, Kieren (as cited in Huinker, 2002, pp. 73–74) found that premature experience with formal procedures may lead to symbolic knowledge that is not based on understanding or connected to the real world. This may impede students' number and operation sense. Recall the student solutions from Chapter 1 (Figure 1.7) that used inappropriate whole number reasoning to add numerators or denominators rather than considering fractions as quantities. Thus, the effective use of visual models is essential but not sufficient. Instruction should focus on using visual models to help students develop generalizable and efficient strategies.

The *OGAP Fraction Progression* and Equivalence

Fractional Strategies related to equivalence include the use of common denominators when ordering and comparing fractions, adding and subtracting fractions with unlike denominators, and when using the common denominator strategy for division.

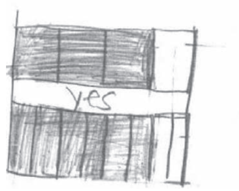
Corey's response in Figure 7.3 is an example of a fractional strategy. His solution suggests an understanding of the relationship between fifths and tenths.

Transitional Strategies for equivalence are evidenced by use of visual models when comparing or ordering fractions or when demonstrating that two fractions are equivalent. Daniel's solution in Figure 7.14 is an example of effectively using a model to determine whether two fractions are equivalent.

Another example of a transitional strategy is Adam's response, shown in Figure 9.15 in Chapter 9, Number Lines and Fractions. Adam used a number line to show that $\frac{8}{12}$ and $\frac{2}{3}$ are equivalent and are thus located at the same point on the number line.

Figure 7.14 Daniel's solution contains evidence of effectively using a visual model to determine whether $\frac{3}{4}$ is equivalent to $\frac{6}{8}$.

Kim said that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$ and to $\frac{12}{16}$. Is Kim correct?



Student understanding of equivalence is dependent upon the concepts developed at the *Transitional Strategies* level through students' use of number lines and other visual models. This work helps students understand that equivalent fractions are different names for the same value, located at the same location on a number line, and represent the same area in an area model. As student understanding of equivalence deepens, they can use what they learned from working with visual models to identify equivalent fractions and develop fluency with strategies at the fractional level.

Early Transitional strategies are characterized by use of a visual model in which there is an error, often related to the size of the wholes or with partitioning. Kyle's response in Figure 7.13 is an example of this. His model accurately shows the relationship between fourths and eighths; however, partitioning errors led Kyle to state that $\frac{12}{16}$ is "way more than $\frac{3}{4}$."

Equivalence has a dual role on the progression. It is an important component at the *Fractional Strategies* level: Students find equivalent fractions, use equivalence to solve problems, and use efficient algorithms based on equivalent fractions. In addition, as shown in the vertical arrow on the right side of the *OGAP Fraction Progression*, understanding of equivalence is an essential concept for moving students from *Transitional* to *Fractional Strategies*.

Go To

See Chapter 11, Addition and Subtraction of Fractions, for more information on developing student procedural and conceptual knowledge of addition and subtraction of fractions.

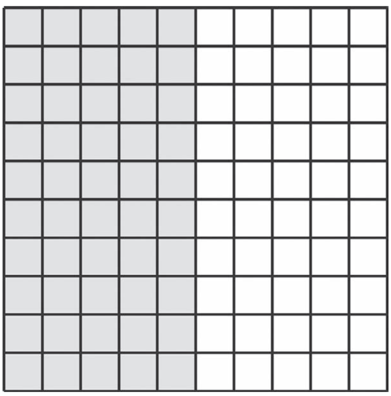
Fraction-Decimal Equivalence

Fractions can also be expressed as decimal fractions. Decimals are a way to write fractional amounts in our base-ten number system, with the whole number units to the left of the decimal point and the fractional amounts to the right of the decimal point ($3.8 = 3\frac{8}{10}$; 62.35 is $62\frac{35}{100}$).

In the base-ten number system, the number of digits to the right of the decimal point indicates the power of ten (10, 100, 1000, etc.) in the denominator of an equivalent fraction (e.g., $0.8 = \frac{8}{10}$; $0.85 = \frac{85}{10 \times 10} = \frac{85}{100}$; $0.085 = \frac{85}{10 \times 10 \times 10} = \frac{85}{1000}$).

Conceptual understanding of decimal fractions is built upon an understanding of equipartitioning and equivalence along with the base-ten number system. Visual models, such as the 10×10 grids shown in Figure 7.15, are an important way to connect these concepts. The larger 10×10 square has been equipartitioned into 10 rows or columns and then again into 100 smaller squares. If the 10×10 grid is equal to 1, then each row or column is $\frac{1}{10}$ of 1 and each square is $\frac{1}{100}$ of 1. The shaded portion in Figure 7.15 can be seen as $\frac{1}{2}, \frac{5}{10}$, 0.5, $\frac{50}{100}$, and 0.50.

Figure 7.15 A 10×10 grid can be used to show the equivalence of $\frac{1}{2}, \frac{5}{10}$, 0.5, $\frac{50}{100}$, and 0.50.

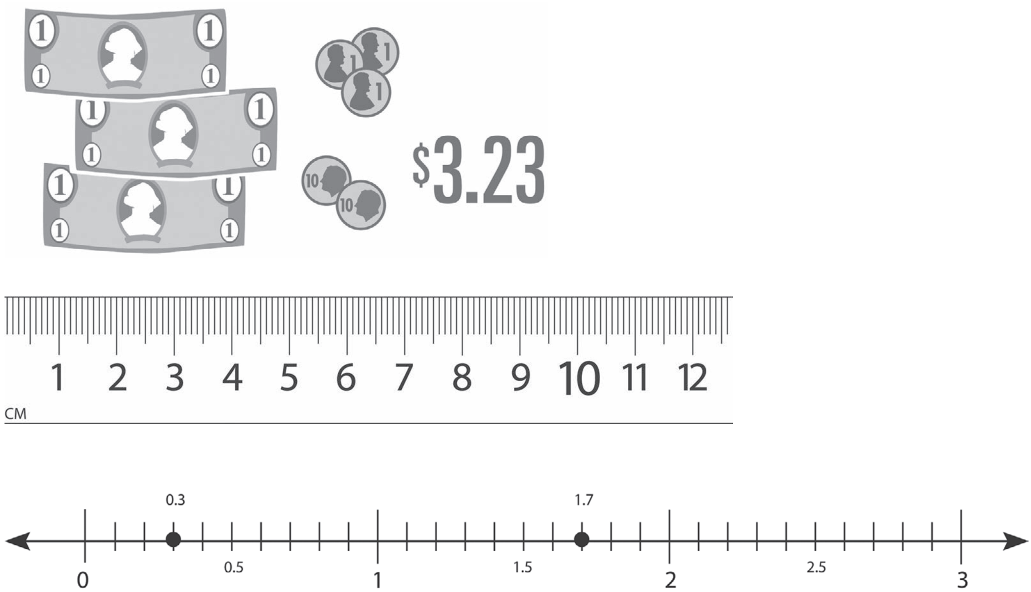


Experience naming these visual models with equivalent forms of both fractions and decimals is important because some students do not believe that a single value can have different symbolic representations (Hiebert & Wearne, 1986). In addition, many students do not believe that numbers such as 0.5 and 0.50 can be equivalent because they overgeneralize the fact that appending a zero to a whole number increases it by a factor of ten (Hiebert & Wearne, 1986; Karp et al., 2014). This concept of equivalence becomes important when zeros need to be appended in order to add or subtract decimals with different number of digits to the right of the decimal point (e.g., 0.3 – 0.156 = 0.300 – 0.156).

There are several additional models (shown in Figure 7.16) that can help students make sense of decimal numbers by drawing on fractional understanding. Number lines, discussed in Chapter 9, are an important visual model to help students make sense equipartitioning whole numbers into tenths.

To understand tenths and hundredths, students can also build on their understanding of money (1 dollar is equivalent to 10 dimes or 100 pennies, so 1 dime is 1/10 of a dollar and 1 penny is 1/100 of a dollar). A meter stick can be used to help students understand thousandths (1 meter = 100 centimeters = 1000 millimeters). In all these models, the repeated partitioning of units into ten equal parts is shown to help students make sense of the equivalence between fractions and decimals and the relative magnitude of decimal amounts.

Figure 7.16 Models for decimals.



CCSSM and Decimals

In the CCSSM, students are introduced to decimal notation in grade 4, when they are expected to express fractions with denominators of 10 and 100 as decimals, and to express fractions with denominators of 10 as equivalent fractions with denominators of 100 (e.g., $\frac{4}{10} = \frac{40}{100}$) and use this to add fractions with different denominators ($\frac{4}{10} + \frac{52}{100} = \frac{40}{100} + \frac{52}{100} = \frac{92}{100}$). This understanding of equivalent fractions sets the foundation for adding and subtracting with decimals (0.4 + 0.52 = 0.40 + 0.52 = 0.92).

In grade 5 students use fraction notation for decimals to the thousandths and use their understanding of equivalence to convert metric units (30 centimeters = $\frac{30}{100}$ meter = 0.30 meter = 0.3 meters). Fraction decimal equivalence provides the foundation for performing operations and understanding the system of rational numbers in grades 6–8 (including the equivalence between fractions, decimals, and percents). See *A Focus on Ratio and Proportion: Bringing Mathematics Education Research to the Classroom* (Petit et al., 2020) for more on developing students' understanding of percent.

Chapter Summary

This chapter focused on the concept of equivalence and research related to developing an understanding of equivalence. Examples and discussions focused on:

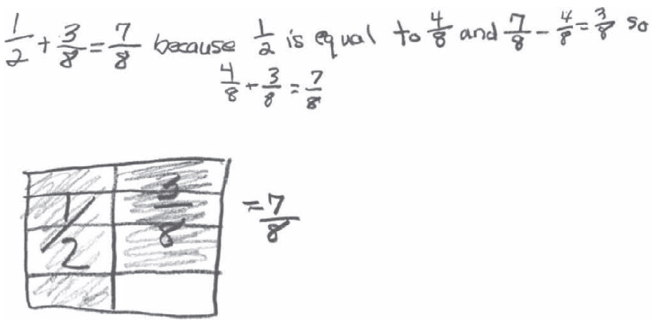
- The meaning of equivalence
- Using models to develop conceptual understanding of equivalence
- The importance of transitioning students by the end of fifth grade to efficient and generalized strategies for finding equivalent fractions, comparing fractions, ordering fractions, and adding and subtracting fractions.

Looking Back

1. What is the evidence in Emma's response in Figure 7.17 that demonstrates an understanding of equivalence in this situation?

Figure 7.17 Emma's response.

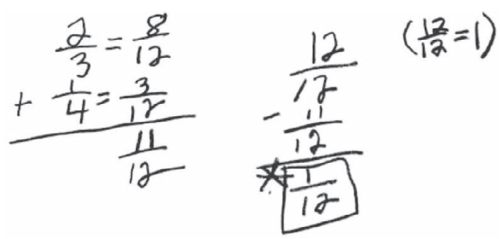
Name one fraction that can be added to $\frac{1}{2}$ to get a sum of $\frac{7}{8}$.



2. Use models to address the following:
   a. Illustrate that $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{8}{12}$ are equivalent using area models, set models, and number lines.
   b. Name one more fraction that is equivalent to $\frac{2}{3}$. Adapt one of your models in part a to show that the fraction is equivalent to $\frac{2}{3}$.
3. Review Kenny's response (Figure 7.18) and then answer the questions that follow.

Figure 7.18 Kenny's response.

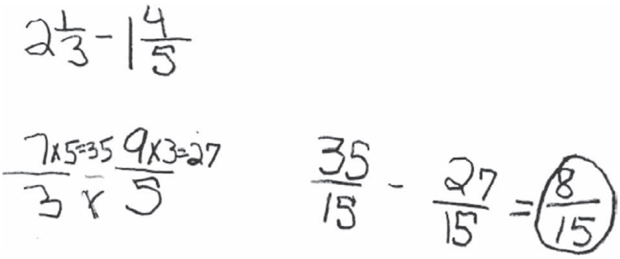
Tina ate $\frac{2}{3}$ of her candy and gave $\frac{1}{4}$ of her candy to her sister. She saved the rest of her candy. What is the fractional part of the candy that Tina saved?



   a. What is the evidence in Kenny's response that demonstrates an understanding of equivalence in this situation?
   b. How might a student select a model to use in solving this problem? Show how the model you select can help build an understanding of equivalence.
4. Chris accurately calculated the distance in the problem in Figure 7.19.
   a. What is the evidence in Chris's response that he understands equivalence?
   b. What concerns do you have about his solution?

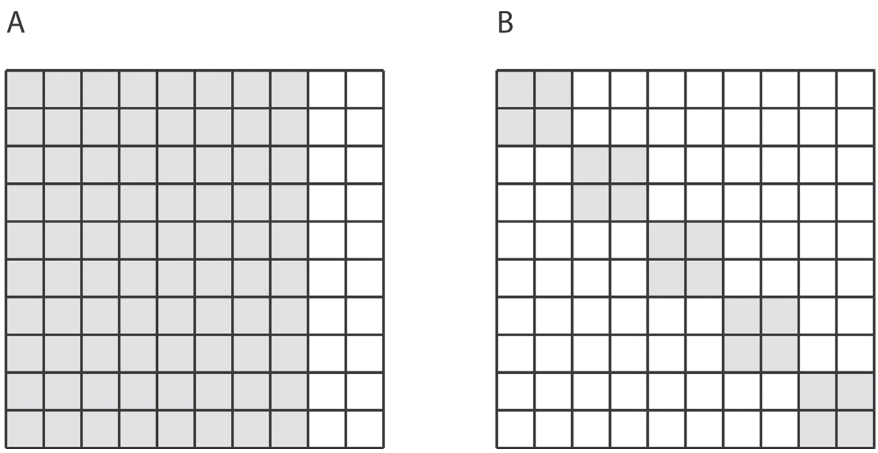
Figure 7.19 Chris's response.

Billy drove $2\frac{1}{3}$ miles from his home to work. His car broke down $1\frac{4}{5}$ miles from work. How far was he from home?



5. Name the portion of each square in Figure 7.20 that is shaded, using two fractions and two decimals.

Figure 7.20 Identify the shaded portion of the 10 × 10 grids.



6. Case Study: Building understanding of equivalence by facilitating a discussion using student solutions.

Ms. Smith, a fourth grade teacher, has been working with her class on fraction equivalence. To date her instruction has focused on the meaning of equivalence and using context to help students identify fractions equivalent to $\frac{1}{4}$ and $\frac{1}{2}$.

Ms. Smith's goals for today's lesson were for students to use fraction visual models to:

- Develop understanding that equivalent fractions are different names for the same value/quantity.
- Lay the foundation for understanding the multiplicative relationship between equivalent fractions.

To accomplish these goals, her lesson included opportunities for students to work with equivalent fractions (e.g., $\frac{1}{3} = \frac{2}{6}$). Students were asked to draw visual models to represent the different pairs of equivalent fractions and describe patterns and relationships they noticed (See Figures 7.8 and 7.9 for examples).

To gather evidence of her students' understandings related to the two lesson goals, Ms. Smith administered the following exit question at the end of the lesson. Together with what she observed as the students were working in small groups during the lesson, the evidence from the exit question will inform her plan for tomorrow's lesson.

6a) Study and solve the exit question. Consider the different solutions you would expect to see from students.

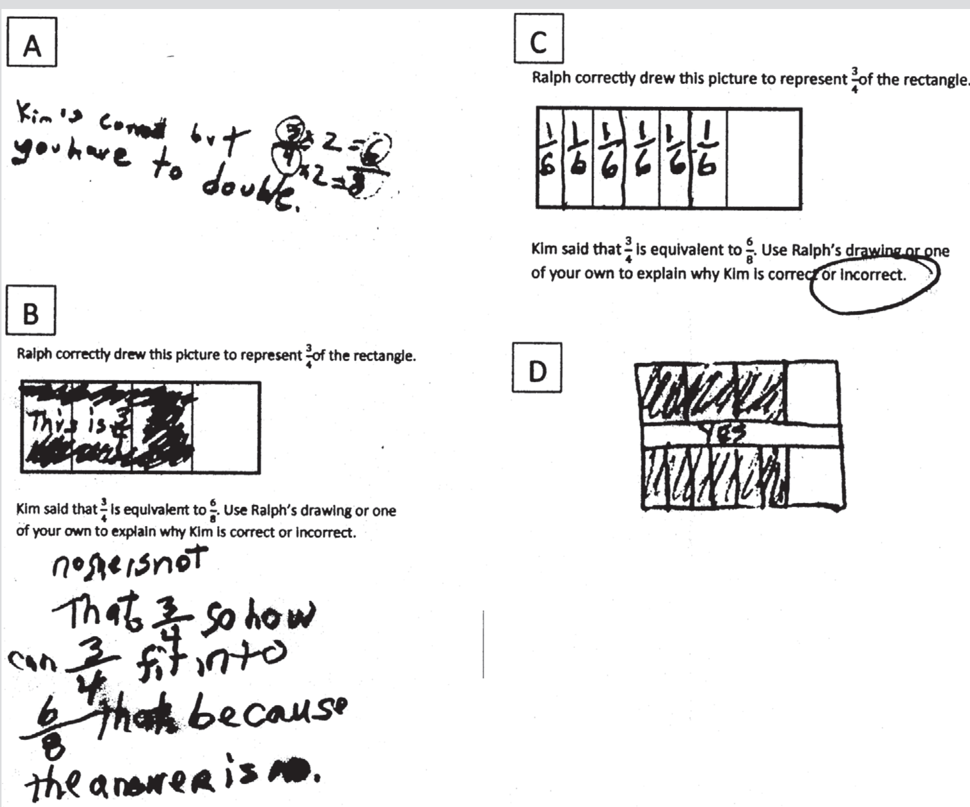
Figure 7.21 Ms. Smith's exit question.

Ralph correctly drew this picture to represent $\frac{3}{4}$ of the rectangle.

Kim said that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$. Is Kim correct? Use Figure 7.21, one of your own, or an explanation to explain why Kim is correct or incorrect.

6b) Analyze Evidence in Student Responses using the *OGAP Sort*. Study the four sample solutions to the exit question found in Figure 7.22. What does each solution

Figure 7.22 Four sample student responses to Ms. Smith's exit question.



suggest about the student's understanding that equivalent fractions are different names for the same number and the multiplicative relationships that exist between equivalent fractions?

6c) Record your analysis of each solution in Table 7.1.

- What level on the progression best represents each solution?
- List any errors and/or underlying issues evidenced in each student response.

Table 7.1 Record sheet for progression levels for student solutions in Figure 7.22

Fractional	*Transitional*	*Early Fractional*	*Non-Fractional*
Underlying Issues?			

6d) Use the evidence you recorded in Table 7.1 **to answer the questions**:

1. What is evidence of developing understanding that future instruction can build upon?
2. What are some errors or misconceptions that future instruction needs to address?
3. Identify potential next instructional steps?

Ms. Smith's response to the evidence she collected in the *OGAP Sort*

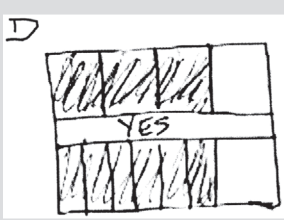
Ms. Smith decided to use student solutions from the exit question to begin the next day's lesson. She wanted to continue working on the goals of the lesson, that equivalent fractions are different names for the same value and to understand the multiplicative relationships that exist between equivalent fractions. Ms. Smith plans to project a pair of carefully chosen student solutions that will form the basis for a targeted class discussion. She decided to begin the discussion with solutions D and G.

Analyze Ms. Smith's lesson plans by answering the questions:

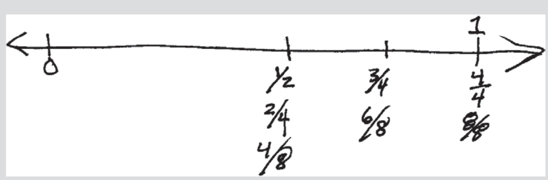
6e) Why might Ms. Smith have chosen Solutions D and G as the basis for a class discussion focused on the concept that equivalent fractions are different names for the same value?

Figure 7.23 Solutions D and G.

Solution D



Solution G



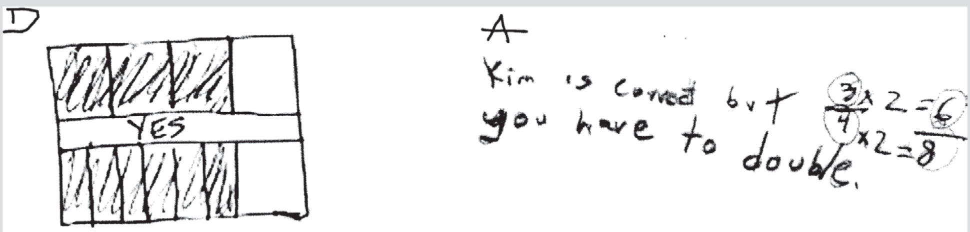
6f) Write two or more questions Ms. Smith could ask about Solutions D and G that focus on the idea that equivalent fractions are different names for the same value. How might these questions engage all students including those whose responses to the exit question were early or non-fractional.

Next Ms. Smith focused a class discussion on Solutions A and D in Figure 7.24. She wanted to use this pair of fractions to focus student thinking on the multiplicative relationships that exist between equivalent fractions.

6g) Why might this pair of student solutions be a good choice for this purpose?

Figure 7.24 Solutions D and A.

Solution D Solution A



6h) Using solutions D and A write two or more questions to ask students that would help them see the multiplicative relationship between these equivalent fractions.

Instructional Link—Your Turn

Use the prompts in Table 7.2 to help you think about how your instruction and mathematics program provide students with the opportunity to develop understandings of equivalence.

Table 7.2 Instructional Link—Strategies to Support Development of Concepts Related to Equivalence

Do you or does your program:	*Yes/No*
1. Use visual models to build understanding of equivalence?	
2. Use visual models to build understanding of decimals and fraction-decimal equivalence?	
3. Transition from visual models to a generalized understanding of equivalence for comparing and ordering fractions and decimals?	
4. Transition from visual models to a generalized understanding of equivalence for adding and subtracting fractions?	

Identify any gaps between your instruction (including what your program offers) and what should be addressed based on the research from this chapter.

8
Comparing and Ordering

Big Idea

- Students should develop a range of strategies for comparing and ordering fractions.
- Students often misapply whole number and/or fractional reasoning when comparing and ordering decimals.
- Comparing and ordering decimals should be built upon an understanding of fractions as numbers, visual models, and relative magnitude.

This chapter focuses on comparing and ordering positive fractions and decimal fractions: fractions written with a numerator and a denominator that are both positive. Ultimately, however, students should also be able to:

- Compare a positive fraction with a negative fraction (any positive fraction is greater than any negative fraction)
- Compare and order negative fractions and decimals $\left(\text{e.g.,} -\frac{3}{4} < -\frac{1}{3}\right)$.

See Chapter 9, Number Lines, for more discussion of comparing and ordering negative fractions using number lines.

Note that many of the examples in this chapter involve making comparisons that are not in a context, $\left(\text{e.g., which fraction is closest to 1}: \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}\right)$. Unless otherwise stated, it is assumed that the fractions being compared are associated with the same-sized whole. See Chapter 3, What Is the Whole?, for more about fractions and their associated wholes.

Comparing two fractions involves determining the relative magnitude of the two fractions; that is, are they equal to each other, or is one fraction less than or greater than the other? (See Figure 8.1.)

Figure 8.1 Examples of comparisons of fraction pairs.

$$\frac{3}{4} > \frac{1}{3}$$

$$2\frac{1}{3} < 2\frac{3}{4}$$

$$\frac{3}{4} = \frac{6}{8}$$

Ordering fractions involves putting a set of fractions in order from the least to the greatest or the greatest to the least. (See Figure 8.2.)

DOI: 10.4324/9781003185475-8

Figure 8.2 An example of fractions ordered from least to greatest.

$$\frac{1}{6}, \frac{1}{3}, \frac{3}{8}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$$

"Fraction order and equivalence ideas are fundamentally important concepts. They form the framework for understanding fractions and decimals as quantities that can be operated on in meaningful ways" (Post et al., 1993, p. 15).

Rewriting fractions as equivalent fractions with common denominators is one way to compare and order fractions. Students sometimes learn to perform this procedure, however, without considering the meaning or relative magnitude of the fractions they are comparing (See Figure 1.1 in Chapter 1). In addition to using common denominators, students should develop a variety of reasoning strategies to compare and order fractions.

Study the fraction pairs in Figure 8.3. Explain which fraction is larger by using reasoning rather than rules or procedures.

Figure 8.3 Which fraction is larger? Use reasoning rather than rules or procedures to justify.

1. $\frac{5}{6}\ \frac{1}{6}$
2. $\frac{3}{10}\ \frac{3}{4}$
3. $\frac{7}{8}\ \frac{5}{6}$
4. $\frac{9}{16}\ \frac{3}{8}$

To determine which fraction is greater, you may have thought about the size of the parts or pieces, real life contexts like cooking or measuring, unit fractions, equivalent fractions or decimals, or common benchmark fractions (more fully explained later in the chapter). These reasoning strategies are based on understanding of the meaning of fractions and are important for understanding fraction magnitude and order.

Classes of Fractions

The relationship between the numerators and denominators of the fractions being compared or ordered may have impacted the strategy you chose to use. In fact, there are three different classes of fraction relationships, and these relationships impact the types of reasoning that students use as they compare and order fractions:

- Fractions with different numerators, but same denominators $\left(\text{e.g., for } \frac{3}{6} \text{ and } \frac{5}{6}, \frac{3}{6} < \frac{5}{6}\right)$
- Fractions with the same numerators, but different denominators $\left(\text{e.g.,for } \frac{3}{8} \text{ and } \frac{3}{5}, \frac{3}{8} < \frac{3}{5}\right)$
- Fractions with different numerators and different denominators $\left(\text{e.g., for } \frac{3}{8} \text{ and } \frac{5}{6}, \frac{3}{8} < \frac{5}{6}\right)$.

(Behr et al., 1984)

One can find examples of each of these classes of fractions in Figure 8.3. Number 1 involves comparing fractions with the same denominators. Number 2 has fractions with the same numerators but different denominators. Numbers 3 and 4 involve comparing fractions with different numerators and denominators. As you read this section on the development of students' reasoning strategies as they compare and order fractions, think about the role these numerator-denominator relationships have on student strategies.

Developing a Range of Reasoning Strategies When Comparing and Ordering Fractions

Researchers have found that students use five types of reasoning when they successfully compare and order fractions.

1. Reasoning with unit fractions (e.g., $\frac{1}{4} < \frac{1}{3}$ because fourths are less than thirds)
2. Extension of unit fraction reasoning to non-unit fractions (e.g., $\frac{7}{8} > \frac{4}{5}$ because $\frac{1}{8}$ [the distance $\frac{7}{8}$ is away from a whole] is smaller than $\frac{1}{5}$ [the distance $\frac{4}{5}$ is away from a whole])
3. Reasoning based on equipartitioning visual models (OGAP, 2005)
4. Reasoning by using a common benchmark or reference fraction, such as $\frac{1}{2}$
5. Reasoning involving equivalence. (Behr & Post, 1992)

Reasoning with Unit Fractions

Many students learn to compare fractions with common denominators, such as $\frac{5}{6}$ and $\frac{1}{6}$ in number 1 of Figure 8.3, by following a rule. That is, when the denominators are the same number, the fraction with the larger numerator is the larger fraction. While this rule works when the denominators are the same, it is not built on a unit fraction understanding of fractions that is important for understanding the meaning of a fraction. An alternative approach to comparing these fractions, developed through early experiences with equipartitioning (see Chapter 4), involves using unit fraction understanding: $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} > \frac{1}{6}$ or $5\left(\frac{1}{6}\right) > \left(\frac{1}{6}\right)$.

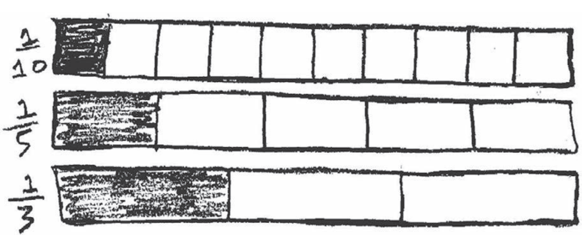
Why is this distinction important? One reason is that the comparing numerator rule may reinforce inappropriate whole number reasoning by focusing simply on the numerator, not on the fraction as a quantity. Additionally, some researchers indicate that "a child's understanding of the ordering of two fractions needs to be based on an understanding of the ordering of unit fractions" (Behr & Post, 1992, p. 21).

Recall from Chapter 1 that a *unit fraction* is defined as a fraction with numerator of 1 and with a denominator that is any positive whole number $\left(\text{e.g.,} \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{50}, \ldots \frac{1}{128}, \ldots\right)$. Unit fraction reasoning involves using an understanding of the relative size of unit fractions to solve problems. An important aspect of unit fractions is the fact that every fraction $\frac{a}{b}$ is composed of a copies of $\frac{1}{b}$ $\left(\text{e.g.,} \frac{5}{6} \text{ is 5 copies of } \frac{1}{6}\right)$.

Students gain an understanding of the relative magnitude of unit fractions when they interact with manipulatives and draw visual models to solve problems that involve comparing unit fractions.

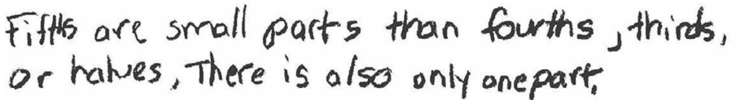
Figure 8.4 shows the response from a student who equipartitioned rectangles to compare the fractions $\frac{1}{10}$, $\frac{1}{5}$, and $\frac{1}{3}$. Teachers can capitalize on student-generated visual models like these by asking about the relationships in visual models that can lead to an understanding of the size of the parts in a whole based on the magnitude of unit fractions (e.g., "What do you notice about the size of the pieces in relationship to the size of the denominator in each of your diagrams?")

Figure 8.4 Visual area models partitioned into tenths, fifths, and thirds with $\frac{1}{10}$, $\frac{1}{5}$, and $\frac{1}{3}$ of each shaded.



When comparing $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ (Figure 8.5), Kayla used unit fraction reasoning. She described the relative sizes of the unit fractions and then related that information to the magnitudes of the fractions by saying that "fifths are smaller parts than fourths, thirds, or halves." Her comment that "there is also only one part" probably refers to the fact that each of the rectangles has one part shaded, that is, the fractions are unit fractions.

Figure 8.5 Kayla's response. Kayla used unit fraction reasoning when comparing $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.



Extension of Unit Fraction Reasoning to Non-Unit Fractions

Extending unit fraction reasoning means using what is known about unit fractions to compare and order fractions with:

- The same numerator but different denominators. For example, $\frac{1}{7} < \frac{1}{5}$, therefore $\frac{2}{7}$ must be less than $\frac{2}{5}$.
- Different numerators and different denominators, where the difference between the numerator and the denominator in each fraction is the same. For instance, the fractions $\frac{5}{7}$ and $\frac{3}{5}$ are $\frac{2}{7}$ and $\frac{2}{5}$, respectively, away from the whole. Five-sevenths is two-sevenths less than one whole $\left(1-\frac{5}{7}=\frac{2}{7}\right)$ and three-fifths is two-fifths less than one whole $\left(1-\frac{3}{5}=\frac{2}{5}\right)$. Because $\frac{2}{7} < \frac{2}{5}$, subtracting $\frac{2}{7}$ is "taking away" less than subtracting $\frac{2}{5}$. The less one subtracts from 1, the closer to 1 the difference will be, thus $\frac{5}{7} > \frac{3}{5}$.

One can also compare and order fractions with the same numerators but different denominators, for example $\frac{3}{8}$ and $\frac{3}{5}$, by extending unit fraction reasoning. Using the representations of $\frac{3}{8}$ and $\frac{3}{5}$ in Figure 8.6, one can see that eighths are smaller than fifths (given the same-sized whole). Because the number of eighths (three) and fifths (three) are the same (the number of parts shaded in each of the area visual models in Figure 8.6), and $\frac{1}{8} < \frac{1}{5}$, it follows that $\frac{3}{8} < \frac{3}{5}$.

Mike's solution (Figure 8.7) contains evidence of extended unit fraction reasoning. He describes the impact of the magnitude of the denominator on the size of the pieces. He then uses the size of $\frac{1}{7}$ and $\frac{1}{5}$ to compare $\frac{3}{7}$ and $\frac{3}{5}$. Finally, he concludes that $\frac{3}{7}$ is smaller than $\frac{3}{5}$ "because they have the same numerators."

Figure 8.6 The number of equal parts shaded is the same, but the size of each of the parts is not.

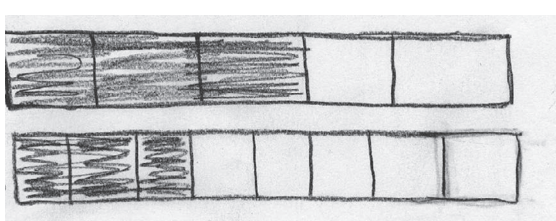
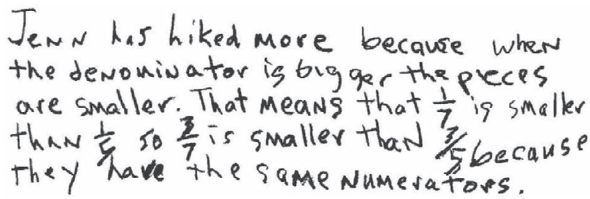


Figure 8.7 Mike's example of extending unit fraction understanding to non-unit fractions.

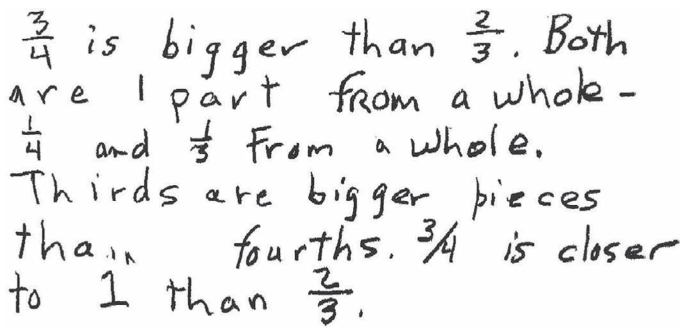
Linda hiked $\frac{3}{7}$ of the way up Mt. Mansfield. Jenn hiked $\frac{3}{5}$ of the way up Mt. Mansfield. Who hiked farther?



When comparing $\frac{3}{4}$ to $\frac{2}{3}$, Sam (Figure 8.8) extends his unit fraction reasoning. He compares fractions with different numerators and different denominators where the difference between the numerators and the denominators is the same. His explanation that "both are one part from a whole" can be rewritten as $1-\frac{3}{4}=\frac{1}{4}$ and $1-\frac{2}{3}=\frac{1}{3}$. He continues his explanation that "thirds are bigger pieces than fourths," that is, since thirds are bigger pieces than fourths and each fraction is one part from the whole, Sam concludes that "$\frac{3}{4}$ is closer to 1 than $\frac{2}{3}$."

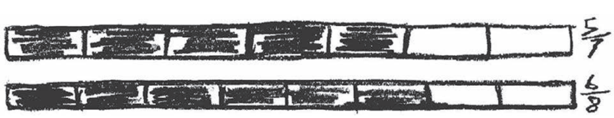
Figure 8.8 Sam's response. Sam's response shows evidence of using extended unit fraction reasoning by first recognizing that each fraction is one fractional part from a whole and then using the relative distance each fraction is from 1.

Mary is comparing the fractions $\frac{2}{3}$ and $\frac{3}{4}$. Mary thinks that both fractions are equal because they are both one part away from 1. Do you agree with Mary? Explain your thinking.



The same reasoning can be used when comparing fractions in which both fractions are two or more parts away from the whole. Using the area visual models in Figure 8.9 comparing $\frac{6}{8}$ to $\frac{5}{7}$, one can see that eighths are smaller pieces than sevenths. Since $\frac{1}{8}<\frac{1}{7}$, then $\frac{2}{8}<\frac{2}{7}$. Therefore, $\frac{6}{8}$ is closer to 1 than $\frac{5}{7}$.

Figure 8.9 Visual area model comparing $\frac{6}{8}$ to $\frac{5}{7}$.

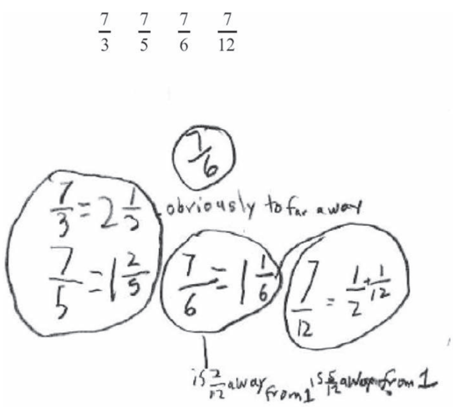


Comparing to Benchmarks

Another reasoning strategy that students can use to compare and order fractions is to use reference points or benchmarks. Reference points such as 0, $\frac{1}{2}$, and 1 can be useful benchmarks for students to consider as they order and compare fractions. To compare a fractional amount to a benchmark, students integrate their understanding of equivalence, unit fractions, and the underlying relationships between the numerator and the denominator. For example, in Tyler's response (Figure 8.10) there is evidence of using equivalence when converting improper fractions into mixed numbers and again when $\frac{7}{6}$ is referred to as "$\frac{2}{12}$ away from 1." In addition, there is evidence of understanding the contribution of the numerator and denominator by recognizing that $\frac{7}{12}$ or $7\left(\frac{1}{12}\right)$ is not 5 away from one, but rather "$\frac{5}{12}$ or $5\left(\frac{1}{12}\right)$ away from 1."

Figure 8.10 Tyler's response shows evidence of applying understanding of equivalence, unit fractions, and the underlying relationships between the numerator and the denominator.

Which fraction is closest to 1? Show your work.



As students become flexible in using benchmarks and combining their understanding of equivalence and unit fractions, they will be able to compare and order unfamiliar fractions such as $\frac{13}{24}$ and $\frac{24}{50}$, noticing that $\frac{13}{24}$ is $\frac{1}{24}$ greater than $\frac{1}{2}$ and $\frac{24}{50}$ is less than $\frac{1}{2}$ by $\frac{1}{50}$.

Equivalence: Common Denominators

Although Tyler could have relied on unit fraction reasoning without the use of equivalence to solve the problem in Figure 8.10, there are some fraction combinations that are difficult to compare using reasoning strategies alone. In addition, some fraction comparisons lend themselves to using common denominators for easy comparisons. For example, question 4 in Figure 8.3 can easily be solved using either benchmark reasoning ($\frac{9}{16} > \frac{3}{8}$ because $\frac{9}{16}$ is $\frac{1}{16}$ greater than $\frac{1}{2}$ while $\frac{3}{8}$ is $\frac{1}{8}$ less than $\frac{1}{2}$) or equivalence ($\frac{9}{16} > \frac{3}{8}$ because $\frac{9}{16} > \frac{6}{16}$).

Behr and Post (1992) found that "ultimately the problem of ordering two general fractions (different numerators and denominators) rests on considerable knowledge of fraction equivalence" (p. 23).

See Chapter 7, Equivalence, for more discussion on developing understanding of equivalence.

Comparing Decimals

Using reasoning strategies to compare decimals involves reasoning about their relative magnitude and considering the place value of each digit. Many of the errors students make when comparing and ordering decimals come from overgeneralization from whole numbers:

- Some students compare the digits to the right of the decimal point as if they are whole numbers (e.g., 0.23 is greater than 0.8, because 23 is greater than 8; or 15.6 is smaller than 15.45 because 6 is smaller than 45) (Sackur-Grisvard & Léonard, 1985).
- Some students select the number with more digits as the larger number (e.g., 0.257 is greater than 0.6) (Desmet et al., 2010; Hiebert & Wearne, 1986; Karp et al., 2014).
- Some students believe that annexing a zero makes the number larger (e.g., 0.560 is greater than 0.56) (Hiebert & Wearne, 1986)

Figure 8.11 shows an example of overgeneralizing from whole numbers to compare decimals. To order the given numbers, Connor appears to have thought about the quantities as whole numbers (8, 9, and 55).

Figure 8.11 Connor uses whole number reasoning to compare decimals and percents.

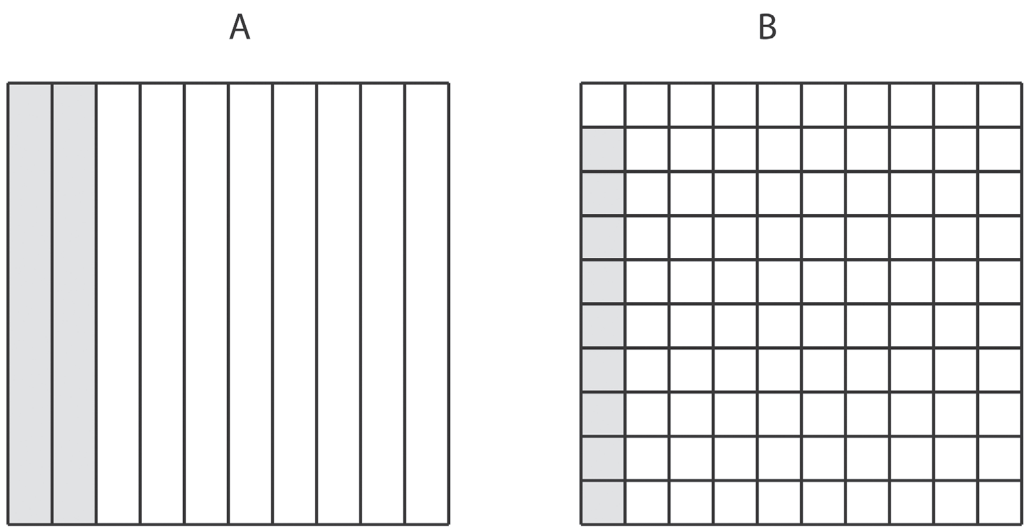
Trevor ordered the following numbers from smallest to largest. Is Trevor correct? Why or why not?

Trevor's Order
0.8 9% 0.55

yes I think the. takes place of %

In all these cases, the overgeneralization can be addressed by using visual models to make sense of decimal place value and relative magnitude. For example, Figure 8.12 shows how the area model can be used to help students compare decimals by reasoning about their size. The square on the left has been partitioned into 10 equal parts, so if the large square represents 1, the shaded region represents $\frac{2}{10}$ or 0.2 of 1. The square on the right has been partitioned into 100 equal parts, so the shaded region represents $\frac{9}{100}$ or 0.09. By comparing the shaded regions, one can see that 0.2 is larger than 0.09, because hundredths are smaller than tenths.

Figure 8.12 The area model can be used to show relative magnitude of decimal fractions.



Students are often taught a procedure for comparing decimals where they add zeros to the shorter quantity so that the numbers have the same number of digits (e.g., changing 0.2 to 0.20 to compare it to 0.09 by comparing 20 and 9). While this leads them to the correct answer, it can reinforce the misconception that the amount to the right of the decimal point can be treated like a whole number (Resnick et al., 1989). In contrast, anchoring students' understanding in visual models supports reasoning about relative magnitude.

CCSSM At grade 3, students coordinate unit fraction understanding with work on number lines and other visual models as they compare and order fractions and recognize that equivalent fractions name the same quantity and are found at the same location on a number line. In addition, students at grade 3 recognize fractions that are equivalent to whole numbers $\left(\text{e.g.,} \frac{5}{5} = 1; \frac{4}{1} = 4\right)$.

At grade 4, students extend this to comparing fractions with unlike denominators by creating common denominators or numerators and by comparing fractions to a benchmark. They justify their solutions with reasoning strategies and visual models. They also compare two decimals to hundredths by reasoning about their size and using visual models to justify conclusions.

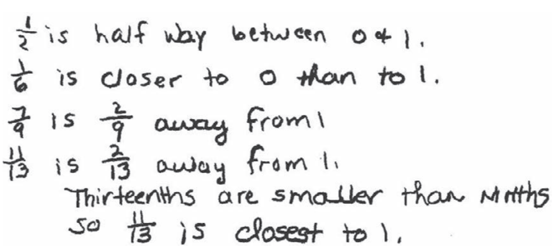
At grade 5, students apply their understanding of comparisons and equivalence to operations with fractions. They also compare two decimals to thousandths based on the meanings of the digits in each place.

Flexibility Over Time

With experience using visual models to help generalize ideas, students will move beyond hand-drawn visual models to use mental models as they generalize ideas to flexibly compare and order fractions and decimals. For example, Nicholas used a range of reasoning strategies in the solution shown in Figure 8.13. He first compared each fraction $\left(\frac{1}{2}, \frac{1}{6}, \frac{11}{13}, \frac{7}{9}\right)$ to the benchmark 1. He then eliminated $\frac{1}{2}$ and $\frac{1}{6}$ as being farther from 1 than $\frac{11}{13}$ or $\frac{7}{9}$. Finally, he used his extended unit fraction reasoning to compare $\frac{11}{13}$ to $\frac{7}{9}$: "thirteenths are smaller than ninths so $\frac{11}{13}$ is closest to 1."

Figure 8.13 Nicholas's response. Nicholas used the benchmark 1 and extended unit fraction reasoning to identify $\frac{11}{13}$ as the largest fraction in the set.

Which fraction is closest to 1? $\frac{1}{2}$, $\frac{1}{6}$, $\frac{11}{13}$, or $\frac{7}{9}$



In Figure 8.14, Shaniqua determined the correct order of the numbers by using her understanding of equivalence and place value to express each amount as a fraction out of 100. Terrence shows understanding of relative magnitude, place value, and benchmarks, reasoning that 0.55 is close to one-half and 0.8 is close to 1. There is, however, a misconception about percent (that 9% is like 9 ones rather than 9 hundredths).

Figure 8.14 Shaniqua's and Terrence's Responses.

Trevor ordered the following numbers from smallest to largest. Is Trevor correct? Why or why not?

<u>Trevor's Order</u>

0.8   9%   0.55

Shaniqua's Response

$\frac{80}{100}$ .09 $\frac{55}{100}$

$\frac{9}{100}$

9%   .55   .8

tenths

hundres

Percent

(Per hundred)

Terrence's Response

no because .55 is just about half and .8 is almost full and 9% is like 9 1's

.55 .8 .9%

Teachers Need Flexibility, Too

The evidence in the 2005 OGAP Study indicated that the major strategy that students in grades 3 through 5 used on the post-assessment to compare and order fractions involved the use of visual models. While the use of visual models to compare and order fractions appeared to result in a decrease in the use of whole number reasoning as compared to the pre-assessment, researchers saw few examples of students using unit fraction reasoning, extended unit fraction reasoning, equivalence ideas, or benchmarks (OGAP, 2005). Researchers suspected that the students' focus on visual models over other reasoning strategies was directly related to instruction, perhaps because teachers themselves may not have fully understood a range of strategies to solve fraction comparison problems. To test the conjecture and measure the impact of the professional development focused on ordering and comparing fractions, teachers involved in the OGAP Fraction

Scale-Up (2007) solved the problem found in Figure 8.15 as a part of the teacher pre-/post-assessment in that study.

Figure 8.15 OGAP teacher pre-assessment question.

Which fraction is closest to 1? $\frac{1}{2}, \frac{7}{9}, \frac{11}{13}, \frac{1}{6}$
Solve this problem using three different strategies.

This problem was specifically designed to provide opportunities to be solved using any of the five strategies described in this chapter, with the use of a common denominator or visual models as the least efficient ways to solve this problem. Yet, two-thirds of the teacher pre-assessments sampled ($n = 67$) showed evidence of using only a visual model or common denominators as exemplified in the teacher response in Figure 8.16.

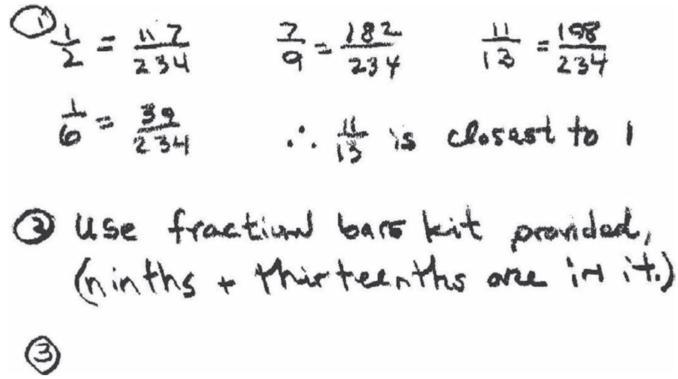
As a result of these findings, the OGAP professional development for teachers began to emphasize the use of a range of strategies described in this chapter.

To analyze the teacher post-assessment responses associated with the pre-assessment response in Figure 8.16, go to question 6 in Looking Back.

The *OGAP Fraction Progression* and Comparing and Ordering Fractions

Fractional Strategies for comparing and ordering fractions show evidence of students internalizing the use of benchmarks, unit fraction and extended unit fraction reasoning, and equivalence without having to draw visual models. At this level, students show flexibility across a range of problems and over time show evidence of selecting strategies based on the fractions being compared and the context for the comparison. Sam's response in Figure 8.8 is an example of a fractional strategy. Sam used extended unit fraction reasoning to determine that $\frac{3}{4} > \frac{2}{3}$.

Figure 8.16 Sample OGAP teacher pre-assessment, in which common denominators and a general description about the use of fractions bars were the only solutions used to identify the fraction closest to 1.



① $\frac{1}{2} = \frac{117}{234}$ $\frac{7}{9} = \frac{182}{234}$ $\frac{11}{13} = \frac{198}{234}$

$\frac{1}{6} = \frac{39}{234}$ $\therefore \frac{11}{13}$ is closest to 1

② Use fractional bars kit provided, (ninths + thirteenths are in it.)

③

Transitional Strategies show evidence of effectively using visual models and number lines to compare and order fractions. The solution in Figure 8.9 is an example of a transitional strategy. The two visual models used to compare $\frac{6}{8}$ to $\frac{5}{7}$ utilize the same size whole and are equipartitioned. The models clearly show that $\frac{6}{8} > \frac{5}{7}$.

Early Fractional Strategies are often characterized by attempts at using visual models to compare and order fractions where the wholes are either not equipartitioned or are not the same size. Kim's solution in Figure 2.21 in Chapter 2 is an example of an early fractional strategy. Kim used circles to compare $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$. Her visual models were not accurate enough to determine that $\frac{2}{3} > \frac{3}{5}$. Due to a difference in the size of the wholes and perhaps to inaccurate partitioning, Kim erroneously concluded that $\frac{2}{3} = \frac{3}{5}$.

Non-Fractional Strategies for comparing and ordering fractions are often evidenced by inappropriate whole number reasoning and by the application of rules without understanding. Figure 1.5 in Chapter 1, Understanding a Fraction as a Number, exemplifies a nonfractional strategy that utilizes inappropriate whole number reasoning.

Chapter Summary

This chapter focused on developing a range of strategies to order and compare different classes of fractions:

- reasoning with unit fractions (e.g., $\frac{1}{4} < \frac{1}{3}$ because fourths are smaller than thirds)
- extension of unit fraction reasoning to non-unit fractions (e.g., $\frac{7}{8} > \frac{4}{5}$ because $\frac{1}{8}$ away from a whole is smaller than $\frac{1}{5}$ away from a whole)
- reasoning based on visual models
- reasoning through the use of benchmark fractions such as $\frac{1}{2}$
- reasoning involving equivalence

Looking Back

1. Review Ted's response in Figure 8.17. While we cannot be sure, it is possible that Ted relied on a rule to compare $\frac{5}{6}$ to $\frac{3}{6}$. What are two reasoning strategies that Ted could have used to decide who planted more corn in their garden? Describe each.

Figure 8.17 Ted's response.

Sam and Don each have a garden. The gardens are the same size. $\frac{5}{6}$ of Don's garden is planted with corn. $\frac{3}{6}$ of Sam's garden is planted with corn. Who has more corn in his garden?

When the denominators are the
same just look at the numerators.
5 is bigger than 3. That means
that $\frac{5}{6}$ is bigger than $\frac{3}{6}$.

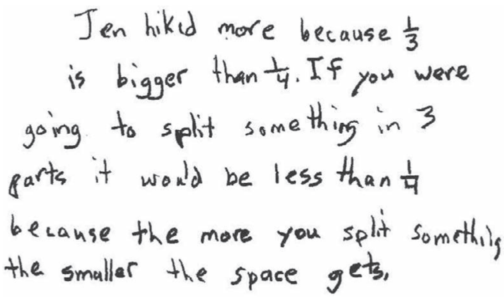
2. Review the fraction pairs in Figure 8.3 and answer the following questions.
   a. Which fraction pairs or sets of fractions provide the opportunity to use benchmarks to compare the fractions? Explain your choices.
   b. Which fraction pairs or sets of fractions provide the opportunity to use unit fraction reasoning to compare the fractions? Explain your choices.

3. Read through Mark's response to the problem in Figure 8.18.
   a. Why did Mark's reasoning result in a correct solution to the problem?
   b. Under what conditions would Mark's reasoning not work? Explain your answer with examples.

c. Provide a couple of examples of pairs of fractions you might ask Mark to compare to determine whether he can extend his unit fraction understanding to comparing other fractions. Provide a rationale for each of the fraction pairs.

Figure 8.18 Mark's response.

Linda hiked $\frac{1}{4}$ of the way up Mt. Mansfield. Jenn hiked $\frac{1}{3}$ of the way up Mt. Mansfield. Who hiked farther? Explain your answer using words and diagrams.

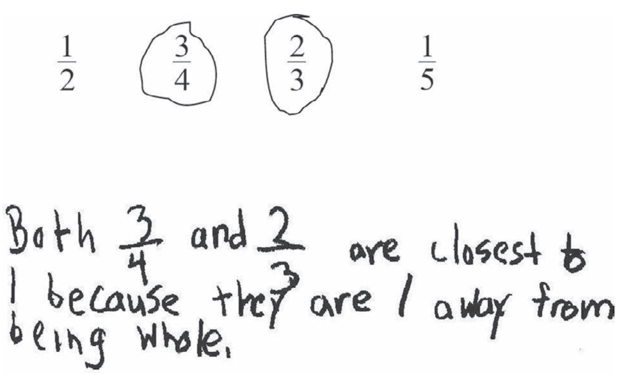


Jen hikd more because $\frac{1}{3}$ is bigger than $\frac{1}{4}$. If you were going to split something in 3 parts it would be less than $\frac{1}{4}$ because the more you split somethig the smaller the space gets,

4. Read through Tom's response to the problem in Figure 8.19 and answer the following questions.

Figure 8.19 Tom's response.

Which fraction is closest to 1?



$\frac{1}{2}$ $\frac{3}{4}$ $\frac{2}{3}$ $\frac{1}{5}$

Both $\frac{3}{4}$ and $\frac{2}{3}$ are closest to 1 because they are 1 away from being whole.

a. What misunderstanding led Tom to conclude that both $\frac{3}{4}$ and $\frac{2}{3}$ are closest to 1?
b. What additional questions might help Tom to understand why $\frac{3}{4}$ and $\frac{2}{3}$ are not the same distance from 1 even though they are both "1 away" from a whole?

5. Read through Kim's and Bob's responses to the same problem in Figures 8.20 and 8.21.

Figure 8.20 Kim's response.

Susan ate $1\frac{1}{2}$ cupcakes and Billy ate $\frac{9}{8}$ cupcakes.

Who ate more?

Use words or diagrams to explain your answer.

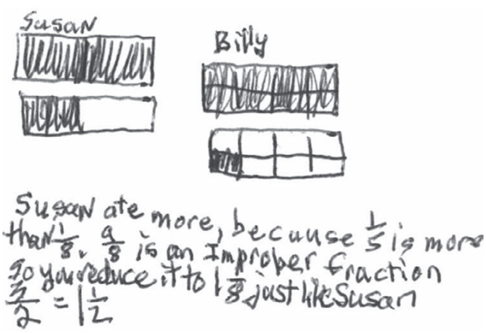
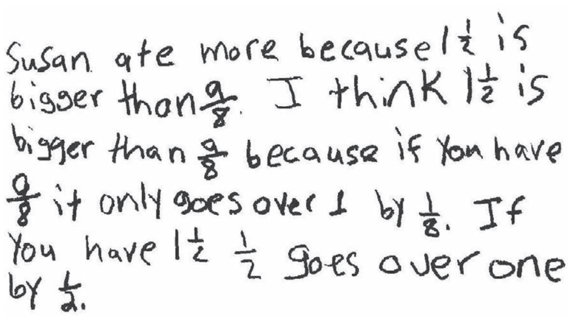


Figure 8.21 Bob's response.

Susan ate $1\frac{1}{2}$ cupcakes and Billy ate $\frac{9}{8}$ cupcakes. Who ate more?

Use words or diagrams to explain your answer.



a. How did Kim and Bob use their knowledge of comparing proper fractions when they compared a mixed number to an improper fraction? Explain.
b. Identify some mixed numbers/improper fractions that can be compared using benchmark reasoning to halves. Explain your choices.
c. Identify some mixed numbers/improper fractions that can be compared using unit fraction understanding. Explain your choices.

6. Figure 8.22 is the post-assessment response associated with the pre-assessment response shown in Figure 8.16. Identify the strategies used in Figure 8.22. Identify where each of the strategies is found on the *OGAP Fraction Progression*. Explain each of your choices.

7. Order these decimals from greatest to least, using reasoning or visual models to defend your answer.

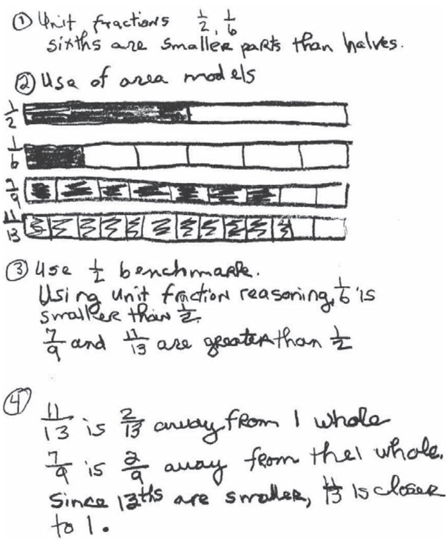
   23.4   21.493   23.034   23.12

   What are some errors that you anticipate students might make when ordering these decimals?

Figure 8.22 Teacher post-assessment response.

Which fraction is closest to 1? $\frac{1}{2}, \frac{7}{9}, \frac{11}{13}, \frac{1}{6}$?

Solve this problem using three different strategies.



8. Case Study—Facilitate a discussion using student solutions: Developing Unit Fraction Reasoning

Part I: Ms. Smith's Lesson Background

Goal: To use visual models to develop understanding of unit fractions and benchmark reasoning when comparing fractions.

To accomplish the goal Ms. Smith asked students to compare a variety of familiar fractions (e.g., Which is greater? $\frac{1}{3}$ or $\frac{3}{4}$?) by creating visual models. She then had students compare unit fractions and fractions in which unit fraction reasoning could be used (e.g., Which is closest to 0? $\frac{1}{3}$ or $\frac{1}{4}$? $\frac{4}{5}$ or $\frac{3}{4}$?) using visual models. After solving these and other similar problems students were asked to make observations about their visual models (both number lines and area models). By the end of the lesson Ms. Smith noticed that many students appeared to be moving away from relying on visual models to using benchmark and unit fraction reasoning when comparing fractions.

Part II: Exit question

At the end of the lesson Ms. Smith administered these exit questions. She planned to use the evidence in the student work to help plan her lesson for the next day.

Q1: Which fraction is closest to 1? $\frac{5}{6}$ or $\frac{7}{8}$

Q2: Which fraction is closest to $\frac{1}{2}$? $\frac{3}{4}$ or $\frac{5}{12}$

8a) Solve Q1 and Q2 by thinking about different solutions you would expect to see from students.

8b) Q1 was like the comparisons students made during the class, but with different fractions. However, the students did not make comparisons to the benchmark $\frac{1}{2}$. Why do you think Ms. Smith chose to give her students both questions?

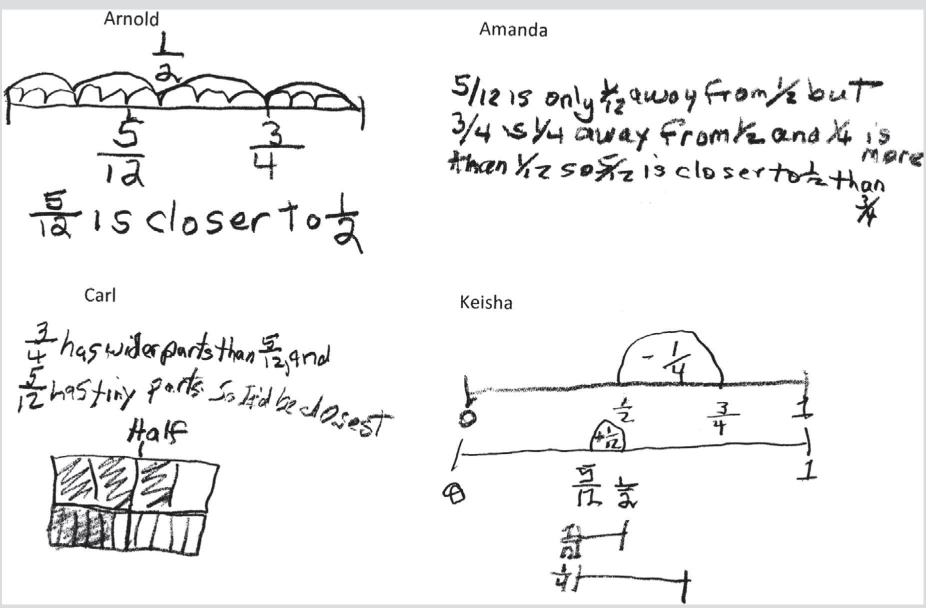
Part III: Analyze evidence

Ms. Smith quickly reviewed the student responses using the *OGAP Sort* for Q1 and found all but one student used unit fraction reasoning in their solution. Study the four solutions to Q2 in Figure 8.23. All but Amanda's solution is typical of the responses of the whole class to Q2.

8c) Where on the OGAP Fraction Progression is the evidence in these solutions? Record the level, strategy, and any underlying issues in Table 8.1.

Figure 8.23 Arnold's, Amanda's, Carl's, and Keisha's solutions to Q2.

Q2: Which fraction is closest to $\frac{1}{2}$? $\frac{3}{4}$ or $\frac{5}{12}$



8d) Planning Questions:

i) What evidence of developing understandings can be built upon?
ii) What are some errors or issues that need to be addressed?
iii) What are potential next instructional steps?

Table 8.1 Evidence Collection Table

Fractional	**Transitional**	**Early Fractional**	**Non-Fractional**
Underlying Issues?			

Part IV: Planning Based on Evidence

The fact that most students fell back on using visual models to solve Q2 didn't surprise Ms. Smith. She decided to use student solutions from Q2 to help move students to use unit fraction reasoning without having to draw visual models.

8e) Choose two solutions that can be used to help students move from visual models to using unit fraction reasoning.

8f) Why did you choose the solutions that you did?

8g) Explain how you would engage students in a discussion using the solutions you chose to help them see how unit fraction reasoning is represented in the visual models. (e.g., What questions will you ask?)

Instructional Link: Your Turn

Use the guiding questions in Table 8.2 to help you think about your instruction and your math program. How do they support students as they are learning to compare and order fractions?

Table 8.2 Instructional Link—Strategies to Support Development of Concepts Related to Comparing and Ordering Fractions

Do you (or does your program):	*Yes/No*
1. Provide opportunities for your students to compare fractions with different numerator and denominator combinations?	
2. Use models to compare and order fractions and decimals?	
3. Use models to develop and generalize reasoning strategies for comparing and ordering fractions and decimals?	
4. Encourage your students to use a variety of reasoning strategies when comparing or ordering fractions and decimals?	
5. Encourage and build on foundational skills to develop conceptual understanding and procedural fluency?	

Based on this analysis, identify gaps in your instruction or mathematics program. How might you address these gaps?

9
Number Lines and Fractions

Big Ideas

Number lines can be used to help students build an understanding of:

- the relative magnitude of fractions
- unit fractions
- equivalence
- operations with fractions
- the density of rational numbers

Researchers suggest that number lines can be used to help build an understanding of the magnitude of fractions as well as concepts of equivalence and the density of rational numbers (Behr & Post, 1992; Saxe et al., 2007). Teachers in the OGAP studies found this as well. In particular, teachers found that using number lines helped students think about a fraction as a number, allowing them to order, compare, and find equivalent fractions, and to move away from using whole number reasoning as they worked with fractions (OGAP, 2005, 2007).

CCSSM

The CCSSM and Number Lines

The CCSSM emphasizes the use of the number line to build understanding of a fraction as a number, unit fractions, equivalent fractions, and operations with fractions and other rational numbers (Common Core Standards Writing Team, 2013a, 2013b).

At grade 3, the number line is used to develop understanding of a fraction as a quantity, to provide a foundation for understanding equivalent fractions, and to solidify work with unit fractions. To ensure that the number line is used to develop understanding of a fraction as quantity, number lines should extend beyond one unit (e.g., 0–3). Extending beyond one unit helps ensure that students are locating fractions on a number line relative to the defined unit rather than finding the fractional part of the whole number line (Common Core Standards Writing Team, 2013a). In addition, it allows for representing fractions greater than one. OGAP studies have shown that students who encounter only number lines from 0 to 1 often only consider the length of the line, not the length of the unit when the number line involves more than one unit (see Figure 9.6 later in the chapter).

At grade 4, the number line is used to understand equivalence and the relationship between unit fractions and addition and subtraction of fractions. See Chapter 11, Addition and Subtraction of Fractions, for more discussion of this.

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At grade 5, students use number lines to extend their understanding of addition and subtraction of fractions as well as to build understanding of multiplication of fractions. (See Chapter 11, Addition and Subtraction of Fractions, and Chapter 12, Multiplication and Division of Fractions.) The use of the number line is extended to the set of rational numbers in grade 6, which includes negative fractions in grade 7.

This chapter builds on the introductory discussion in Chapter 2, Visual Models. The chapter also contains descriptions of difficulties students encounter when using a number line as well as instructional strategies that have been effective in helping students use a number line to develop an understanding of the important fraction concepts contained in the CCSSM.

Characteristics of the Number Line

The following are four important characteristics that distinguish the number line from other fraction models (Bright et al., 1988):

1. The unit is represented by a length as opposed to an area or a set of objects.
2. A number line requires symbols to define the unit, whereas the unit in an area or set of objects is implied in the model.
3. There are no visual separations between iterations of the units; that is, the units are continuous, unlike an area or set model in which the units are physically divided.

Figure 9.1 provides an example of how lengths are identified using points on a number line.

Figure 9.1 Number lines indicate lengths. On a number line, a defined length represents the unit. A point on the line identifies a distance or length from 0. The point at 3 represents a length or distance of three units from 0.

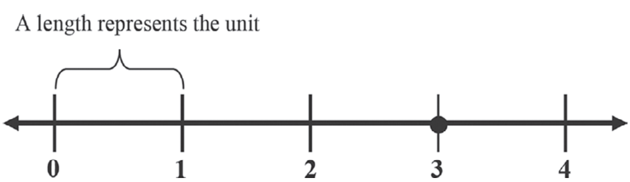


Figure 9.2 illustrates how the location of a fraction on a number line is dependent upon the symbols that define the unit.

Figure 9.2 Symbols define the unit. The accurate location of another number (e.g., $\frac{1}{2}$) on a number line is dependent on the symbols that define the unit. In contrast, no symbols are required to communicate $\frac{1}{2}$ in either the set or area models.

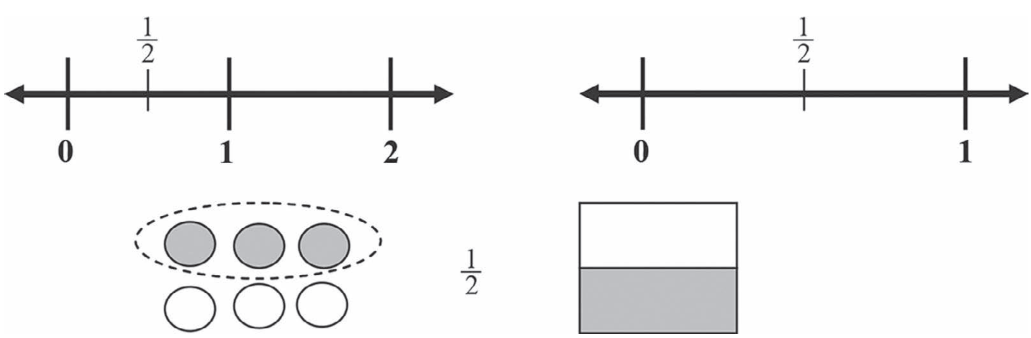


Figure 9.3 contrasts the continuous iteration of the units on a number line to visual area models.

Figure 9.3 Continuous model. There are no visual separations between iterations of the units on a number line—the units are continuous. In contrast, the units are physically separate in an area or set model.

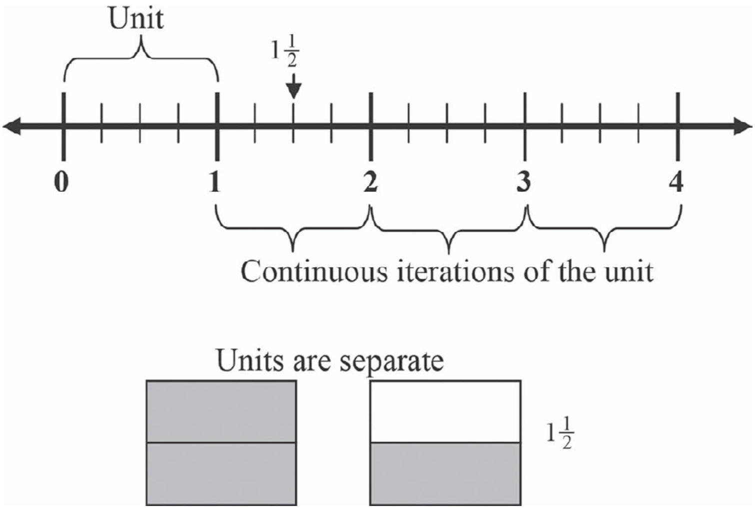
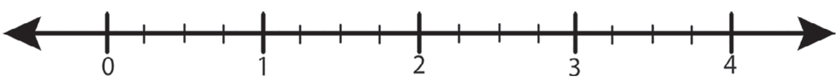


Figure 9.4 illustrates the subdivision of units. One can imagine subdividing the number line into 8ths, then 16ths, then 32nds, and so on, with the only limitation being the accuracy of the tools used to partition the line.

Figure 9.4 Subdivisions. The units on a number line can be subdivided into equal subunits. Pictured is a number line consisting of four units, each subdivided into fourths and illustrating the representation of fourths greater than 1.



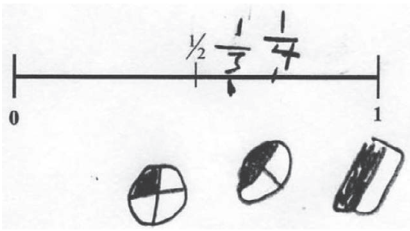
Difficulties Students Encounter When Using Number Lines

Teachers reported that when students first interacted with number lines, they often reverted to whole number reasoning (Figure 9.5) and placed fractions on the number line in order of the magnitude of their numerators or denominator (OGAP, 2005, 2007).

These observations were supported by data from the OGAP student work substudy, which found about 59% $\left(\frac{23}{39}\right)$ of the student pre-assessment responses in the sample placed the fractions $\frac{1}{4}$ and $\frac{1}{3}$ as Ken did in his solution in Figure 9.5 (OGAP, 2005).

Figure 9.5 Ken's response. Ken used visual area models (circles) to represent $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$. However, when Ken placed $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$ on the number line, he incorrectly ordered them by the magnitude of the denominators.

Place $\frac{1}{3}$ and $\frac{1}{4}$ on the number line.



When students first encounter number lines with multiple units, it is not uncommon for them to find the fractional part of the whole line instead of locating the fraction relative to the defined unit (Mitchell & Horne, 2008; OGAP, 2005). Peter's response in Figure 9.6 exemplifies this error.

These errors may, in part, be explained by research findings related to student difficulties with the use of number lines.

- Students have difficulty integrating the visual model (line) and the symbols necessary to define the unit. The symbols and the tick marks that define the units and subunits can act as distractors (Behr et al., as cited in Bright et al., 1988).
- Students have a difficult time locating fractions on number lines that have been marked to show multiples of the unit or to show marks that span from negative numbers to positive numbers (Novillis-Larson, as cited in Behr & Post, 1992).
- Students don't always understand that the numbers associated with points on a number line tell how far the points are from 0 (Petitto, 1990). For example, the two points marked 3 and −3 on a number line are both three units from 0.
- Researchers also "hypothesize [that] as long as partitioning and unpartitioning are difficult for children, number line representations of fractions may not be easily taught" (Bright et al., 1988, p. 17).

Before discussing the research cited, it is important to realize that "students in the first three grades shift from sequential to proportional strategies to place numbers on a number line" (Petitto, 1990, p. 57). This suggests that before asking students to locate fractions on a number line, teachers should be sure that students are thinking proportionally about a number line, not just sequentially (see Figures 9.7 and 9.8).

Figure 9.6 Peter's response. Peter found $\frac{1}{3}$ of the line and ignored the defined units (−1 to 0 and 0 to 1).

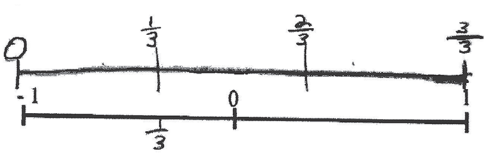


Figure 9.7 Sequential thinking. Whole numbers are placed on the number line considering only sequence, not the proportional distance between units that can define a length. The points at 2, 3, and 4 represent the second, third, and fourth numbers in the sequence, not the distance of 2, 3, and 4 equal units based on a defined unit 0 to 1.

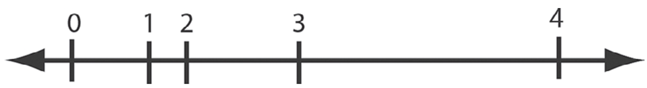
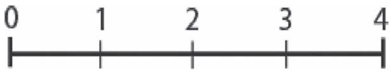


Figure 9.8 Proportional thinking. Whole numbers are placed on the number line proportionally (equal intervals). The point at 4 represents a length or distance of 4 equal units from 0.

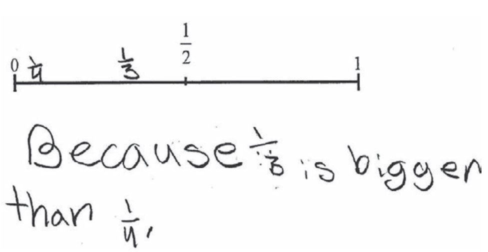


Any number can be placed proportionally on a number line once the unit (the distance from 0 to 1) is established. (Note that the distance between any two whole numbers could serve just as well to establish the unit.) The numbers on the number line in Figure 9.8 are spaced proportionally, that is, 2 is twice as far from 0 as 1; 3 is three times as far from 0 as 1; and 4 is four times as far from 0 as 1. A firm understanding of the proportional nature of a number line allows any fraction to be placed on a number line once the unit is established. For example, the number $\frac{1}{2}$ is half as far from 0 as the number 1, $2\frac{1}{4}$ is $2\frac{1}{4}$ as far from 0 as 1, and so forth.

Students who are thinking sequentially, not proportionally, may place fractions on a number line as, Judy did in the response in Figure 9.9.

The location of $\frac{1}{4}$ suggests that Judy may not have progressed beyond sequential thinking. The mark showing her location of $\frac{1}{4}$ is not $\frac{1}{4}$ of the distance from 0 to 1. More evidence that Judy is thinking sequentially is her justification for the locations of her fractions—that they belong where she placed them "because $\frac{1}{3}$ is bigger than $\frac{1}{4}$." Although her comment is correct, it suggests that the magnitude of those two fractions is her only consideration in locating them on the number line.

Figure 9.9 Judy's sequential response. The placement of $\frac{1}{4}$ and $\frac{1}{3}$ on this number line is sequential, not proportional. It appears the student understands that $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$, but she did not place the fractions on the number line proportionally.



Students who are thinking proportionally will show evidence of partitioning proportionally regarding the defined units on the number line. See Figures 9.10 and 9.11 for examples of this.

Figure 9.10 Proportional thinking. Visual area models equal to length of the number line were partitioned proportionally and then used to locate the fractions on the number line.

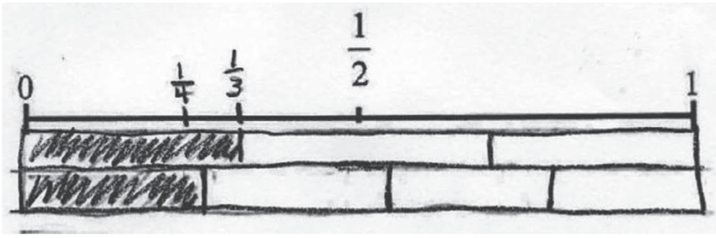
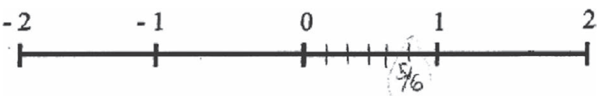


Figure 9.11 Proportional thinking. Five-sixths is placed on the number line proportional to the distance of 0 to 1. That is, $\frac{5}{6}$ is located about five-sixths of the way from 0 to 1.

Locate $\frac{5}{6}$ on the number line.



Introducing Number Lines into Instruction

"Although the number line is introduced to students in elementary school textbooks, its potential for students' learning has not been exploited by educators or researchers" (Saxe et al., 2007, p. 1). Teachers in the OGAP (2007) study began to embed the number line into instruction beyond what was provided in their text materials, as suggested by Saxe and colleagues.

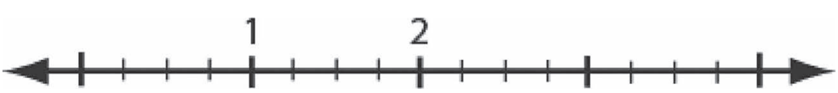
First, research implies that strategies for engaging students in using number lines may vary across grades. For example, when first introducing young students to number lines involving fractions, the research suggests that teachers should ensure that students are thinking about number lines with whole numbers proportionally, not sequentially. See *A Focus on Addition and Subtraction* (Ebby et al., 2021) for more on how to develop understanding of whole numbers on a number line.

Second, when moving from number lines with only whole numbers to using number lines to locate fractions, researchers suggest that teachers use number lines with full knowledge of the difficulties that students may encounter (Behr & Post, 1992). Some teachers found that engaging students intentionally in the features of a number line that may later cause students difficulty made the use of the number line a more valuable instructional tool (OGAP, 2005).

For example, teachers might use the number line in Figure 9.12 along with questions such as these to guide a discussion on the features of a number line:

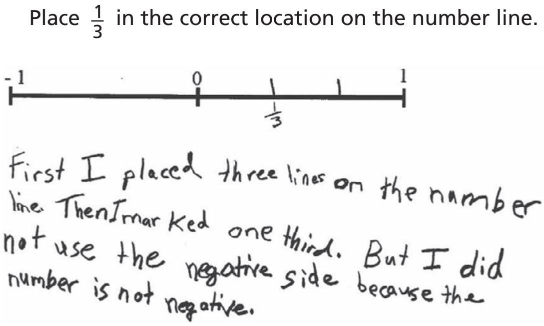
1. Make a list of everything you notice about the number line. (The teacher uses the lists to guide a whole-class discussion.)
2. Identify where the number 4 is on this number line. What defined where the number 4 is located? What whole numbers are represented on this number line?
3. What do the tick marks between the numbers 1 and 2 indicate?
4. What numbers are represented on the number line for the tick marks to the left of 1? Are there other numbers between the tick marks on the number line? How could you determine what those numbers are? (See Chapter 10, The Density of Fractions.)

Figure 9.12 Sample number line to use with guided questions.



Another strategy adopted by some teachers and implied in the research is to vary the number lines presented to students. Number lines might contain single or multiple units, span from negative to positive numbers, or require partitioning or repartitioning. Figures 9.13 to 9.15 provide examples from students who successfully located fractions on number lines with different structures.

Figure 9.13 Asher's response. One-third is placed in the correct location on the number line that spans −1 to 1, providing evidence that he integrated the visual model with the symbols.



In the problem in Figure 9.14, Kim was given a number line partitioned into sixths and was asked to locate the fractions $\frac{5}{12}$ and $\frac{3}{4}$. To solve the problem, Kim repartitioned the number line into twelfths and then recognized that $\frac{9}{12}$ was equivalent to $\frac{3}{4}$.

In Figure 9.15, Adam was given a number line that spans from −1 to 3 and used equipartitioning to correctly locate $\frac{8}{12}$, $\frac{8}{3}$, and $\frac{2}{3}$ on the number line.

Figure 9.14 Kim's response. The student repartitioned the number line into twelfths and marked the location of $\frac{5}{12}$. Recognizing the equivalence of $\frac{3}{4}$ with $\frac{9}{12}$ allowed the student to accurately locate $\frac{3}{4}$ on the number line.

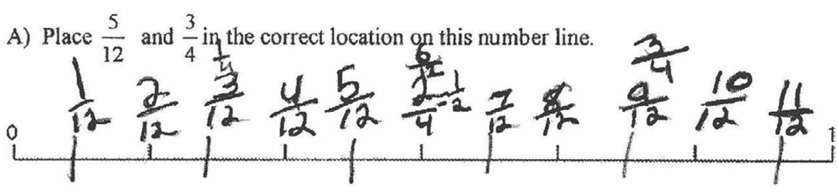
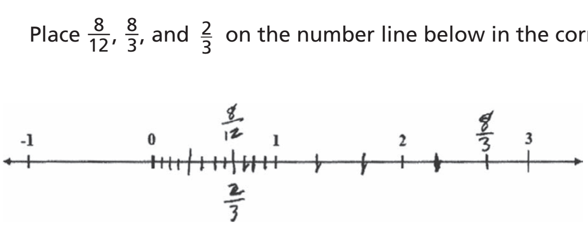
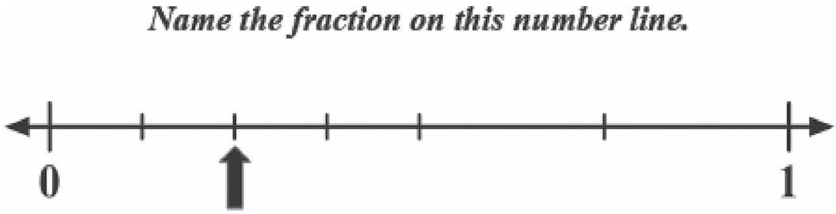


Figure 9.15 Adam's response. To correctly locate $\frac{8}{12}$, $\frac{8}{3}$, and $\frac{2}{3}$ on the number line, Adam partitioned from 0 to 3 into thirds and from 0 to 1 into twelfths.



The number line in Figure 9.16 was used in a study conducted by Saxe et al. (2007). The problem includes a complete equipartitioning of the unit into fourths and includes an incomplete partitioning of the unit into eighths. Saxe and colleagues found that this number line with its missing partitions created challenges for students who lacked a strong conceptual sense of the magnitude of a fraction or who thought sequentially instead of proportionally. For these reasons, number lines such as this one, which force students to use their proportional sense of the distance from 0 to the tick mark with the arrow in relation to the defined unit 0 to 1, can be a valuable instructional or assessment task.

Figure 9.16 A non-routine problem with incomplete partitioning into eighths (Saxe et al., 2007).



Evidence shows that translating the linear feature of visual area models or tape diagrams to number lines may help to explain improvement in the use of number lines (Bright et al., 1988; OGAP, 2005). This was a strategy adopted by many students and teachers in the 2005 OGAP study and evidenced in Laura's and Marko's responses found in Figures 9.17 and 9.18 (OGAP, 2005).

Figure 9.17 Laura's response. Laura drew visual area models under the number line, then used her partitioning of the linear feature of the area models to correctly locate the fractions on the number line.

Place $\frac{8}{12}$, $\frac{8}{3}$, and $\frac{2}{3}$ on the number line in the correct position.

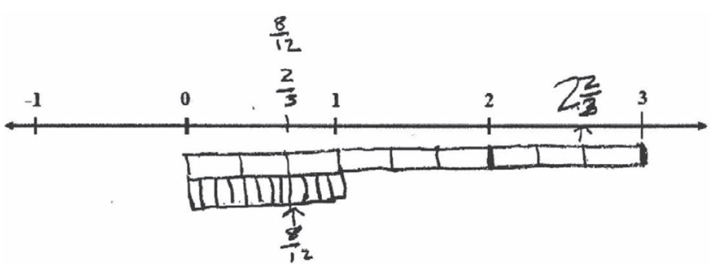
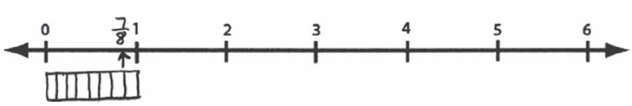


Figure 9.18 Marko's response. Marko partitioned a visual area model equal to one unit on the number line and then located $\frac{7}{8}$ relative to the defined unit.

Place $\frac{7}{8}$ on the number line.



To reinforce the underlying structure of a number line and to strengthen understanding of the relative magnitude of fractions and decimals, some teachers in the OGAP studies used a classroom-sized number line. The design allowed teachers from day-to-day to change the focus from one aspect of the number line to another (e.g., change the size of the unit or the number of units) (see Figure 9.19) (OGAP, 2005, 2007). These number lines can allow students to use their understanding of unit fractions, equipartitioning, equivalence, and relative magnitude to place both fractions and decimals in relation to the given units.

Number Lines and Decimal Fractions

Number lines can also be used to develop the concept of the magnitude of decimal fractions and equivalence between fractions and decimal fractions. In the number line from 0 to 3 in Figure 9.20, each unit is partitioned into tenths and shows that 0.1 is the distance from zero when the unit is divided into 10 equal parts and that 0.4 is four times the distance of 0.1 from 0, and 1.2 is 12 times the distance of 0.1 from 0.

Figure 9.19 Two different classroom-sized number lines are displayed on a blackboard. Each number line has a different-sized unit and a different number of units.

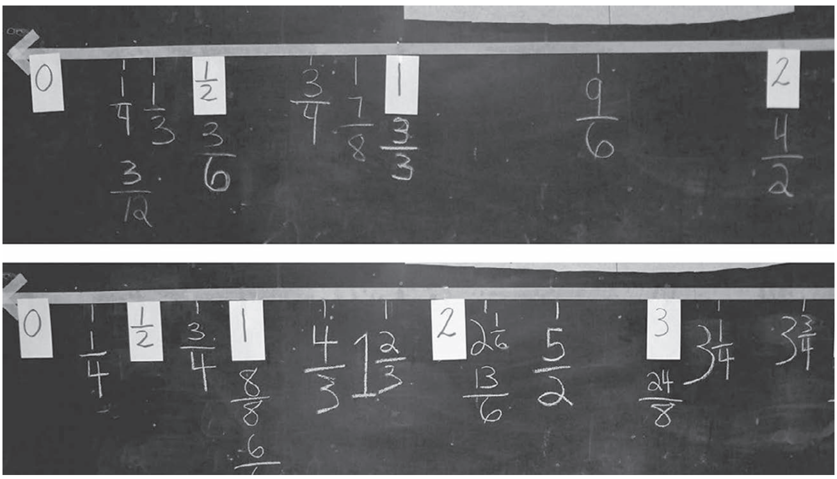
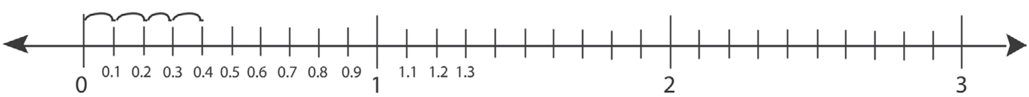
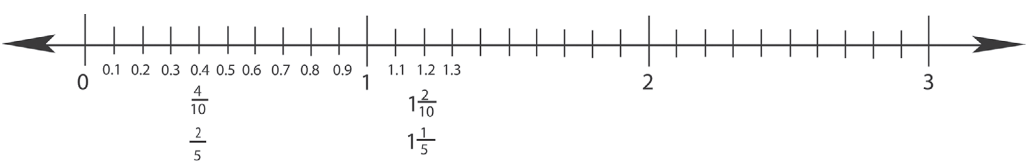


Figure 9.20 Number line representing the location of 0.4.



As discussed in Chapter 7, some students have a hard time believing that a single value can be expressed in different symbolic formats (Hiebert & Wearne, 1986). The number line can be used to develop the understanding of the equivalence of fractions and decimal fractions. In Figure 9.21, fractions equivalent to 0.4 are represented on the number line. The visual cue that they are indeed equivalent is the fact that they are located at the same point on the number line. Also identified are fractions equivalent to 1.2. Decimal fractions, like fractions, are a natural extension of whole numbers, and using the number, line reinforces this concept as well as helps students understand the relative magnitude of decimal fractions to each other and to whole numbers.

Figure 9.21 Number line with equivalent fractions and decimal fractions.



The Number Line and Negative Fractions

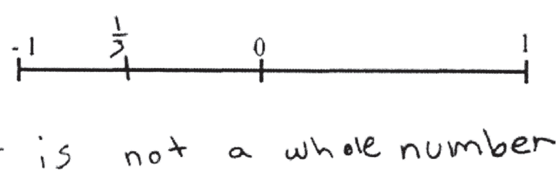
A central notion in this chapter and in other chapters throughout this book is the vital role the number line plays in developing students' understanding of fractions as quantities, unit fractions, equivalence, and operations with fractions. It is not a surprise that in sixth grade, the number line also takes a prominent instructional role as students begin to compare, order, and operate with rational numbers, first with integers and then with positive and negative fractions and decimals in grade 7. Since negative fractions (e.g., $-\frac{1}{3}$) cannot be represented using visual area models or sets of objects, the number line takes a prominent instructional role when developing understanding of integers and negative fractions as quantities. This section focuses on potential problems students may encounter when comparing and using number lines with negative fractions.

The issues that students encounter while making sense of positive fractions tend to resurface as they begin working with negative fractions. That is, foundational misunderstandings about the magnitude of positive fractions can significantly affect student's learning of negative fractions.

Review Asher's and Nick's responses to the same problem in Figures 9.13 and 9.22, respectively. Based on evidence in Asher's work, it appears he understands that $\frac{1}{3}$ is not a negative number; therefore, he focuses his attention on the positive side of the number line. In contrast, Nick appears to believe that the fraction $\frac{1}{3}$ is located on the negative side of the number line. Although we do not know for sure, it appears that Nick has a fragile understanding of positive fractions, perhaps believing that because a fraction is not a whole number, it is less than 0.

Figure 9.22 Nick's response to placing $\frac{1}{3}$ on a number line between −1 and 1.

Place $\frac{1}{3}$ on the number line in the correct location. Explain your answer using words or diagrams.



This example illustrates how important it is for students to encounter multiple opportunities to place and make sense of positive fractions on number lines that extend from negative to positive numbers before considering negative fractions on a number line. Nick's response is an example of a foundational misunderstanding about the magnitude of a positive fraction that will undoubtedly make it more difficult for him to conceptualize negative fractions.

Earlier in this chapter, we briefly discussed the importance of students understanding that points on a number represent a distance from 0. This concept is central when thinking about the magnitude of negative fractions. To understand the importance of this concept, contrast Eli's and Evan's responses in Figures 9.23 and 9.24 to a question related to placing $-\frac{2}{3}$ on a number line that is marked with only 0 and 1.

Figure 9.23 Eli's response.

Locate and label a point on the number line to show where $-\frac{2}{3}$ is located.

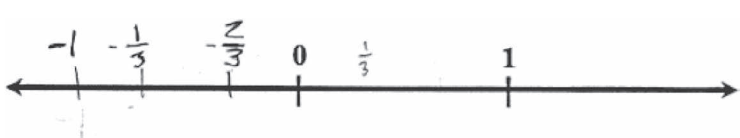
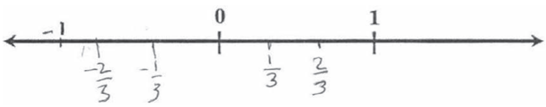


Figure 9.24 Evan's response. Evan correctly located $-\frac{2}{3}$ on the number line.

Locate and label a point on the number line to show where $-\frac{2}{3}$ is located.



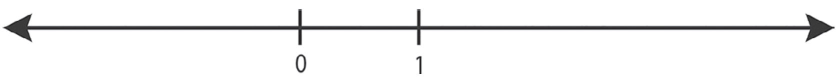
Both responses show evidence of understanding the magnitude of the unit by locating −1 and 1 as the same distance from 0. Where the solutions differ is in the placement of $-\frac{1}{3}$ and $-\frac{2}{3}$. Evan correctly located $-\frac{1}{3}$ and $-\frac{2}{3}$ as the same distance from 0 as $+\frac{1}{3}$ and $+\frac{2}{3}$.

In contrast, notice that in Eli's response the distance from 0 to $-\frac{1}{3}$ is not equal to the distance from 0 to $\frac{1}{3}$, nor is the distance from 0 to $-\frac{2}{3}$ the same distance as 0 to $\frac{2}{3}$. It seems he incorrectly interpreted both negative fractions originating from −1 rather than from 0.

The mathematical concept described here is central to understanding negative fractions. That is, two fractions that are additive opposites, such as $\frac{1}{3}$ and $-\frac{1}{3}$ or $\frac{2}{3}$ and $-\frac{2}{3}$, are the same absolute distance from 0. Building this idea in the context of placing fractions on a number line will later help students use the number line to conceptualize operations with negative fractions. One way to help students develop this idea is to build upon their understanding of integers on a number line.

The number line in Figure 9.25 and the question sequence that follows provide an example of some questions that build understanding of negative fractions on a number line from students' prior work with integers on a number line.

Figure 9.25 Use this number line for accompanying questions 1 through 9.



Example Question Sequence

1. Use what you know about number lines to place other positive integers on this number line. How did you do this? What limits the number of positive integers that can be placed on this number line?
2. Use what you know about the placement of the positive integers to locate the negative integers. (Students should develop the understanding that the distance from 0 to 1 is the same as the distance from 0 to −1. Therefore, to locate −1, one can begin at 0 and move to the left the distance from 0 to 1. In the same way −2 can be located using the distance from 0 to 2).
3. Can other numbers be placed on this number line? Explain why or why not. (An infinite number of fractions, both positive and negative, could be placed on the number line, limited only by the ability and space to physically partition the number line.)
4. Is $-\frac{2}{3}$ closer to 0 or 1? Explain your answer.
5. Is $-\frac{2}{3}$ closer to 0 or −1? Explain your answer.
6. Place $\frac{2}{3}$ and $-\frac{2}{3}$ on the number line. What do you notice about the placement of these two fractions? (Both are the same absolute distance from 0.)
7. Place one negative fraction on the number line that is located between 0 and $-\frac{2}{3}$.
8. Place one negative fraction on the number line that is less than −1.
9. Place one negative fraction on the number line that is located between −1 and $-\frac{2}{3}$.

Note how these questions help to move beyond partitioning to focus students on the distance from a given fraction to 0 on the number line.

Student Use of Number Lines to Solve Problems

Many OGAP teachers in the 2005 study encouraged students to use number lines to solve problems. A preliminary analysis of 39 fourth-grade OGAP pre-assessments $\left(\frac{39}{229}\right)$ illustrates this point. Approximately 41% $\left(\frac{16}{39}\right)$ of the students used number lines to solve problems in the post-assessment. In contrast, only 8% $\left(\frac{3}{39}\right)$ of the students used the number line to solve problems in the pre-assessment (see Table 9.1) (OGAP, 2005).

Table 9.1 Grade 4 OGAP—Use of Number Lines Pre- to Post-Assessment (OGAP, 2005)

	Pre-assessment	*Post-assessment*
Percentage of students (n = 39)	8%	41%
Number of responses with number lines used to help solve problems	3	43

Examples of some ways that students used number lines to solve fraction problems are shown in Figures 9.26–9.28. In Figure 9.26, Kaitlyn used a number line to solve a problem involving the density of fractions.

Go-To See Chapter 7, The Density of Fractions, for more examples on how number lines can be used to develop understanding of the density of rational numbers.

Notice that Juan effectively used a number line to find the difference between the sum of $\frac{2}{3}+\frac{1}{4}$ and 1. He partitioned the number line into twelfths. He then used the partitioned number line to: (a) identify the distance from 0 to $\frac{1}{4}$ on the number line, (b) represent the addition of $\frac{1}{4}$ and $\frac{2}{3}$ by placing a segment equivalent to the length of $\frac{1}{4}$ at $\frac{2}{3}$, and (c) identify the difference left as $\frac{1}{12}$ (circled on his solution).

Figure 9.26 Kaitlyn's response. Kaitlyn created a number line that she partitioned into twelfths and then she used the partitions to identify fractions between $\frac{1}{3}$ and $\frac{5}{6}$.

Name two fractions that are between $\frac{1}{3}$ and $\frac{5}{6}$.

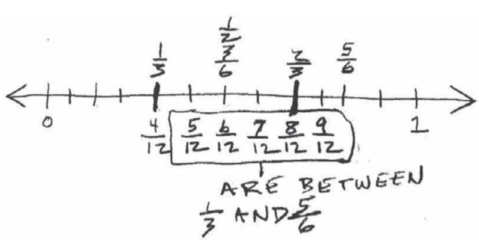


Figure 9.27 Juan's response. Juan effectively used a number line to solve a problem involving addition and subtraction of fractions.

Tina ate $\frac{2}{3}$ of her candy and gave $\frac{1}{4}$ of her candy to her sister. She saved the rest of her candy. What is the fractional part of the candy that Tina saved?

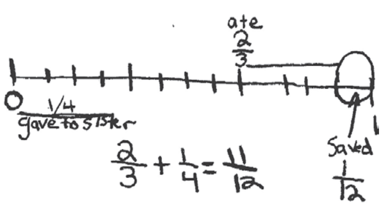


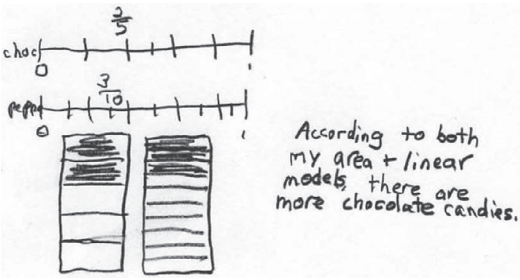
Figure 9.28 Mathew's response. Mathew effectively used a number line (and visual area model) to compare $\frac{2}{5}$ and $\frac{3}{10}$.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies in the dish?



While it is important that students understand and use number lines to solve problems, students need to develop more generalized understandings of fraction concepts. An overreliance on visual models can sometimes interfere with students gaining a deeper, more abstract understanding of concepts. The case study that follows involves a student, Maria, who consistently uses number lines to solve problems even though her teacher, Mr. King, is confident the student can use more efficient strategies.

Case Study—Provide Actionable Feedback

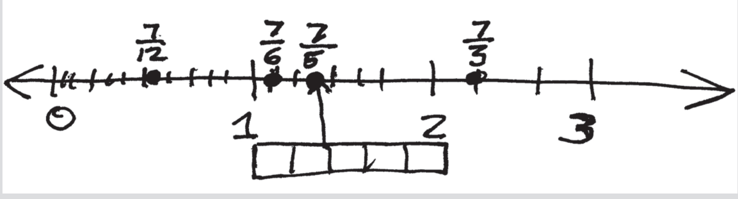
Mr. King regularly provides actionable feedback to his students, that is, feedback that will advance student thinking and deepen understanding. He understands that comments like "think," "try again," or "good work" do not result in increased motivation or achievement (Wiliam, 2011). Sometimes Mr. King's feedback is provided orally, either to the whole class or to individual students. Other times he provides written feedback to all or some of his students.

Study Maria's solution in Figure 9.29. What do you notice about her solution?

Figure 9.29 Maria's solution to the problem. One can see that Maria successfully used the number line to determine which fraction was closest to 1.

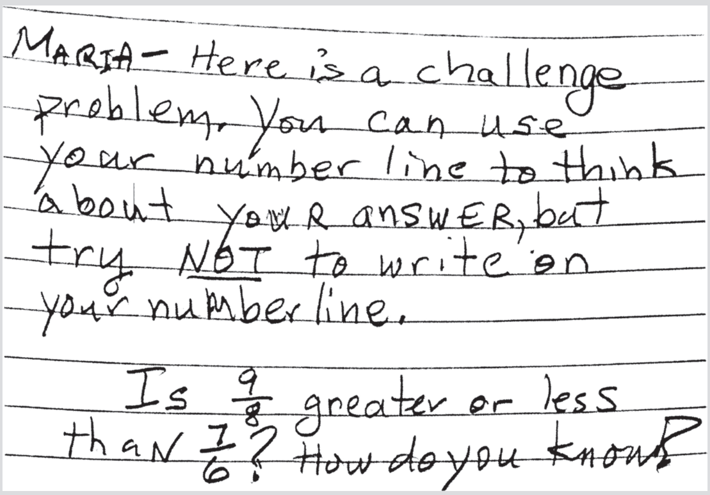
Which fraction is closest to 1?

$\frac{7}{3}$ $\frac{7}{5}$ $\frac{7}{6}$ $\frac{7}{12}$



You probably noticed that Maria effectively partitioned the number line into 3rds, 6ths, and 12ths, and used an area model to help her locate $\frac{7}{5}$ on the number line. This evidence suggests that Maria can construct and partition a useable number line to solve problems. Knowledge of and flexibility with visual models, including number lines, are important aspects of understanding fraction concepts. Yet Mr. King knows that effective use of visual models alone reflects an incomplete understanding of fraction comparison. Students must also develop reasoning strategies that do not require the creation of a visual model to compare fractions. To help Maria and other students develop reasoning strategies beyond visual models, Mr. King included a challenge problem in his written feedback. The feedback and problem is shown in Figure 9.30.

Figure 9.30 Mr. King's feedback to Maria

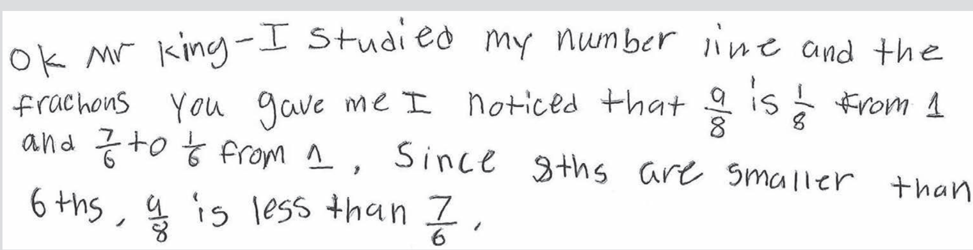


MARIA – Here is a challenge problem. You can use your number line to think about your ANSWER, but try NOT to write on your numberline.

Is $\frac{9}{8}$ greater or less than $\frac{7}{6}$? How do you know?

By asking Maria to compare $\frac{9}{8}$ and $\frac{7}{6}$ without writing on her number line, Mr. King was nudging Maria to use what she has learned from her experiences constructing number lines to reason about the two fractions. Study Maria's response to Mr. King's challenge problem in Figure 9.31.

Figure 9.31 Maria's response to challenge problem.



ok Mr King - I Studied my number line and the frachons you gave me I noticed that $\frac{9}{8}$ is $\frac{1}{8}$ from 1 and $\frac{7}{6}$ to $\frac{1}{6}$ from 1, Since 8ths are smaller than 6ths, $\frac{9}{8}$ is less than $\frac{7}{6}$.

After receiving the student responses to the challenge problem in Figure 9.30, Mr. King decided to pose the question in Figure 9.32 to the whole class. Why do you think Mr. King gave his students this problem?

Figure 9.32 Exit question.

Isaac said that $\frac{1}{129} > \frac{1}{59}$. Is Isaac correct? Why or why not?

You probably noticed that the numbers in this problem make the use of a visual model unreasonable. In this way Mr. King engineered a problem that supports the use of unit fraction reasoning rather than the creation of a visual model. As you have seen in this case sometimes the instructional response based on the evidence is to use another formative assessment item to move student thinking forward on the *OGAP Fraction Progression*.

Chapter 6 Using the *OGAP Fraction Progression* for more on instructional response to evidence.

Math Programs and Number Lines

To capitalize on the potential power of using the number line, OGAP teachers found themselves supplementing their mathematics program (OGAP, 2005, 2007). This is consistent with the observations of Saxe et al. (2007):

> We argue that number lines can support students' understanding of important properties of fractions. Fifth- and sixth-grade students can use the number line as a vehicle for understanding ideas like numerical units, relations between whole numbers and fractions, the density of the rational numbers (there are infinitely many rational numbers between any two), and although every number is unique, the number can be named in infinitely many ways (equivalence).
>
> (p. 1)

In US textbooks, area models are the most prominent representation used for fractions. In Japanese textbooks, the primary representation is the number line and linear models (Watanabe, 2007). With the focus on linear models in the CCSSM, this is likely to change as new textbooks are produced. To examine how your program uses number lines in instruction, complete the Instructional Link activity at the end of this chapter.

Measurement: A Direct Application of the Number Line

Some teachers have told us how surprised they are after completing a fraction unit to find that students have difficulty measuring with rulers. It appears that the students had not made the connection between the fraction concepts they had learned and their application to linear measurement.

Bright et al. indicated that "the number line can be treated as a ruler" (1988, p. 1). This appears to be a question of which comes first: understanding of the number line and application to measurement, or measuring with an application to the number line? While one can argue for a particular order, one teacher who used the number line first experienced huge payoffs:

> My students were surprised to learn that the tick marks on a ruler seemed to be related in size to their value just like a number line. The tick mark for $\frac{1}{2}$ was half of the distance from 0 to 1. The tick mark for $\frac{1}{4}$ was quarter the distance from 0 to 1. The tick mark for $1\frac{1}{2}$ is $1\frac{1}{2}$ times the distance from 0 to 1.
>
> (OGAP, 2007)

Experience has shown that students have difficulty both measuring and using the number lines. Both, however, have the same conceptual foundation. To help think through the concept related to number lines and measurement, go to question 1 in Looking Back.

The *OGAP Fraction Progression* and Number Lines

The number line is an important visual model to help build understanding of fractions as quantities. Therefore, it has a prominent place on the *OGAP Fraction Progression*.

Fractional Strategies include accurately locating fractions on a number line provided without using other visual models, as Evan did in Figure 9.24. *Transitional Strategies* are evidenced by effective use of a visual model to solve a problem (e.g., Figures 9.26 and 9.27) or by locating fractions on number lines using another visual model (Figure 9.17). The use of the number line as a model, like other visual models, can also be a transitional strategy to help build understanding of the magnitude of fractions, equivalence, and operations with fractions.

Chapter Summary

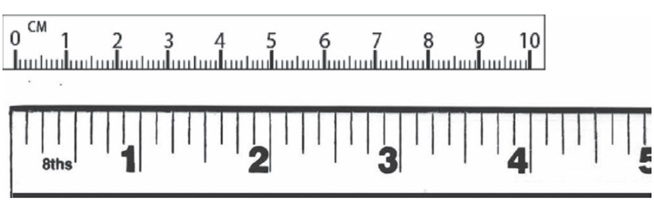
This chapter presented ideas related to the use of number lines that research has shown help students as they develop their understanding of fraction concepts. In particular, the chapter focused on:

- the potential instructional benefits of using number lines as students develop an understanding of fraction concepts
- the difficulties that students may encounter as they begin using number lines
- instructional strategies using number lines
- the number line and negative fractions
- examples of how students use number lines to solve problems
- the importance of reviewing your mathematics program to assure that both the material and your instruction maximize the power of the number line to aid students as they develop their understanding of fractions.

Looking Back

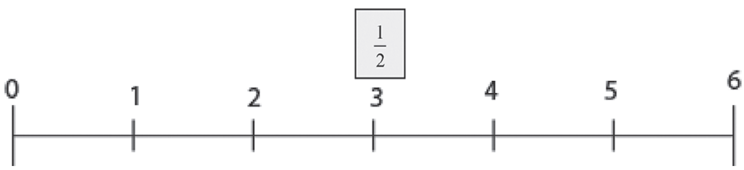
1. To help explore the relationship between measurement and number lines, respond to the following questions.
   a. What are at least three important properties that number lines and measurement tools (such as rulers) share that have the potential to facilitate students' understanding of the connections between rulers and number lines?
   b. What are two important differences between number lines and scales on measuring tools?
   c. You provide your students with inch rulers, centimeter rulers, and strips of paper to measure. Their task is to measure each of the strips to the nearest eighth of an inch and tenth of a centimeter. Before your students begin using the rulers to measure the strips, identify three similarities and three differences between an inch ruler and a centimeter ruler that will allow students to measure strips accurately or identify challenges to accurate measurements. Use Figure 9.33.

Figure 9.33 Centimeter and inch rulers.



2. Mr. Brown had a large number line in the front of his classroom (Figure 9.34). On the first day that he used the number line, he asked some students to place $\frac{1}{2}$ on the number where they thought it belonged. Mr. Brown had done no prior instruction with number lines, but he thought this would be a good way to get information about what instructional issues he might face as students began using number lines to solve fraction problems.
   a. The students were unsure where to locate $\frac{1}{2}$, but decided to locate it at the 3 on the number line. Is this correct or incorrect? Explain.
   b. What feature(s) of the number line may have been ignored by these students?

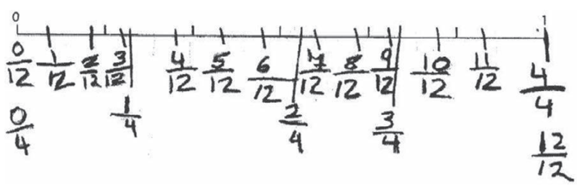
Figure 9.34 The number line in Mr. Brown's class.



3. Look at Matt's response to the problem in Figure 9.35. What understandings and misunderstandings are evidenced in Matt's response? Describe the evidence.

Figure 9.35 Matt's response.

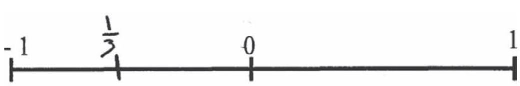
Place $\frac{5}{12}$ and $\frac{3}{4}$ on the number line.



4. Nick (Figure 9.36) placed $\frac{1}{3}$ to the left of 0 on the number line.
   a. What reasoning did Nick use to solve the problem? Describe the evidence.
   b. What are some potential next instructional steps for Nick given the evidence in his work?

Figure 9.36 Nick's response.

Place $\frac{1}{3}$ on the number line in the correct location.



Explain your answer using words or diagrams.

$\frac{1}{3}$ is not a whole number

5. *OGAP Fraction Progression*: Review student work in Figures 9.9, 9.10, 9.11, 9.13, 9.15, 9.17, and 9.26 in this chapter using the *OGAP Fraction Progression*. For each piece of student work:
   a. Use the evidence in the student work to locate the work on the framework.
   b. Provide a rationale for your decision.
   c. Discuss some possible next instructional steps or feedback you might give to the student based on the evidence in the student work.

Instructional Link: Your Turn

Use the prompts in Table 9.2 to help you think about how your instruction and mathematics program provide students the opportunity to develop understandings of number lines.

Table 9.2 Strategies that support development of reasoning with fractions as quantities with an emphasis on instructional strategies that include number lines.

Do you (or does your program):	*Yes/No*
1. Use number lines to build concepts of magnitude, equivalence, and the density of rational numbers?	
2. Provide opportunities for students to regularly engage in problems involving number lines?	
3. Engage students in understanding the features of number lines?	
4. Provide opportunities for students to place fractions and decimals on number lines with more than one unit?	
5. Provide opportunities for students to place fractions and decimals on number lines with units of different sizes?	
6. Provide opportunities for students to place fractions on number lines that are already partitioned?	
7. Provide opportunities to solve nonroutine problems involving number lines?	
8. Encourage students to use number lines to solve problems?	

Based on this analysis, what gaps in your instruction or mathematics program did you identify? How might you address these gaps?

10
The Density of Fractions

Big Ideas

- There are an infinite number of fractions between any two fractions.
- Number lines can help students understand the *betweenness* of rational numbers.

When studying fractions, students encounter a feature of fractions that is different from whole numbers; between any two fractions there are an infinite number of unique fractions.

To understand this concept, let us investigate a TV remote control. A remote control for a television set has sound settings (volume) that show on the television screen as shown in Figures 10.1 and 10.2. The volume indicator on the television screen works much like counting whole numbers. In these examples, there are 10 settings for volume, from no volume to the loudest volume.

Figure 10.1 The volume indicator showing no volume.

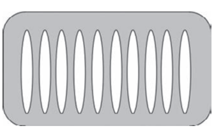
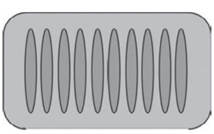


Figure 10.2 The volume indicator showing the loudest setting.



There is no compromise sound setting that is louder than setting 4 (Figure 10.3) but quieter than setting 5 (Figure 10.4). The whole numbers have this property; that is, there is no whole number between 4 and 5.

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Figure 10.3 The volume set at 4.

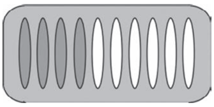
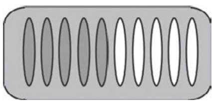


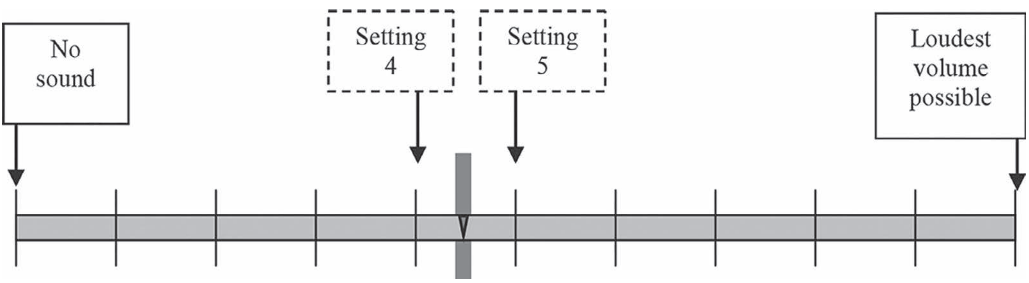
Figure 10.4 The volume set at 5.



Fractions are different: Between any two fractions, there are many different fractions. In fact, there are an infinite number of fractions between any two fractions. Students encounter this property in a number of situations when working with fractions. In mathematics class, for instance, students may be asked to name three fractions between $\frac{1}{4}$ and $\frac{3}{8}$.

However, this idea is more than a mathematical exercise. It is encountered in everyday situations, often without people realizing it. For example, unlike the television with a limited number of settings, imagine a volume control that can be placed at any position along a slide, like the example in Figure 10.5.

Figure 10.5 Volume control on a slide.



Unlike the previous volume indicator, there are many positions between the fourth and fifth settings that are slightly louder than the fourth setting but softer than the fifth setting. For instance, the sound setting of $4\frac{1}{4}$ delivers a level that is louder than setting 4 and softer than level 5. The setting $4\frac{1}{7}$ provides a louder sound than 4, but a softer sound than $4\frac{1}{4}$. Every position on the slide can be approximated by a fraction $\left(e.g., 4\frac{30}{100}, 4\frac{31}{100}\right)$.

The volume control on a slide helps one begin to conceptualize the idea of density. However, to truly comprehend the density property, we can start with two different fractions, such as $\frac{1}{4}$ and $\frac{1}{2}$ and see how to develop an infinite number of fractions that are between them. As a start, it should be clarified what it means to say "an infinite number of fractions" between $\frac{1}{4}$ and $\frac{1}{2}$; this means that a list of all the different fractions between these two is infinite—no matter how many fractions one identifies, there are more to be found.

Density of Fractions: The Mathematics

One way to mathematically demonstrate the density of fractions is by looking at averages. The average of any two different numbers results in another number halfway between the two numbers (see Figure 10.6).

Figure 10.6 The average of two different numbers results in another number halfway between the two numbers. Start with 60 and 100 and use averages to generate an endless list of numbers, all of them between 60 and 100.

The average of 60 and 100 is 80 $\left(\frac{60+100}{2}=80\right)$.

The average of 80 and 100 is 90 $\left(\frac{80+100}{2}=90\right)$.

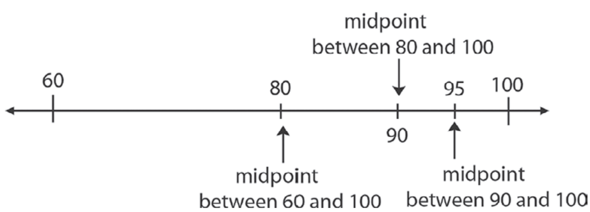
The average of 90 and 100 is 95 $\left(\frac{90+100}{2}=95\right)$.

The average of 95 and 100 is 97.5 $\left(\frac{95+100}{2}=97.5\right)$.

These averages give us the start of a list: 60 < 80 < 90 < 95 < 97.5 < . . . < 100. Note that the ellipsis in the previous sentence indicate that the list we are generating is an endless list (an *infinite* list) of numbers that are between 60 and 100.

The number line allows us to understand the average of two numbers geometrically. For two different numbers, their average is located on the number line exactly midway between them. (See the location of 80 midway between 60 and 100 in Figure 10.7.) Starting at 80 and going halfway to 100 brings us to 90, the average of 80 and 100. Next, the average of 90 and 100 is 95, and the average of 95 and 100 is 97.5 (which is halfway between 95 to 100). Notice when we start at the previous average and go halfway to 100, we can never reach 100. We have an endless list of averages, and these averages form an infinite list of numbers. (Note that this averaging process will be referred to in the rest of the chapter as *successive averages*.)

Figure 10.7 The average of 60 and 100 is 80 or the midpoint between 60 and 100. The average of 80 and 100 is 90 or the midpoint between 80 and 100. The average of 90 and 100 is 95 or the midpoint between 90 and 100.



We can continue with the idea of successive averages to investigate the infinite number of fractions between $\frac{1}{4}$ and $\frac{1}{2}$. Figure 10.8 illustrates how halving the distance between $\frac{1}{4}$ and $\frac{1}{2}$ (averaging the two) results in the fraction $\left(\frac{3}{8}\right)$ that is exactly midway on the number line between $\frac{1}{4}$ and $\frac{1}{2}$. While successive averages $\left(\frac{3}{8},\frac{5}{16},\frac{9}{32},\ldots,>\frac{1}{4}\right)$ get closer and closer to $\frac{1}{4}$, the average will never reach $\frac{1}{4}$ because there will always be a fraction halfway between $\frac{1}{4}$ and the fraction being averaged.

The density of fractions property applies to mixed numbers as well (see Figure 10.9). Through successive averaging starting with $1\frac{1}{4}$ and $1\frac{1}{2}$, we will find an infinite list of mixed numbers that are between $1\frac{1}{4}$ and $1\frac{1}{2}$, but that never reach $1\frac{1}{4}$ $\left(1\frac{3}{8}>1\frac{5}{16}>1\frac{9}{32},\ldots,>1\frac{1}{4}\right)$.

The number lines in Figures 10.8 and 10.9 help us understand an important idea about how density between proper fractions is related to density between mixed numbers. What we can see is that each fraction on the number line in Figure 10.9 is exactly one more than a corresponding fraction on the number line in Figure 10.8. For the infinite list of fractions we generated that are between $\frac{1}{4}$ and $\frac{1}{2}$, adding 1 to each of these gives us an infinite list of mixed numbers that are between $1\frac{1}{4}$ and $1\frac{1}{2}$.

Figure 10.8 Three-eighths is the midpoint between $\frac{1}{4}$ and $\frac{1}{2}$. Five-sixteenths is the midpoint between $\frac{3}{8}$ and $\frac{2}{8}$. The midpoint between $\frac{4}{16}$ and $\frac{5}{16}$ is $\frac{9}{32}$.

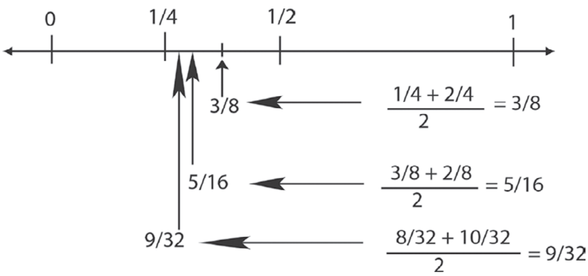
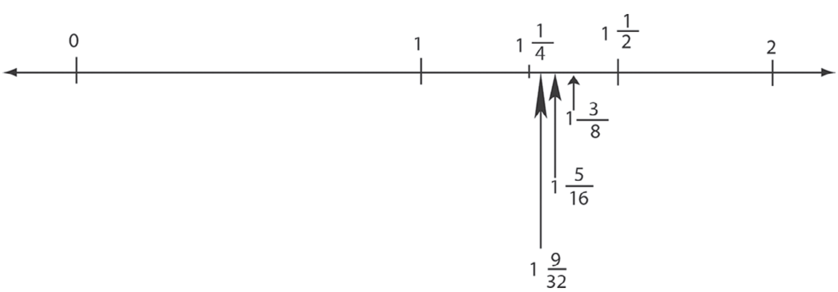


Figure 10.9 The fractions on the number line illustrate that the same rational numbers that can be mapped between 0 and 1 can be mapped onto the interval between 1 and 2.



To accept as true that there are an infinite number of fractions between *any* two fractions requires integrating these ideas:

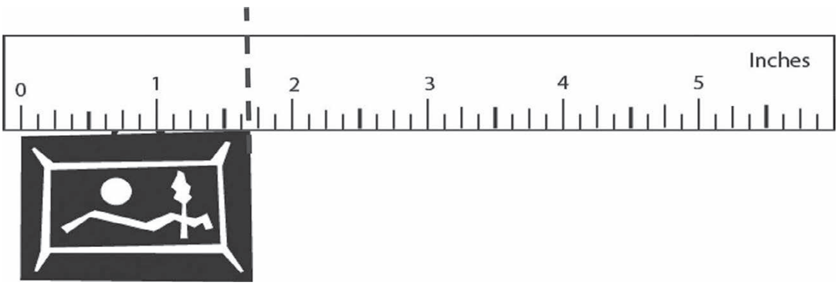
1. The successive averaging of fractions in Figure 10.8 (starting with averaging of $\frac{1}{4}$ and $\frac{1}{2}$) illustrates the use of averages to generate an unending list of different fractions, each average closer to $\frac{1}{4}$ than the previous average. Because all of these averages are between $\frac{1}{4}$ and $\frac{1}{2}$, we have generated an infinite list of fractions that are between $\frac{1}{4}$ and $\frac{1}{2}$.
2. While the example used focused on the fractions $\frac{1}{4}$ and $\frac{1}{2}$, successive averages between any two different fractions would have produced the same effect: Between any two different fractions, there is an infinite number of fractions between them.
3. Any fraction that is located between 0 and 1 can be mapped onto any other consecutive whole number interval as was previously explained and modeled in Figure 10.9.
4. Because there are an infinite number of proper fractions between any two fractions, and they can be mapped to other intervals, there is an infinite number of fractions between any two numbers.

The next section illustrates how the density of rational numbers property is applied in the design of measurement tools.

The Connection of Density of Fractions to Accuracy in Measurements

Measurement is another common area in which people encounter this *betweenness* property of fractions and rational numbers. As you read this section, think about how the halving process (successive averaging) described previously might be used to manufacture measurement instruments of varying accuracy. For example, one may use a ruler (or other scale) to measure something that requires greater accuracy than the tool allows. In Figure 10.10, the object being measured has a length between $1\frac{5}{8}$ inches and $1\frac{3}{4}$ inches. To determine a more exact length of the object would require an understanding that there are fractions between $1\frac{5}{8}$ inches and $1\frac{3}{4}$ inches. In fact, a new ruler that measures in sixteenths of an inch could allow us to decide that the length is about $1\frac{11}{16}$ inches.

Figure 10.10 The picture is about $1\frac{11}{16}$ inches long.



To connect this directly to the concept of density, each mark on the new ruler to sixteenths would be based on averaging adjacent numbers on the pictured ruler. For example, $1\frac{11}{16}$ is the average of $1\frac{5}{8}$ and $1\frac{3}{4}$.

While the example in Figure 10.10 may seem simplistic, the idea and its applications are not. The concept of the density of rational numbers is applied when making measuring tools designed for different purposes. For example, a carpenter building a barn would be content with accuracy to the nearest eighth of an inch. However, a cabinet maker would not. The cabinet maker may need accuracy to 32nds of an inch to assure that components of the cabinet fit together. However, 32nds of an inch would be wholly inadequate for someone making electronic components because electronic components need to be accurate to microns (40-millionths of an inch). Imagine the repartitioning necessary to go from a 32nd of an inch to a 40-millionth of an inch!

To engage students in the concept, you may want them to interview people who use measurement in their work or have students research the accuracy needed to produce different products, such as an iPod, a bicycle, a car, an airplane, or another other item.

CCSSM The CCSSM and Density

Although there are no standards that relate specifically to density of fractions, the concept of density as related to decimal expansion is a central notion in grade 7, "Number System." Here students see partitioning of 10ths into 100ths and 100ths into 1,000ths, as an infinite process, and they use the concepts related to this process to understand finite expansion (decimals that reside on a 10th, 100th, 1,000th, etc. mark on the number line, such as the number 0.235) and infinite expansion (decimals that always reside between the 10th, 100th, 1,000th, etc. mark on the number line, such as the number 0.3) (Common Core Standards Writing Team, 2013b).

Multiple opportunities to locate fractions on number lines, compare and order fractions, and find fractions between fractions provide students the foundations needed to use density concepts in the context of decimals in seventh grade.

Developing Students' Understanding of the Concept

Despite the importance of this concept, researchers indicate that students have a difficult time understanding and applying the concept of the density of rational numbers (Orton et al., 1995). This finding was supported in an assessment of prospective teachers' knowledge of rational numbers involving 147 first-year elementary majors. Tirosh and colleagues (1998) found that "only 24 percent knew there was an infinite amount of numbers between $\frac{1}{5}$ and $\frac{1}{4}$, 43 percent claimed that there are no numbers between one-fifth and one-fourth, and 30 percent claimed that one-fourth is the successor of one-fifth" (pp. 8–9).

These findings are also supported with data from the OGAP (2005) study based on two questions in the OGAP pre- and post-assessments assessing fifth-grade students' understanding of the density of fractions. The first question asked students to name two fractions that are between one-third and three-fourths. The second asked the more general question about the concept: Do you think there are any other fractions besides the ones you identified that are between one-third and three-fourths? Only about 46% $\left(\frac{15}{35}\right)$ of the fifth-grade students in the sample correctly identified two fractions between one-third and three-fourths in the pre-assessment and 60% $\left(\frac{21}{35}\right)$ in the post-assessment (see Table 10.1) (OGAP, 2005).

Richard's response (Figure 10.11) is typical of solutions in which only one fraction was named. While we don't know for sure, Richard may have used his benchmark understanding to recognize that $\frac{1}{3} < \frac{1}{2} < \frac{3}{4}$, but he was unable to apply another strategy or understanding to naming another fraction between $\frac{1}{3}$ and $\frac{3}{4}$.

While Richard seemed limited in his understanding, other students, like Madison (Figure 10.12), either misunderstood the problem or the concept. While Madison is correct that $\frac{2}{3}$ and its equivalent, $\frac{4}{6}$, are between $\frac{1}{3}$ and $\frac{3}{4}$, they are not different fractions, just different names for the same fraction (see Chapter 7, Equivalence). Researchers indicate that using number lines has the potential to help build an understanding of the density of rational number concept (Saxe et al., 2007).

Table 10.1 Percentage of Sample That Identified No Fractions, One Fraction, or Two Fractions in the Grade 5 OGAP Pre- and Post-Assessments

Number of Fractions Identified	*OGAP Pre-Assessment (n = 35)*	*OGAP Post-Assessment (n = 35)*
0	26%	11%
1	29%	29%
2	46%	60%

Figure 10.11 Richard's response—Richard's response leads one to believe that he assumed there were only 2 fractions between $\frac{1}{3}$ and $\frac{3}{4}$.

Name two fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.

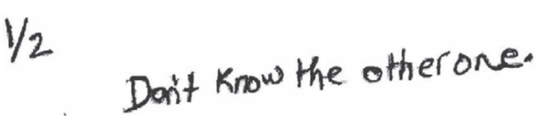
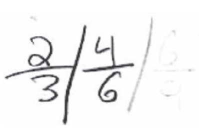


Figure 10.12 Madison's response. Madison identified two equivalent fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.

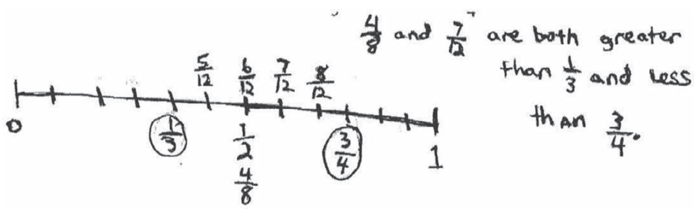
Name two fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.



Todd's (Figure 10.13) and Kaitlyn's (Figure 10.14) responses show some ways in which students use number lines to solve problems involving "betweenness." Todd partitioned a number line into twelfths. This enabled him to identify the twelfths found between $\frac{1}{3}$ and $\frac{3}{4}$.

Figure 10.13 Todd's response. Todd partitioned a number line into twelfths, which enabled him to identify four fractions between $\frac{1}{3}$ and $\frac{3}{4}$.

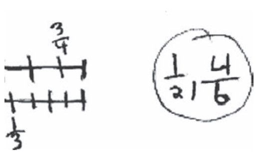
Name two fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.



Kaitlyn (Figure 10.14) partitioned two number lines of equal length: one into fourths and the other into sixths. This enabled her to see the "betweenness" involved in the problem and identify two different fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.

Figure 10.14 Kaitlyn's response. Kaitlyn partitioned two number lines and successfully identified $\frac{1}{2}$ and $\frac{4}{6}$ as fractions between $\frac{1}{3}$ and $\frac{3}{4}$.

Name two fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.

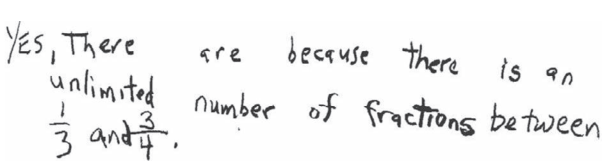


Based on the results from the OGAP pre- and post-assessment, most students were not successful with the second, more general density of fractions question: Do you think there are any other fractions besides the ones you identified that are between $\frac{1}{3}$ and $\frac{3}{4}$? Only one student response in the OGAP pre-assessment sample $\left(\frac{1}{35}\right)$ and two student responses $\left(\frac{2}{35}\right)$ in the post-assessment showed evidence of understanding the infinite nature of the density of rational numbers (OGAP, 2005).

Ava's response (Figure 10.15) illustrates a developing understanding that there are an infinite number of fractions between $\frac{1}{3}$ and $\frac{3}{4}$.

Figure 10.15 Ava's response. Ava recognized that there are unlimited fractions between $\frac{1}{3}$ and $\frac{3}{4}$.

Do you think there are any other fractions besides the ones you identified that are between $\frac{1}{3}$ and $\frac{3}{4}$?



The most common (57% of the sample $\left[\frac{20}{35}\right]$) misconception evidenced for the more general question involved students explicitly stating or alluding to there being a finite set of fractions between the two fractions (OGAP, 2005). This type of response represents a developing understanding—that is, they are able to name other fractions, but have not generalized the concept to recognize that there are infinite fractions between any two fractions.

Todd's response (Figure 10.16) to the more general question about the density of fractions is an example of this developing understanding. Todd explicitly states that "I only named 2 of 5 numbers," inferring that there is a finite set of fractions between the two fractions.

Figure 10.16 Todd's response. Todd's response suggests that he believes there is a limited number of fractions between $\frac{4}{12}$ and $\frac{9}{12}$.

Do you think there are any other fractions besides the ones you identified that are between $\frac{1}{3}$ and $\frac{3}{4}$?

Yes, because $\frac{1}{3}=\frac{4}{12}$ and $\frac{9}{12}=\frac{3}{4}$ and there are 5 numbers in between. I only named 2 out of 5 numbers.

Renee's response in Figure 10.17 illustrates the second most common error found in the OGAP post-assessment sample related to the general question about the density of fractions. Some 26% of the sample indicated that there are more fractions between $\frac{1}{3}$ and $\frac{3}{4}$, but identified equivalent fractions, not different fractions (OGAP, 2005).

Figure 10.17 Renee's response. Renee's response describes finding equivalent fractions "forever."

Do you think there are any other fractions besides the ones you identified that are between $\frac{1}{3}$ and $\frac{3}{4}$? Explain why or why not.

Yes because you can' go on forever looking for equalvilent factions.

Because number lines provide a visual picture of the "betweenness" of fractions, they can be used to extend the developing understanding of each of these students: Richard (Figure 10.11), who named only one fraction; Madison (Figure 10.12) and Renee (Figure 10.17), who named equivalent fractions instead of different fractions; and Todd (Figures 10.13 and 10.16), who used a number line to identify some fractions, but who did not extend that to a more general understanding.

Question 2 in Looking Back provides an opportunity for you to think about how number lines can be used to extend each of these student's developing understanding of the density of fractions.

Case Study—Whole Class Instruction

After reading this chapter Ms. Cane, a sixth-grade teacher, administered the question in Figures 10.11–10.17 and shown in the following. She was surprised by the findings in the study and interested to see how her students responded to these questions.

1. Name two fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.
2. Do you think there are any other fractions besides the ones you identified that are between and $\frac{1}{3}$ and $\frac{3}{4}$? Why or why not?

Consistent with the findings of the OGAP study described earlier, she found that about half of her students identified two fractions between $\frac{1}{3}$ and $\frac{3}{4}$. She also found that a quarter of her students could name one or more additional fractions between $\frac{1}{3}$ and $\frac{3}{4}$. About a third named equivalent fractions like Renee in Figure 10.17. No students indicated that there are an infinite number of fractions between any two numbers. She decided to design a lesson to help her students gain a better understanding of this concept by asking them to respond to a task that involved a fictional student, Sam, making a conjecture about the density of fractions:

> Sam said, "there are an unlimited number of numbers between any two other numbers."
> Is Sam correct? Why or why not?

After posing the task, Ms. Cane decided to use a think, pair, share strategy to gather additional evidence by listening to conversations and probing deeper when the opportunity arose. As she walked around the room listening, she realized that her question was too abstract. Students were choosing whole numbers and stopping there (e.g., deciding that 2, 3, and 4 were the only numbers between 1 and 5). She stopped the class and had some students share what they found and write them on the white board, as shown in Figure 10.18. She asked the students to study the responses on the board and then asked, "What do you notice? What do you wonder?"

Ms. Cane wrote their thoughts on the white board shown in Figure 10.19. Study the students' comments. What do you observe about the student's thoughts?

Figure 10.18 Ms. Cane's recording of student solutions on the board.

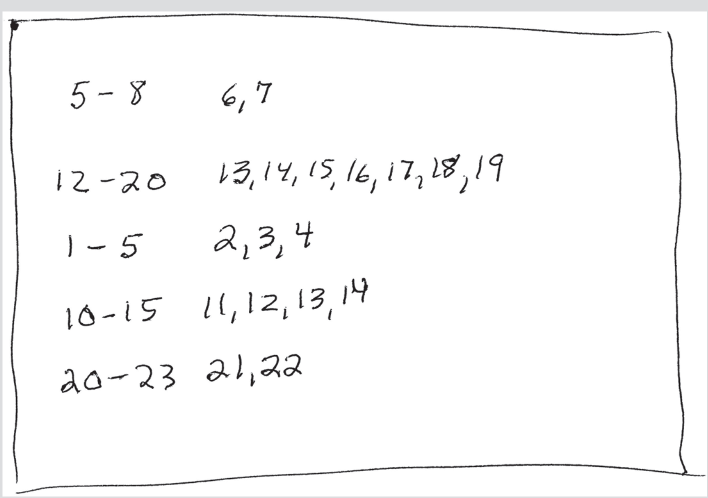
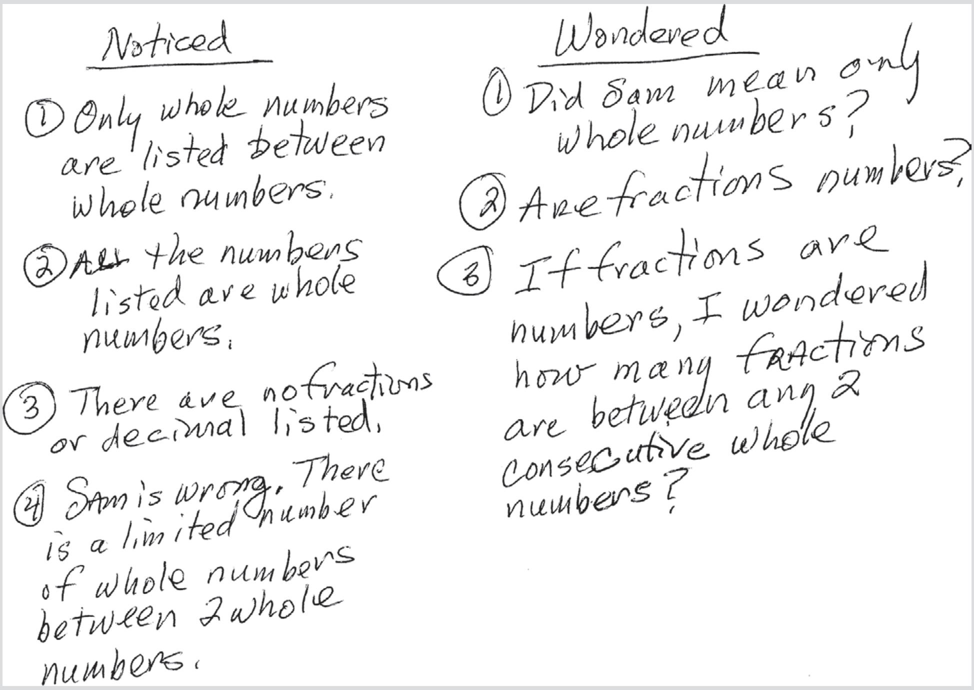


Figure 10.19 Students comments—What do you notice? What do you wonder?


Noticed

1. Only whole numbers are listed between whole numbers.
2. ~~All~~ the numbers listed are whole numbers.
3. There are no fractions or decimal listed.
4. Sam is wrong. There is a limited number of whole numbers between 2 whole numbers.

Wondered

1. Did Sam mean only whole numbers?
2. Are fractions numbers?
3. If fractions are numbers, I wondered how many fractions are between any 2 consecutive whole numbers?

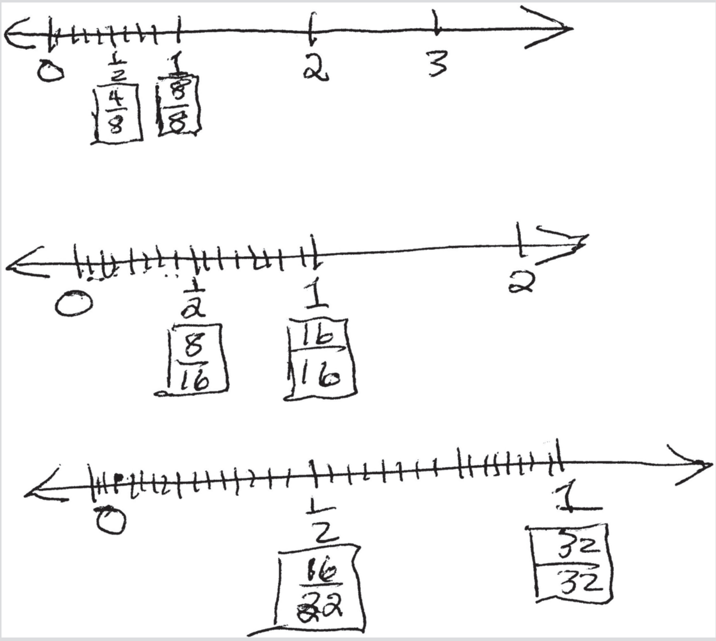
As you probably observed the students also recognized that everyone was only thinking about whole numbers. Some students wondered if Sam meant only whole numbers or wondered if fractions were numbers. After having a discussion that led to students agreeing that fractions are numbers, the class decided to explore Sam's conjecture by looking at numbers between two consecutive whole numbers. Everyone agreed they knew there were some fractions between consecutive whole numbers but still weren't sure if Sam's conjecture—"there are an unlimited number of numbers between any two numbers"—was true.

Ms. Cane asked each pair of students to pick two consecutive whole numbers. She wrote the consecutive whole numbers on the board and asked each pair to choose their number pair or someone else's pair and identify as many fractions as they can between the two numbers. Some pairs picked low numbers like 0 and 1 and 1 and 2 while others picked larger numbers like 56 and 57, and one pair picked 199 and 200.

Arnold and Izola decided to explore the conjecture using the numbers 0–1. They made a couple of number lines that led to agreeing with Sam's conjecture. Study Arnold and Izola's number lines in Figure 10.20. What do you think Arnold and Izola discovered?

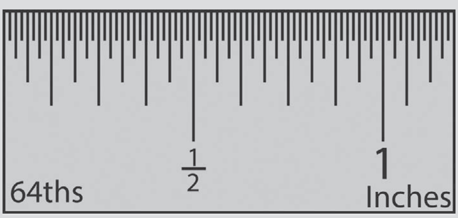
You probably noticed that Arnold and Izola kept making the number line 0–1 longer, realizing that they could partition the number line further. During the group discussion they said, "We first partitioned the number line into 8ths but could not fit any more partitions, so we made the number line longer. We could partition the second longer number line into 16ths and the third longer number line into 32nds. We decided that Sam was right. If our pencil was sharper or our number line was drawn longer and longer, we could go on forever partitioning with no end in sight." Other students had similar experiences and made similar

Figure 10.20 Arnold and Izola's number lines.



conclusions working with their own numbers. One student went online looking for rulers and found the ruler in Figure 10.21 that measures to 32nds and 64ths and stated, "You don't have to make the number line longer, you just need to be more precise in your partitioning which I could never do with my pencil, but machines can."

Figure 10.21 Ruler partitioned to the 32nds and 64ths.



Following the class discussion, Ms. Cane administered the exit question shown in Figure 10.22.

Figure 10.22 Ms. Cane's exit question.

1. Name two fractions that are between $12\frac{1}{4}$ and $12\frac{3}{4}$.
2. Do you think there are any other fractions besides the ones you identified that are between and $12\frac{1}{4}$ and $12\frac{3}{4}$? Why or why not?

Why do you think Ms. Cane administered this exit question?
This question, as you can see, asks students to identify other numbers between two mixed numbers instead of just between two whole numbers or between 0 and 1, as in the original problem and the focus of the class discussion. By using an exit question, Ms. Cane hoped to gather evidence of developing understanding from each student.

This case study illustrates an important point about the OGAP cycle—you can use the evidence from a formative assessment item to inform whole class instruction when you notice that most students do not show understanding of an important concept. Ms. Cane allowed students to grapple with this idea, rather than just telling them that there are an infinite amount of numbers between any two numbers. She recognized that students needed time to explore with number lines to really think about this concept. She then used another formative assessment item to gather evidence of their developing understanding after the lesson.

Density at the Elementary and Middle School Levels

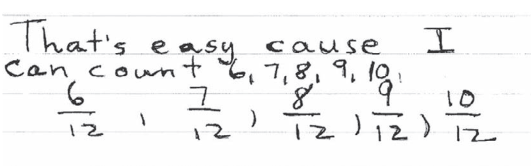
Most elementary and middle-school students' experience with density is limited to identifying fractions between fractions. Teachers should carefully select pairs of fractions that strategically and thoughtfully expand their students' abilities to identify fractions between fraction pairs and to develop their understanding of the generalized concept.

For example, one might provide students fraction pairs with common denominators, such as $\frac{5}{12}$ and $\frac{11}{12}$. Kim's counting strategy in Figure 10.23 is effective in identifying several fractions between $\frac{5}{12}$ and $\frac{11}{12}$. Notice however, that this strategy is limiting in that it does not help one see that that are an infinite number of fractions between $\frac{5}{12}$ and $\frac{11}{12}$, not just five.

On the other hand, choosing a pair of fractions based on students' experience is important. For younger students, asking them to find two fractions between $\frac{1}{3}$ and $\frac{1}{4}$ may be unreasonable because the fractions are very close to each other.

Figure 10.23 Kim's response. Kim may have used her whole number understanding of counting numbers.

Name 3 fractions that are between $\frac{5}{12}$ and $\frac{11}{12}$.



You may want to start with fractions such as $\frac{1}{3}$ and $\frac{3}{4}$. These fractions are not so close together, are familiar fractions, and are on either side of the benchmark $\frac{1}{2}$. Over time, however, students should be able to use their understanding of partitioning and equivalence to identify fractions between a variety of given fractions. Older students should solve density of rational number problems involving fraction pairs that are very close, such as $\frac{1}{10}$ and $\frac{1}{11}$, and fraction pairs that include mixed numbers and improper fractions.

Finding fractions between fractions that are "very close" requires a more generalized understanding and greater flexibility than partitioning a number line, as Todd did in Figure 10.13, or using benchmark fractions, as Richard did in Figure 10.11.

Question 3 in Looking Back provides an opportunity for you to determine fractions between pairs of fractions using a range of reasoning strategies.

The *OGAP Fraction Progression* and Density

Students at the *Fractional Strategies* level are able to explain why there are an infinite number of fractions between any two numbers. An explanation at the *Fractional Strategy* level should express this understanding. *Transitional Strategies* will be evidenced using visual models to identify fractions between fractions, as seen in the student work in Figures 10.13 and 10.14. Strategies at the *Early Fractional Strategies* level suggest that there are a limited number of fractions between the numbers being considered. Richard's response in Figure 10.11 is an example of an early fractional strategy. Solutions that indicate there are no fractions between other fractions or that rely only on whole number reasoning are considered *Non-Fractional Strategies*.

Chapter Summary

This chapter focused on the concept of the density of fractions with an emphasis on:

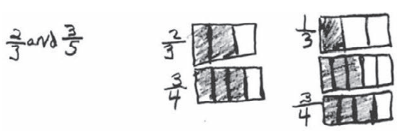
- the concept of density of fractions
- misunderstandings that students have as they are identifying fractions between fraction pairs and developing an understanding of the generalized concept of density
- the role number lines can play in helping build students' understanding of the *betweenness* of fractions.

Looking Back

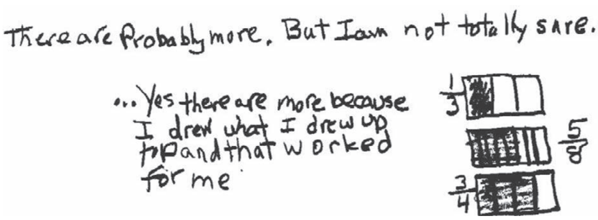
1. Review Seth's response in Figure 10.24 and then answer the following questions.

Figure 10.24 Seth's response.

A) Name two fractions that are between $\frac{1}{3}$ and $\frac{3}{4}$.



B) Do you think there are any other fractions besides the ones you identified that are between $\frac{1}{3}$ and $\frac{3}{4}$?



a. How did Seth use his understanding of partitioning to answer part A of the question?
b. Based on Seth's response to part B, what are the strengths and limitations of his partitioning strategy?

2. It was suggested in the chapter that number lines could be used to extend the developing understanding of Richard (Figure 10.11), Madison (Figure 10.12), and Todd (Figures 10.13 and 10.16). Review each response and then answer the following questions.

a. Richard (Figure 10.11) named only one fraction. Provide an example of a way the number line could extend his understanding to identifying different fractions between $\frac{1}{3}$ and $\frac{3}{4}$ besides $\frac{1}{2}$.
b. Madison (Figure 10.12) named equivalent fractions instead of other fractions. Provide an example of a way the number line could extend her thinking beyond equivalent fractions.
c. Todd (Figures 10.13 and 10.16) used a number line to identify some fractions, but did not extend that to a more general understanding. Provide an example of a way the number line could extend his thinking beyond equivalent fractions.

3. Find three different fractions between the following fraction pairs using two different strategies for each fraction pair. Then answer the questions that follow.

$\frac{4}{10}$ and $\frac{7}{10}$

$\frac{1}{8}$ and $\frac{1}{4}$

$\frac{1}{10}$ and $\frac{1}{9}$

a. What difficulties do you think students might encounter as they solve these problems?
b. What kinds of errors might result from these difficulties?
c. As a set of questions, what information can the student work provide that the evidence from a single question might not provide?

Instructional Link: Your Turn

Use the guiding questions in Table 10.2 to help you think about how your instruction or mathematics program provides students with the opportunities to solve problems involving the density of rational numbers.

Is there anything in your instruction or mathematics program (activities, games, lessons, problems) that *intentionally* provides opportunities for students to transfer their knowledge of partitioning and equivalence to identifying a fraction or fractions that are between any two given fractions?

Table 10.2 Strategies to Support Development of Concepts Related to Density of Fractions

Do you (or does your program):	*Yes/No*
1. Provide opportunities for students to use number lines to develop and expand their understanding of density?	
2. Make a connection between density of fractions and accuracy in measurements?	
3. Provide students opportunities to solve problems that promote a clear conceptualization of the density of fractions?	

Based on this analysis, what gaps in your instruction or mathematics program did you identify? How might you address these gaps?

11
Addition and Subtraction of Fractions

Big Ideas

- Procedural fluency and conceptual understanding work together to deepen student understanding of fraction addition and subtraction.
- Conceptual understanding of addition and subtraction of fractions is built using visual models, estimation, unit fraction understanding, equivalence, and properties of operations.

Conceptual Understanding and Procedural Fluency

An important goal of fraction instruction is to ensure that students develop procedural fluency when adding and subtracting fractions. "Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (National Research Council [NRC], 2001, p. 121).

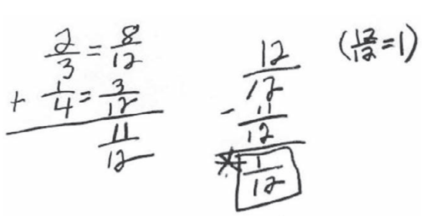
It is important to understand, however, that procedural fluency alone is not sufficient to ensure proficiency with addition and subtraction of fractions. Procedural fluency works together with conceptual understanding, each contributing to a deeper understanding of the other.

> Conceptual understanding refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful.
>
> (NRC, 2001, p. 118)

Figure 11.1 shows Kenny's strategy for solving a multistep fraction operation problem. Kenny finds and uses common denominators to add $\frac{2}{3}+\frac{1}{4}$, then uses his understanding that $\frac{12}{12}$ = 1 to find the missing fractional part.

Figure 11.1 Kenny's response. The response shows an understanding of equivalent fractions.

Tina ate $\frac{2}{3}$ of her candy and gave $\frac{1}{4}$ of her candy to her sister. She saved the rest of her candy. What is the fractional part of the candy that Tina saved?



DOI: 10.4324/9781003185475-11

To determine Kenny's overall proficiency regarding adding and subtracting fractions, however, one would need to consider Kenny's understanding of addition and subtraction across contexts and with a variety of fractions (e.g., fractions with the same and different denominators, mixed numbers). That said, the evidence suggests that Kenny is on his way to becoming both procedurally fluent and conceptually sound with fraction addition and subtraction.

Fraction instruction that includes thoughtful use of visual models and reasoning strategies based on an understanding of the magnitude of fractions can lead to both procedural fluency and conceptual understanding of addition and subtraction of fractions. Researchers have found that students who can translate between various fraction representations "are more likely to reason with fraction symbols as quantities and not as two whole numbers" (Towsey, as cited in Huinker, 2002, p. 4) when solving problems.

The solutions in Figures 11.2 and 11.3 are examples of using visual models (an area model and a number line) to solve addition and subtraction problems.

Juan did not explicitly use common denominators to find the amount of candy that Tina saved, but evidence on his number line reveals a developing understanding of equivalence. For example, Juan partitioned his number line into thirds and twelfths. He used the partitioning to locate $\frac{1}{4}$ and $\frac{2}{3}$ on the number line, and then recognized that $\frac{1}{12}$ was left.

Figure 11.2 Felicia's response. Felicia used an area model to find the sum of $\frac{1}{4}+\frac{5}{8}$.

Josh and Alison ordered one pizza. Josh ate $\frac{1}{4}$ of the pizza and Alison ate $\frac{5}{8}$ of the pizza. Did Josh and Alison eat the whole pizza?

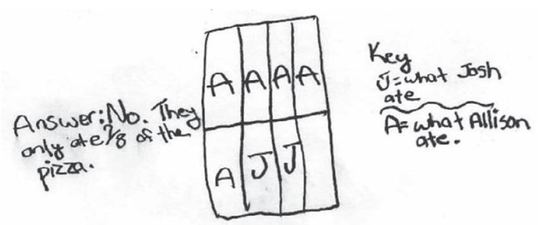
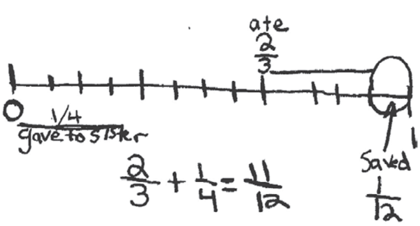


Figure 11.3 Juan's response. Juan used a number line in his solution to find the candy Tina saved.

Tina ate $\frac{2}{3}$ of her candy and gave $\frac{1}{4}$ of her candy to her sister. She saved the rest of her candy. What is the fractional part of the candy that Tina saved?



Both Felicia and Juan appear to be in a good position to move to a more efficient strategy that is grounded in conceptual understanding of equipartitioning, equivalence, and recognition of an additive situation. Future instruction should include opportunities for these students to compare the information presented in a visual model with a calculation to ensure that they can:

- recognize the answer from a visual model
- use an algorithm to solve fraction addition/subtraction problems

- compare answers from calculations with answers from visual models
- complete similar calculations without reference to a visual model

Using a visual model is one way to solve the pizza problem in Figure 11.2. Another way to solve this problem is by reasoning, which involves an understanding of the magnitude of the fractions $\frac{1}{4}$ and $\frac{5}{8}$, particularly because an exact answer is not a requirement of the problem.

Sample reasoning strategies:

1. Five-eighths is three-eighths less than the whole pizza. One-fourth is the same as two-eighths. That leaves one-eighth uneaten.
2. Five-eighths is one-eighth more than one-half $\left(\frac{4}{8}\right)$ of the pizza. One-fourth is the same as two-eighths. One-eighth (the amount five-eighths is over a half) plus two-eighths do not equal one-half of the pizza. So, one-half of the pizza and less than one-half of a pizza is less than a whole pizza.
3. Five-eighths is three-eighths less than a whole pizza. One-fourth of a pizza is less than three-eighths of the pizza. So, $\frac{5}{8}+\frac{1}{4}$ is less than a whole pizza.

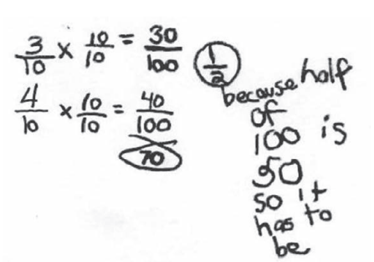
Many times, instruction in adding and subtracting fractions focuses primarily on facility with algorithms and less on the conceptual underpinnings. Premature experience with formal procedures may lead to symbolic knowledge that is not based on understanding or connected to the real world. This may impede students' number and operation sense (Kiernan, as cited in Huinker, 2002).

Figures 11.4 and 11.5 contain pieces of student work that exemplify this point. You will notice that both students utilize an algorithm to solve the problems. The type of partial understandings and errors seen in these examples are typical of students who use algorithms without full understanding of the underlying concepts.

Felix (Figure 11.4) found a common denominator of 100 even though the fractions given in the problem already share a common denominator of 10. In doing this, he may have made the problem unnecessarily harder.

Figure 11.4 Felix's response. Felix found a common denominator even though each fraction has a denominator of 10.

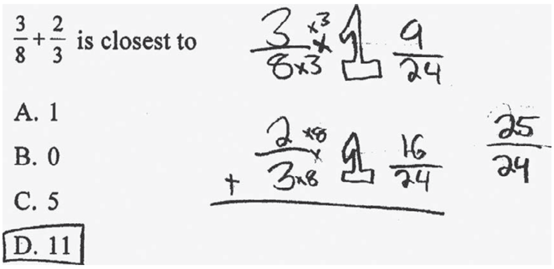
Jill walked her dog $\frac{3}{10}$ of a mile on Saturday and $\frac{4}{10}$ of a mile on Sunday. Is the total distance Jill walked her dog on Saturday and Sunday closer to $\frac{1}{2}$ of a mile or 1 whole mile?



Darcie (Figure 11.5) used an algorithm correctly to convert the given fractions to fractions with a common denominator and found the correct sum. However, her choice of 11 as being closest to $\frac{25}{24}$ provides evidence of a lack of understanding of the magnitude of the fractions in the problem or in the solution.

It is important to note that if Darcie had been asked to only compute $\frac{3}{8}+\frac{2}{3}$, one might have been confident that she is well on her way to developing proficiency with fraction operations.

Figure 11.5 Darcie's response. Darcie calculated the correct sum, but she concluded that $\frac{24}{25}$ is closest to 11, not to 1.



Of course, there is no way of knowing whether Darcie's solution is the result of premature focus on formal algorithms. However, research indicates that students can struggle with the use and understanding of formal algorithms when their knowledge is dependent primarily on memory, rather than anchored with a deeper understanding of the foundational concepts (Kieren, as cited in Huinker, 2002). Solutions like Darcie's serve as a reminder of the importance of focusing on understanding as students are developing efficient procedures.

Using Visual Models, Equipartitioning, and Equivalence to Develop Fraction Addition and Subtraction Concepts

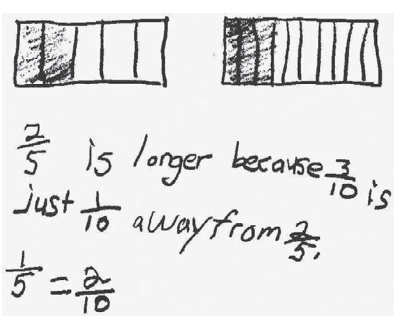
Even though instruction in addition and subtraction of fractions does not formally begin until the upper elementary grades, the roots of these operations begin in earlier grades as students equipartition visual models to represent fractional parts of a whole and compare fractions. Fraction addition and subtraction ideas are not isolated from fraction concepts explored in earlier grades, but rather are a logical continuation of unit fraction, equivalence, and magnitude concepts. As students' understanding of fraction concepts grows, they often move from ordering and comparing fractions and finding equivalent fractions to adding and subtracting fractions (OGAP, 2005).

The examples that follow, like those shown earlier, provide evidence of students' developing understanding of equipartitioning, unit fractions, and equivalence to compare fractions.

In Figure 11.6, Holly states that "$\frac{3}{10}$ is just $\frac{1}{10}$ away from $\frac{2}{5}$." She is describing an additive relationship between $\frac{3}{10}$ and $\frac{2}{5}$. This relationship can be interpreted as $\frac{3}{10}+\frac{1}{10}=\frac{2}{5}$ or $\frac{2}{5}-\frac{1}{10}=\frac{3}{10}$. Holly's solution is an example of how fraction addition and subtraction concepts can logically develop from investigations of equivalence and magnitude.

It is important to note that Holly's ability to use a visual model to represent $\frac{2}{5}$ and $\frac{3}{10}$, including her attention to the size of the whole and accurate equipartitioning of the whole, made

Figure 11.6 Holly's response. Holly created models and used equivalence to compare the relative size of $\frac{3}{10}$ and $\frac{2}{5}$.



it possible for her to consider how much larger $\frac{2}{5}$ is than $\frac{3}{10}$. These ideas, together with her developing understandings of the magnitude of fractions, serve as a foundation upon which she can build additive concepts and skills.

Patrick's visual models in Figure 11.7 show that $\frac{2}{10} = \frac{1}{5}$ and $\frac{4}{10} = \frac{2}{5}$. This can be an important first step leading to an understanding of common denominators, a concept that is integral to developing an efficient algorithm to solve addition and subtraction problems. Although there is no evidence that Patrick contemplated addition or subtraction ideas, this work is ripe for those ideas to take root.

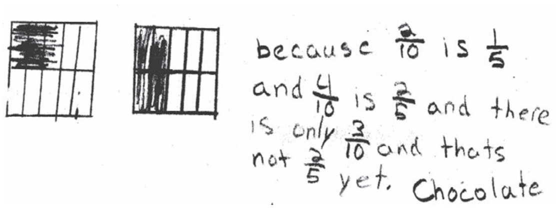
Figure 11.7 Patrick's response. Patrick used models to compare $\frac{2}{5}$ and $\frac{3}{10}$.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies in the dish?



Teachers can facilitate this type of additive reasoning as students solve equivalence and magnitude problems by asking questions such as:

- How much greater or less is one fraction than another?
- How much would you have to add to or subtract from one fraction to equal the other?
- Create and describe a visual model that shows not only which fraction is greater, but also how much greater.
- Use your visual model to show equivalence.

Go-To Chapter 7, Equivalence, and Chapter 8, Comparing and Ordering, for an in-depth discussion of developing understanding of equivalence and magnitude.

The Importance of Estimation When Adding and Subtracting Fractions

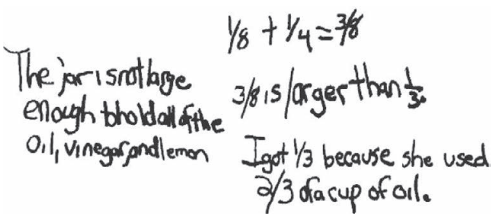
"The development of a quantitative notion, or an awareness of the 'bigness' of fractions is very important" (Bezuk & Bieck, 1993, p. 127). Estimation plays a critical role in students' development of procedural fluency (Siegler et al., 2010) and conceptual understanding with fraction addition and subtraction. Procedural fluency includes the ability to recognize the most efficient way to solve a problem. A student with procedural fluency knows when and how to use a certain strategy depending on the problem and possesses the ability to use estimation to judge the reasonableness of an answer, as Lisa did in Figure 11.8.

Figure 11.8 Lisa's response. Lisa used her understanding of the magnitude of the given fractions to correctly answer the question.

Aunt Sally has a jar that holds one cup of liquid.

Her salad dressing recipe calls for $\frac{2}{3}$ cups of oil, $\frac{1}{8}$ cups of vinegar, and $\frac{1}{4}$ cups of juice.

Is the jar large enough to hold all the oil, vinegar, and juice?



Lisa did not calculate an exact answer (although she did determine the sum of $\frac{1}{8}+\frac{1}{4}$) or create a visual model to represent the situation. Instead, she used her understanding of the magnitudes of $\frac{1}{3}$, $\frac{3}{8}$, and $\frac{2}{3}$ to determine that the jar is not large enough to hold all three liquids.

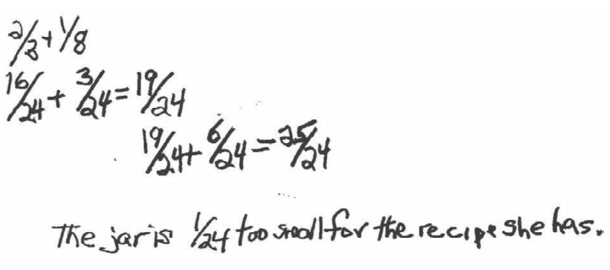
Cody's solution, shown in Figure 11.9, is correct and suggests facility with a fraction addition algorithm. However, it may not be the most efficient solution for this particular problem. Cody's solution illustrates the notion that students often do not apply their understanding of the magnitude (or meaning) of fractions when they operate with them (NRC, 2001).

Figure 11.9 Cody's response. Cody calculated a specific sum even though an exact answer is not needed in this particular context.

Aunt Sally has a jar that holds one cup of liquid.

Her salad dressing recipe calls for $\frac{2}{3}$ cups of oil, $\frac{1}{8}$ cups of vinegar, and $\frac{1}{4}$ cups of juice.

Is the jar large enough to hold all the oil, vinegar, and juice?



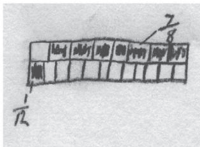
Using visual models to estimate a sum, as Oscar did in Figure 11.10, is a useful step along the continuum that leads to estimating sums in a more abstract manner. His experiences with drawing visual fraction models can help Oscar develop efficient estimation strategies, like the one Lisa (Figure 11.8) used, for determining the relative magnitude of fractions.

An important instructional point is exemplified by Cody's, Oscar's, and Lisa's responses: "Students need facility with a variety of computational tools, and they need to know how to select the appropriate tool for a given situation" (NRC, 2001, p. 122). Although each response includes a different strategy to answer the questions correctly, Lisa's strategy of reasoning about the quantities involved may be the most efficient, given a context that does not require an exact numerical answer. Oscar's solution using a physical visual model might illustrate a developing understanding of estimating fraction sums. The evidence in Cody's response suggests that he is developing fluency with

Figure 11.10 Oscar's response. Oscar did not calculate an exact answer. Instead, he used a model to show the relative size of $\frac{7}{8}$ and $\frac{1}{12}$.

The sum of $\frac{1}{12}+\frac{7}{8}$ is closest to:

a) 20
b) 8
c) $\frac{1}{2}$
d) 1



an algorithm when adding and subtracting fractions. Although this may not be the most efficient strategy for this problem, it may be quite efficient for a problem that requires an exact sum.

Unit Fractions and Fraction Addition and Subtraction

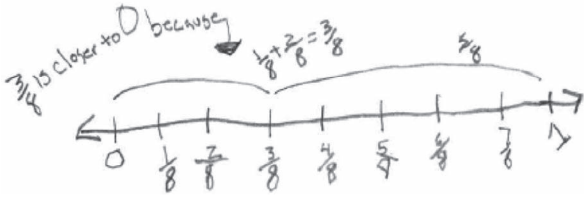
As discussed in Chapter 1, unit fractions are the building blocks for developing understanding of fraction addition and subtraction. Students begin building the understanding that fractions are composed of unit fractions when they use number lines and area models to add fractions with common denominators. Sam's solution in Figure 11.11 provides evidence that he understands fractions are built from unit fractions and that the sum of fractions with like denominators is the sum of the associated unit fractions.

This essential notion that fractions are composed of unit fractions can help students transition from reliance on number lines and other visual models to solve problems involving addition and subtraction of fractions with common denominators $\left(\text{e.g.,} \frac{3}{4}+\frac{2}{4}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{5}{4}\right)$.

Figure 11.11 Sam used a number line and unit fractions to determine that $\frac{2}{8}+\frac{1}{8}=\frac{3}{8}$ and that $\frac{3}{8}$ is closest to 0.

$\frac{1}{8}+\frac{2}{8}$ is closest to

A) 1
B) 0
C) 3
D) 16



Ethan's response in Figure 11.12 suggests an understanding that $\frac{3}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$. He uses this to subtract $\frac{3}{5}$ from $2\frac{1}{5}$.

Figure 11.12 Ethan's response. Ethan used his understanding that fractions can be decomposed into unit fractions to subtract $\frac{3}{5}$ from $2\frac{1}{5}$.

The distance from Billy's house to work is $2\frac{1}{5}$ miles. His car broke down $\frac{3}{5}$ of a mile from work. How far is Billy from his house? Show your work.

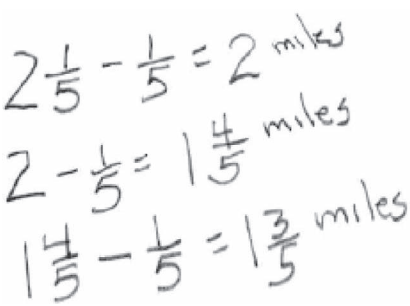
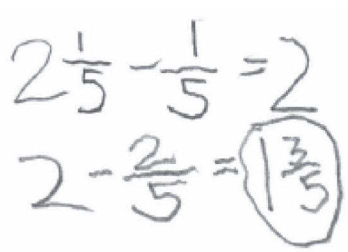


Figure 11.13 Alyssa's response. Alyssa decomposed $\frac{3}{5}$ into $\frac{1}{5}+\frac{2}{5}$ and used the decomposition to determine the distance Billy is from his house.

The distance from Billy's house to work is $2\frac{1}{5}$ miles. His car broke down $\frac{3}{5}$ of a mile from work. How far is Billy from his house? Show your work.



Alyssa's response in Figure 11.13 shows evidence that she used unit fraction understanding to decompose $\frac{3}{5}$ into $\frac{1}{5}+\frac{2}{5}$.

When students solve problems involving addition and subtraction of fractions with unlike denominators, they combine their understanding of addition and subtraction of fractions (developed using unit fraction understanding) with their understanding of equivalence by finding common denominators before adding or subtracting the fractions. Kenny's response in Figure 11.1 illustrates the use of common denominators when adding fractions as well as evidence of unit fraction understanding when he indicates that $\frac{12}{12} = 1$.

Adding and Subtracting Mixed Numbers

Up to this point, the chapter has focused on building conceptual understanding and procedural fluency with addition and subtraction of fractions less than 1. You have seen examples illustrating how this knowledge can develop from visual models, partitioning, reasoning about the magnitude of fractions, and fractions as iterations of unit fractions. It is not surprising that these same ideas are fundamental to students' understanding of adding and subtracting mixed numbers and improper fractions.

We discussed earlier in the chapter how fraction addition and subtraction concepts can build from thoughtful exploration of equivalence and magnitude. Figures 11.14 and 11.15 illustrate this point.

Figure 11.14 Marcus's response. Marcus used the benchmark 1 to compare $1\frac{1}{2}$ and $\frac{9}{8}$.

Susan ate $1\frac{1}{2}$ cupcakes and Billy ate $\frac{9}{8}$ cupcakes.

Who ate more cupcakes?

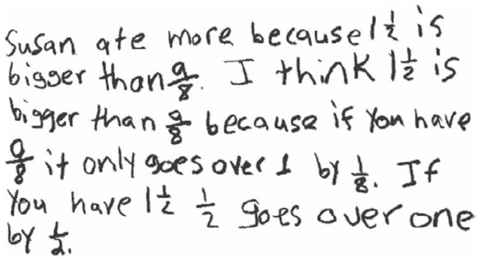


Figure 11.15 Gregory's response. Gregory appears to incorrectly base his decision on whether a whole number is explicit in the fraction notation given in the problem.

Susan ate $\mathbf{1}\frac{1}{6}$ cupcakes and Billy ate $\frac{8}{7}$ cupcakes.

Who ate more cupcakes?

Marcus may be in a good position to add $1\frac{1}{2}$ and $\frac{9}{8}$ because he has the sense of the magnitude of the two numbers relative to 1, and he converted the improper fraction $\frac{9}{8}$ to $1\frac{1}{8}$. Based on the evidence in Marcus's work, one might expect that he would easily recognize the sum to be greater than 2.

Unlike Marcus's solution, the evidence in Gregory's work (Figure 11.15) leads one to believe that he does not understand the magnitude of the improper fraction $\frac{8}{7}$. Gregory ignores the fraction part of the mixed numbers and considers only the whole numbers.

Gregory might benefit from opportunities to represent fractions greater than 1 on number lines and with area visual models, generalizing the relationships between mixed numbers and equivalent fractions. Marcus, on the other hand, appears to be ready to consider questions such as:

- How many cupcakes did Susan and Billy eat together?
- How many more cupcakes did Susan eat than Billy?

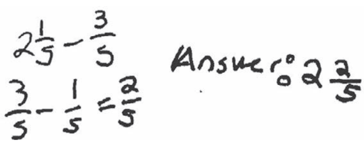
In an error analysis related to addition and subtraction of mixed numbers and improper fractions, Tatsuoka (1984) found that students may subtract the smaller fractional part of one mixed number from the larger fractional part of another regardless of the context. Renee's solution shown in Figure 11.16 provides an example of this error.

Figure 11.16 Renee's response. Renee ignored the whole number part of the mixed number and incorrectly subtracted the smaller fractional part, $\frac{1}{5}$, from the larger fraction, $\frac{3}{5}$.

The distance from Billy's house to work is $2\frac{1}{5}$ miles.

His car broke down $\frac{3}{5}$ of a mile from work.

How far is Billy from his house?



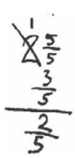
Another common error related to subtracting mixed numbers involves errors in regrouping. Students may borrow by reducing the whole number by one but incorrectly add that amount to the fraction part of the number (e.g., students add 10 to the denominator or ignore the value of the fraction part of the number) (Tatsuoka, 1984). This is exemplified in Shaun's response, shown in Figure 11.17. He subtracted 1 from 2 but incorrectly added this amount to $\frac{1}{5}$. In performing the subtraction, he seemed to ignore the whole number part of the mixed number.

Figure 11.17 Shaun's response. Shaun recognized that the problem could be solved using the operation $2\frac{1}{5} - \frac{3}{5}$. However, when borrowing, he seemed to ignore the fraction part of the mixed number, $2\frac{1}{5}$.

The distance from Billy's house to work is $2\frac{1}{5}$ miles.

His car broke down $\frac{3}{5}$ of a mile from work.

How far is Billy from his house?



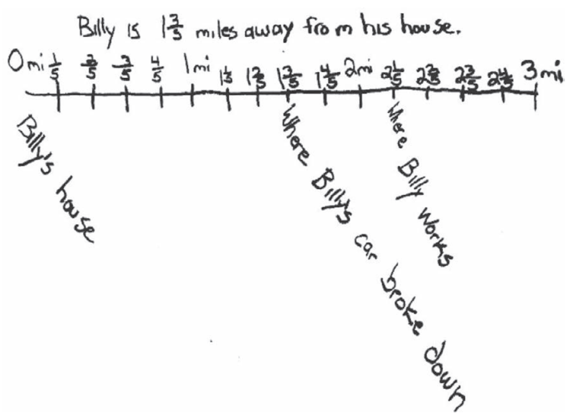
Gregory, Shaun, and Renee all made errors that led to answers with unreasonable magnitudes given the problem (e.g., subtracting $\frac{3}{5}$ from $2\frac{1}{5}$ and obtaining an answer of $\frac{2}{5}$). For this reason, all three students might benefit from representing their problems on a number line, much as Lola did in Figure 11.18.

Figure 11.18 Lola's response. Lola used a number line to successfully solve this problem involving mixed numbers.

The distance from Billy's house to work is $2\frac{1}{5}$ miles.

His car broke down $\frac{3}{5}$ of a mile from work.

How far is Billy from his house?



Notice that Lola's visual model suggests an understanding of the relative magnitude of the fractions presented in the problem. Lola is ready to use this conceptual understanding to develop a more generalized strategy for adding and subtracting mixed numbers.

CCSSM

The CCSSM and Adding and Subtracting Fractions

Procedural fluency with addition and subtraction of fractions is expected by the end of fifth grade. The development of procedural fluency for adding and subtracting fractions is anchored in the use of visual models, unit fraction understanding, and properties of operations in earlier grades.

At grade 3, students investigate unit fractions and understand that fractions are built by iterating unit fractions. For example, through the use of number lines and visual models, students at grade 3 learn that $\frac{3}{4}$ is formed from three units of size $\frac{1}{4}$, thus leading to the understanding that $\frac{3}{4}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$. See Figures 1.2, 1.10, and 1.11 in Chapter 1, Understanding a Fraction as a Number, for examples of visual models that illustrate these concepts.

At grade 4, students extend unit fraction understanding by decomposing fractions into the sum of fractions with the same denominator. This allows them to understand that fractions can be represented in various ways using addition $\left(e.g., \frac{5}{6}=\frac{1}{6}+\frac{4}{6}; \frac{5}{6}=\frac{2}{6}+\frac{3}{6}\right)$. This provides coherence because this concept is closely related to what students have learned in previous grades regarding how whole numbers can be decomposed and represented in multiple ways. This also supports foundational understanding that addition of fractions with like denominators can be thought of as the sum of unit fractions. For example, $\frac{3}{4}+\frac{3}{4}$ can be interpreted as $\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)$. Using visual models, unit fraction understanding, properties of operations, and the relationships between addition and subtraction, students at grade 4 are expected to solve addition and subtraction problems that involve proper fractions and mixed numbers.

At grade 5, students combine their understanding of addition and subtraction of fractions with like denominators together with equivalence concepts to solve problems involving addition and subtraction of proper fractions and mixed numbers with unlike denominators.

The Importance of the Commutative and Associative Properties

Students can use their understanding of the commutative and associative properties for addition of whole numbers to support their development of concepts related to adding fractions. In particular, these properties can help build a deeper understanding of the concepts underlying addition of mixed numbers. To begin this discussion, it is important to understand that a mixed number is the sum of a whole number and a fraction less than 1. Thus, the number $3\frac{1}{4}$ means $3 + \frac{1}{4}$.

Consider the following problem:

> Richard worked on a project due in science class. During the first week he worked for $2\frac{3}{5}$ hours. During the second week he worked for $2\frac{1}{4}$ hours. How many hours did Richard work on his project during these two weeks?

Students may learn how to follow a procedure to solve mixed-number addition problems like this one. For many students, the procedure is not coherently linked to properties of operations that can bring conceptual meaning to the procedure. Figure 11.19 is an example of a common procedure for adding mixed numbers.

Figure 11.19 A typical algorithm for adding mixed numbers.

$$\begin{array}{rr} & 3\frac{3}{5} = \frac{12}{20} \\ + & 2\frac{1}{4} = \frac{5}{20} \\ \hline & 5\frac{17}{20} \end{array}$$

In this procedure, students learn to set up the mixed numbers so that the two whole numbers and the two fractions are aligned. The next step is to find equivalent fractions with common denominators and to notate them using equal signs. Finally, students learn to add the whole numbers, add the fractions, and write the sum as a mixed number. This procedure will result in a correct sum, but for many students it is a collection of steps to remember rather than a procedure built from meaning.

Although the answer in Figure 11.19 is correct, it is important to point out that the equal signs are not used correctly: $3\frac{3}{5} \neq \frac{12}{20}$ and $2\frac{1}{4} \neq \frac{5}{20}$. While it may not be apparent in the procedure shown in Figure 11.19, this solution is based on important mathematics properties. Figure 11.20 illustrates these concepts.

The relationships conveyed in the solution shown in Figure 11.20, $3\frac{3}{5} + 2\frac{1}{4} = 3 + 2 + \frac{3}{5} + \frac{1}{4}$, present the opportunity for an accurate estimate of the sum of the two mixed numbers. One can reason that the sum is between 5 and 6. In fact, using one's understanding that $\frac{3}{5} > \frac{1}{2}$ and the meaning of the unit fraction $\frac{1}{4}$, one can conclude that the sum of $3\frac{3}{5} + 2\frac{1}{4}$ is closer to 6 than it is to 5. Also notice that the equal signs are used correctly throughout this example.

Figure 11.20 A procedure for adding mixed numbers is built on decomposition, the commutative property, equality, and the associative property.

Original problem ⟶ $3\frac{3}{5} + 2\frac{1}{4} =$

Mixed numbers **decomposed** into additive parts $3 + \frac{3}{5} + 2 + \frac{1}{4} =$

Commutative Property (changed the order of addends) $3 + 2 + \frac{3}{5} + \frac{1}{4} =$

Equality (renamed fractions with common denominators) $5 + \frac{12}{20} + \frac{5}{20} =$

Associative Property (added the fractions first, then added 5) $5\frac{17}{20}$

Case Study—Facilitating a Mini-Lesson

Ms. Pratt, a fifth-grade teacher, had never considered the ways in which a focus on reasoning strategies and the properties of operations could help students more deeply understand addition of mixed numbers. She had just introduced her students to the algorithm for adding mixed numbers as shown in Figure 11.19 and they seemed to be able to use it to add mixed numbers that did not require regrouping and mixed numbers with fractions that had the same denominator. She decided to administer a new formative assessment problem:

> Sylvia had $2\frac{3}{4}$ yards of blue fabric and $3\frac{2}{3}$ yards of red fabric. How much fabric did she have altogether? Show your work.

Why do you think this is a good problem for her to administer to her students at this point in her instruction of mixed number addition?

Ms. Pratt selected this problem because it would provide evidence of her students' ability to solve a mixed number addition problems in context. It would also provide valuable information about her students' understanding of how to deal with fractional amounts that summed to more than one whole when adding mixed numbers.

Ms. Pratt analyzed her student responses by using the *OGAP Sort* and recording the evidence on the *OGAP Evidence Collection Sheet* shown in Figure 11.21.

After completing the *OGAP Sort* and recording the evidence on the *OGAP Evidence Collection Sheet*, she made some observations before addressing the OGAP planning questions. Several students effectively used a visual model to add the mixed numbers. She noted that a couple of students used the algorithm she had taught them to find common denominators and understood what to do with the improper fraction, as exemplified in Natalie's solution in Figure 11.22. However, she also noticed that a couple of students converted both mixed numbers to improper fractions and then applied a common denominator to add them, but then didn't know how to interpret the resulting improper fraction in the context of the problem, as seen in the evidence in Azro's solution. She wondered if students like Azro understood the magnitude of the answer of $\frac{68}{12}$ in the context of yards of fabric. Other students made procedural errors while attempting the use of the algorithm. She also saw several papers like Kim's response with an answer of $5\frac{5}{7}$, showing the inappropriate use of whole number reasoning to add the numerators and denominators separately.

Figure 11.21 *OGAP Evidence Collection Sheet*

OGAP Fraction Evidence Collection Sheet

Content	Fractional Strategies	Transitional Strategies	Early Fractional Strategy	Non-fractional
Sylvia adding $2\frac{3}{4}$ yds. to $3\frac{2}{3}$ yds	Natalie, Grace, Clare – Used decomposition Logan, Azro – Used algorithm note: Did not convert improper fraction to mixed number	Nolan, Ella, Thomas, Asher, Lucas, Patricia – Effectively used visual model	Tyler, Sam, Kelyn – Flaw in model Hank, Tracie, Max – Procedure flaw	Kaitlyn, Kim, Oscar – whole number reasoning

General Errors		
	Rule based, not understanding	Inappropriate whole number reasoning Kaitlyn, Kim, Oscar
	Size of whole error Tyler, Sam	Unit missing or incorrect Almost all students
	Partitioning error Kelyn	Remainder error
	Misinterprets model	Concept error
	Calculation error Hank, Tracie, Max	Incorrect operation
	Property relationship error	Equation error

Figure 11.22 Three student solutions.

Natalie's response

$$3+2=5 \qquad \frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}=1\frac{5}{12} \qquad 5+1\frac{5}{12}=6\frac{5}{12}$$

Azro's response

$$\frac{12}{4} \qquad \frac{8}{3}$$

$$\frac{36}{12}+\frac{32}{12}=\frac{68}{12}$$

Kim's response

$$2\frac{3}{4}+3\frac{2}{3}=5\frac{5}{7}$$

Ms. Pratt addressed the three OGAP questions in preparation for planning.

1. ***What are developing understandings that can be built upon?*** A couple of students successfully found common denominators and correctly added the fractions. Others attempted this procedure but made errors. A few students showed evidence of understanding that they could add the whole numbers (2 and 3) and then add the fractions.
2. ***What issues of concern were in the student work?*** Ms. Smith was surprised so many students reverted to using a visual model to add the fractions. She was also concerned that students were not using an understanding of the magnitude of the fractions to make sense of the answers they were getting (e.g., not recognizing that $\frac{3}{4} + \frac{2}{3}$ must be more than 1). In addition, she was concerned that a few students reverted to inappropriate whole number reasoning.
3. ***What are potential next instructional steps for the whole class, small groups, or individual students?*** Ms. Pratt decided to lead a minilesson at the beginning of the next class focused on using unit fraction and equivalence reasoning to add fractions and mixed numbers. She designed the activity to help students move away from a reliance on visual models as well as solidify understanding of decomposing mixed numbers and using properties of operations to solve problems with mixed numbers.

The Mini-Lesson

To start the next class, Ms. Pratt created the following four problems in which students were to decide if the equations were true or false and to be prepared to justify their answer. Before reading further, work through them for yourself.

1. $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{2}{5}+\frac{4}{5}$ True or false?
2. $\frac{2}{5}+\frac{4}{5}=\frac{6}{10}$ True or false?
3. $1\frac{2}{5}+2\frac{4}{5}=1+2+\frac{2}{5}+\frac{4}{5}=3\frac{1}{5}$ True or false?
4. $1\frac{4}{10}+2\frac{4}{5}=\frac{4}{10}+\frac{8}{10}+1+2$ True or false?

Ms. Pratt engaged all students in the discussion by placing one problem on the board at a time and then using a think, pair, share strategy to have students justify whether each problem was true or false. The first problem was designed to get her students thinking and talking about unit fraction reasoning:

$$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{2}{5}+\frac{4}{5}$$

Once students had justified and agreed that this equation was true, she put up an equation that was deliberately false and reflected some of the inappropriate use of whole number reasoning she had seen on her formative assessment:

$$\frac{2}{4}+\frac{4}{5}=\frac{6}{10}$$

One student was able to reason that $\frac{2}{5}$ and $\frac{4}{5}$ was equal to $\frac{6}{5}$ (from the previous problem). Another used equivalence to reason that it was the same as $\frac{4}{10}+\frac{8}{10}$ or $\frac{12}{10}$. Mrs. Pratt then put up a few more variations of these questions to make sure that students could use reasoning about unit fractions and equivalence. Finally, she presented problems 3 and 4 so that students could think about adding mixed numbers by using their reasoning and understanding of properties of operations.

After this mini-lesson, students appeared to be more comfortable with solving a variety of mixed number addition problems in the math lesson that followed. At the end of the lesson, she decided to administer the following formative assessment question. Why do you think she choose this question?

> Alice had a goal to bike 38 miles one week. On Monday she biked $15\frac{5}{8}$ miles. On Wednesday she biked $11\frac{3}{4}$ miles. On Friday she biked $10\frac{1}{2}$ miles. Did she meet her goal? Show your work.

Ms. Pratt selected this problem because it is an example of a contextual problem in which fractional reasoning and the application of the properties of operations can be applied to estimate the solution. Although the problem can be solved using written procedures, algorithmic strategies are neither necessary nor efficient. For example, since the sum of the whole numbers in the problem is 36, student simply must reason whether the sum of the fractions $\frac{5}{8}$, $\frac{3}{4}$, and $\frac{1}{2}$ is greater than or less than 2 (the difference between the sum of the whole numbers and Alice's goal of 38 miles). Ms. Pratt wanted to see if her students could demonstrate procedural fluency by selecting the best operation or strategy for the problem at hand and performing the operation flexibly, accurately, and efficiently. In this case that meant using fractional reasoning to make an estimate of the solution rather than making an exact calculation.

In this example, Ms. Pratt used evidence from a formative assessment question to design a mini-lesson focused on specific concepts and then gave another exit question that asked students to apply these concepts to a new situation (estimation in context).

The *OGAP Fraction Progression* and Adding and Subtracting Fractions

At the *Fractional Strategies* level, student solutions include the use of properties of operations, reasoning about magnitude, or efficient algorithms when solving addition and subtraction problems. Kenny's response in Figure 11.1 is an example of using an efficient algorithm to solve an addition and subtraction problem. The solution in Figure 11.23 is an example of using the properties of operations to solve an addition of mixed numbers problem.

Student solutions at the *Transitional Strategies* level provide evidence of effective use of a visual model and/or the use of unit fractions when adding and subtracting fractions. Solutions at this level may also show evidence of using inefficient procedures such as repeated addition for multiplication of fractions. Felicia's and Shawn's responses in Figures 11.2 and 11.3 are examples of using effective models at the transitional level.

At the *Early Fractional* level, there is evidence of choosing the appropriate operation given the problem, but the solution contains a conceptual error. In Figure 11.17, Shaun used the correct operation, subtraction, but not the correct procedure to subtract $\frac{2}{5}$ from $2\frac{1}{5}$.

The *Non-Fractional* level is characterized by solutions that use an incorrect operation given the problem, inappropriate whole number reasoning, or use of a rule or procedure without evidence of understanding.

Chapter Summary

This chapter focused on research related to adding and subtracting fractions. Through an examination of pertinent research and examples of student solutions, we demonstrated how:

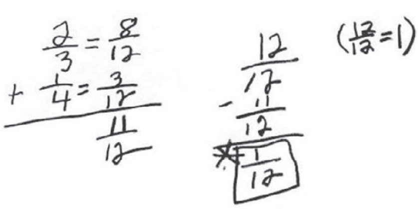
- Procedural fluency and conceptual understanding work together to deepen student understanding of fraction addition and subtraction.
- Fraction addition and subtraction concepts build from, and are dependent on, unit fraction and equivalence, magnitude understanding, and the commutative and associative properties.
- Estimation plays a critical role in students' development of procedural fluency and conceptual und.erstanding related to fraction addition and subtraction.

Looking Back

1. Mrs. Grayson brought Kenny's work (shown first in Figure 11.1 and shown here in Figure 11.23) to a fifth-grade team meeting. She wondered if Kenny simply followed a procedure or if he understood the concepts upon which the algorithm is based.

Figure 11.23 Kenny's response.

Tina ate $\frac{2}{3}$ of her candy and gave $\frac{1}{4}$ of her candy to her sister. She saved the rest of her candy. What is the fractional part of the candy that Tina saved?



Help Mrs. Grayson by answering the following:

a. Describe evidence in Kenny's response that shows understanding of the context of the problem and related fraction concepts.
b. What questions might you ask Mrs. Grayson about her instruction to ensure that Kenny has a foundation for understanding?
c. If Mrs. Grayson wanted to be sure that Kenny understood the algorithm, what else could she ask him?

2. Mr. Benson brought Mathew's response (Figure 11.24) to the team meeting. He felt that this provides evidence that Mathew has a strong conceptualization when comparing $\frac{2}{5}$ to $\frac{3}{10}$ using both a visual area model and a number line.

Answer the following questions:

a. What understandings are evidenced in Mathew's work?

b. What questions could be asked to build on understanding about equivalence and common denominators when comparing or adding and subtracting fractions? Explain how each question might help Mathew to move to a deeper understanding of equivalence and common denominators when comparing or adding and subtracting fractions.

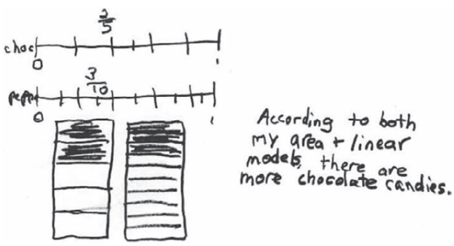
Figure 11.24 Mathew's response.

There are some candies in a dish.

$\frac{2}{5}$ of the candies are chocolate.

$\frac{3}{10}$ of the candies are peppermint.

Are there more chocolate candies or more peppermint candies in the dish?

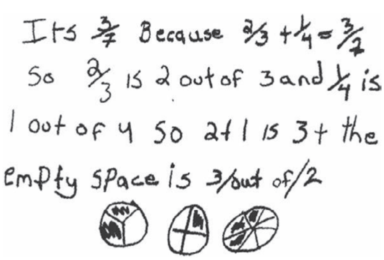


3. Ms. Cunningham shared Kim's work (Figure 11.25) at the math team meeting. She is asking her teammates for advice on how to transition students to accurately using visual models to solve problems involving addition and subtraction.
Help Ms. Cunningham by addressing the following questions:

   a. What was Kim able to do correctly? What is the evidence?
   b. Kim's visual model leads to an incorrect response. What errors did Kim make in her use of a visual model? What is the evidence?

Figure 11.25 Kim's response.

Tina ate $\frac{2}{3}$ of her candy. She gave $\frac{1}{4}$ of her candy to her sister and saved the rest of her candy. What is the fractional part of the candy that Tina saved?



   c. What questions might you ask, or activities might you do, to help Kim understand how to use visual models to solve addition and subtraction problems?

4. Mr. Hill has been spending a lot of time working with his class on estimating sums and differences. Work from three of his students, Willy, Oscar, and Christine, is shown in Figures 11.26, 11.27, and 11.28. Use these three solutions to answer the following questions.
   a. Analyze Willy's, Oscar's, and Christine's work. What strategy did each student use to solve the problem? Locate each solution strategy on the *OGAP Fraction Progression*.
   b. What questions might you ask Christine to help her consider more efficient fractional strategies?
   c. What questions or activities could you propose that would help Oscar move from using a visual model to a mental visual model of the fractions being added? How would the questions or activities you propose help Oscar?
   d. Use the *OGAP Fraction Progression* to help you create other questions or activities that can support or extend Willy's understanding of the relative magnitude of $\frac{1}{12}$ and $\frac{7}{8}$.

Figure 11.26 Willy's response.

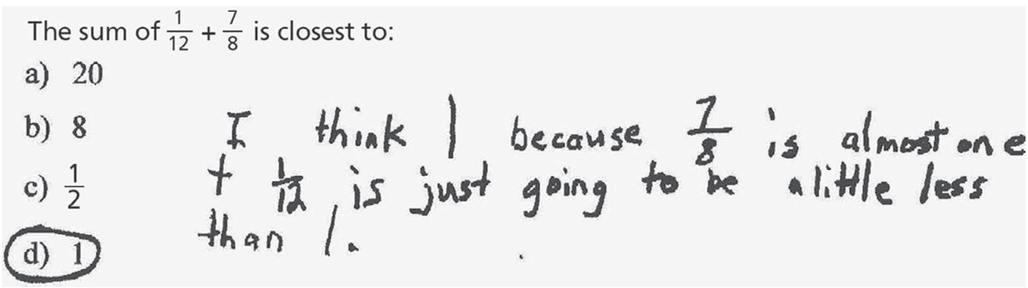


Figure 11.27 Oscar's response.

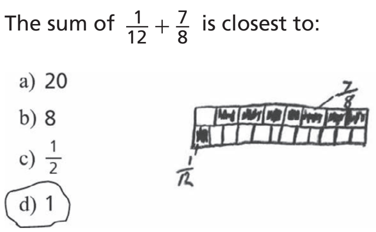
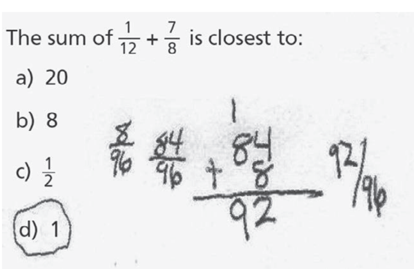


Figure 11.28 Christine's response.



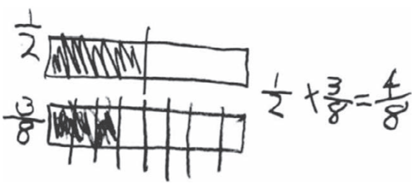
5. Ms. Horton is helping Emanuel (Figure 11.29) understand addition of proper fractions. He is able to draw visual models for most fractions and use visual models to add or subtract fractions with common denominators. However, Emanuel struggles with adding or subtracting fractions with unlike denominators (Figure 11.30). Help Ms. Horton by responding to the following questions and prompts.

Figure 11.29 Emanuel's response for adding fractions with common denominators.

Solve $\frac{1}{8} + \frac{3}{8} =$

Figure 11.30 Emanuel's response for adding fractions with unlike denominators.

Solve $\frac{1}{2} + \frac{3}{8} =$



a. What feature of the visual model in Figure 11.29 allowed Emmanuel to successfully add the two fractions but is not present in Figure 11.30? Locate each solution strategy on the *OGAP Fraction Progression*.
b. What would Emmanuel have to do to the visual model in Figure 11.30 to allow him to effectively use the same strategy for adding fractions as he used in Figure 11.29?
c. Provide a sequence of addition/subtraction problems that would help build this understanding. Describe how the problems you propose can help to build an understanding of the meaning of common denominators.

Instructional Link: Your Turn

Use the questions in Table 11.1 to help you think about how your instruction and mathematics programs provide students the opportunity to develop understandings about addition and subtraction of fractions.

Table 11.1 Strategies to Support Development of Concepts Related to Addition and Subtraction of Fractions

Do you (or your program) provide opportunities for students to:	*Yes/No*
1. Use visual models to build understanding of addition and subtraction concepts?	
2. Solve problems that involve estimating fraction sums and differences?	
3. Build upon other foundational skills such as visual modeling, partitioning, estimating, unit fraction understanding, and equivalent fractions to develop both conceptual understanding and procedural fluency?	

Identify any gaps between your instruction (including what your program offers) and what should be addressed based on the research from this chapter. Consider instructional strategies that you could use to address the gaps you found.

12
Multiplication and Division of Fractions

Big Ideas

- Instructional opportunities for students should foster both conceptual understanding of multiplication and division across a range of contextual situations and procedural fluency built upon conceptual understanding.
- Understanding of and fluency with multiplication and division of fractions should be built upon an understanding of fractions as numbers, visual models, unit fraction understanding, the meaning of the operations, and the properties of operations.

Multiplication and Division of Fractions: Understanding the Concept

Fraction operations in general, and multiplication and division of fractions in particular, are consistently sources of confusion for students. Research suggests that students often have a procedural knowledge of fraction operations but lack understanding of their vital underlying concepts (Mack, as cited in Yetkiner & Capraro, 2009).

A national report summarizing research findings on fraction teaching and learning suggests four key strategies to build procedural fluency based on conceptual understanding (Siegler et al., 2010):

1. Use visual models to "help students gain insight into basic concepts underlying computational procedurals and reason why the procedures work" (p. 28).
2. Use contextual tasks in which the contexts "provide meaning to the fraction quantities involved in a problem and the computational procedure used to solve it" (p. 33).
3. Use estimation to help strengthen understanding of the impact of the operations.
4. Address common misconceptions.

This chapter provides examples of how each of these strategies is applied to building procedural fluency of multiplication and division of fractions with understanding. The chapter begins with an overview of the expectations in the CCSSM for the development of understanding and fluency with multiplying and dividing fractions.

Building on the CCSSM expectations, the chapter explores how the use of visual models, unit fraction understanding, contexts, estimation, and properties of operations can build fluency and conceptual understanding. In this chapter, all fractions used are positive fractions, and the word "fraction" always refers to a positive fraction.

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CCSSM

The CCSSM and Multiplication and Division of Fractions

Students at grade 4 solve problems in context and use visual models and unit fractions to develop understanding of the multiplication of fractions by a whole number. At grade 5, students extend this work to understand multiplication as scaling. For example, a recipe for four people that calls for $\frac{1}{2}$ pound of sugar will need five times as much sugar when the recipe is adjusted for 20 people. Conversely, if the recipe is changed to feed half the number of people, half the amount of sugar would be needed. In the first case, the recipe is scaled up by a scale factor of 5. In the second case, the scale factor is $\frac{1}{2}$. Students at grade 5 also solve problems involving multiplying fractions by both whole numbers and fractions. Division with fractions is formally introduced in grade 5 through problems that involve the interpretation of a fraction as the division of the numerator by the denominator. Additionally, students at grade 5 solve fraction multiplication and division problems involving area and other measurement and data topics.

At grade 6, students solve problems involving the division of fractions by fractions. Students also apply their knowledge of multiplying and dividing fractions to solve problems involving area, volume, data, and equations and expressions. As students develop understanding of the meaning of multiplying and dividing with fractions, they represent the situations with visual models, write equations to represent problem situations, and write story problems to match equations.

At grade 7, fraction concepts are extended to negative fractions. It is important to note that fraction fluency is assumed at grade 7, and students are thus expected to solve problems involving fractions across a range of mathematics topics such as area, volume, surface area, percentages, ratios, proportions, expressions and equations, scaling, similarity, measures of central tendency, probability, and functions. The development of fluency through understanding in earlier grades is therefore critically important.

Extending Understanding of the Properties of Operations and Relationships

Through the early elementary grades, students interact with several important properties and relationships with respect to multiplication and division of whole numbers. These same properties and relationships learned in the context of whole numbers also apply to fractions and are central to reasoning about operations with fractions.

- *Identity property of multiplication*: multiplication of a number by 1 results in a product that is equal to the original number (e.g., $8 \times 1 = 8$ and $\frac{1}{2} \times 1 = \frac{1}{2}$). This also means that division by 1 results in a quotient that is equal to the dividend (e.g., $8 \div 1 = 8$ and $\frac{1}{2} \div 1 = \frac{1}{2}$).
- *Zero property of multiplication*: multiplication of any number by 0 equals 0.
- *Inverse relationship between multiplication and division*: This means that because $4 \times \frac{1}{2} = 2$, then $2 \div \frac{1}{2} = 4$ and $2 \div 4 = \frac{1}{2}$. Students have explored this relationship between multiplication and division with whole numbers. For example, they have learned that since $5 \times 2 = 10$, then $10 \div 2 = 5$ and $10 \div 5 = 2$. Often, students are asked to list members of the same multiplication and division "fact families." Just as $10 \div 2 = 5$ and $5 \times 2 = 10$ are in the same fact family, the equations $2 \div \frac{1}{2} = 4$ and $4 \times \frac{1}{2} = 2$ are in the same fact family.
- *Commutative property of multiplication*: $3 \times 4 = 4 \times 3$ and $\frac{3}{4} \times 20 = 20 \times \frac{3}{4}$ and $\frac{3}{4} \times \frac{4}{5} = \frac{4}{5} \times \frac{3}{4}$.
- *Distributive property of multiplication*: $a(b + c) = ab + ac$ where $a, b,$ and c are real numbers. (e.g., $2 \times 2\frac{1}{2} = (2 \times 2) + \left(2 \times 2\frac{1}{2}\right) = 4 + 1 = 5$). Again, students have seen the distributive property at work with whole numbers. For instance, they have learned that $17 \times 4 = (10 \times 4) + (7 \times 4)$.
- Multiple interpretations of both multiplication (e.g., equal groups, area, scaling) and division (partitive and quotative).

One relationship that many students overgeneralize incorrectly from their work with whole numbers is that multiplication always results in a larger number and division always results in a smaller number (Harel et al., 1994). Doug's response in Figure 12.1 is an example of a student who believes that multiplication must make the result larger.

Figure 12.1 Doug's response. Doug states that the product has to be greater than the given factors.

Stephanie and Paige are discussing the answer to $3\frac{2}{7} \times \frac{5}{9}$.

Stephanie said that the answer is more than $3\frac{2}{7}$.

Paige said the answer is less than $3\frac{2}{7}$.

Who is correct?

if you multplie any thing it has to
be bigger than what you multplie by.
Stephanie is right.

To undo deeply held beliefs, such as Doug's, about the impact of multiplication and division on a product or quotient, instruction needs to focus on developing an understanding of why this result is not always true when operating with fractions. To explore this concept in more depth, we will use the following multiplication problem:

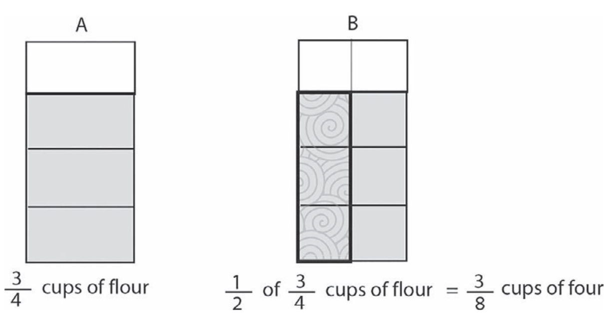
A recipe calls for $\frac{3}{4}$ of a cup of flour. How much flour is needed to make $\frac{1}{2}$ of the recipe?

The solution in Figure 12.2 uses an area model to represent the recipe problem. The visual model and the problem are used to illustrate two important points about multiplication and division of fractions:

1. Multiplication can result in a smaller product.
2. Multiplying by $\frac{1}{2}$ is the same as dividing by 2.

In Figure 12.2, the large rectangle represents a cup of flour. The visual area model A represents $\frac{3}{4}$ of a cup of flour. The visual area model B shows that $\frac{1}{2}$ of $\frac{3}{4}$ of a cup of flour equals $\frac{3}{8}$ of a cup of flour. This logical context and accompanying area models make it clear that a product in multiplication can be less than the initial amount. In this case, the initial amount, $\frac{3}{4}$ of a cup of flour, was scaled by a factor of $\frac{1}{2}$. Thus, half the recipe requires half the amount of flour, or $\frac{3}{8}$ cup of flour. By extending this concept of multiplying a number by $\frac{1}{2}$ to multiplying a number by any fraction between 0 and 1, one sees that multiplication can make the product smaller. When multiplying by a fraction between zero and one, one is always finding a fractional part of a given number.

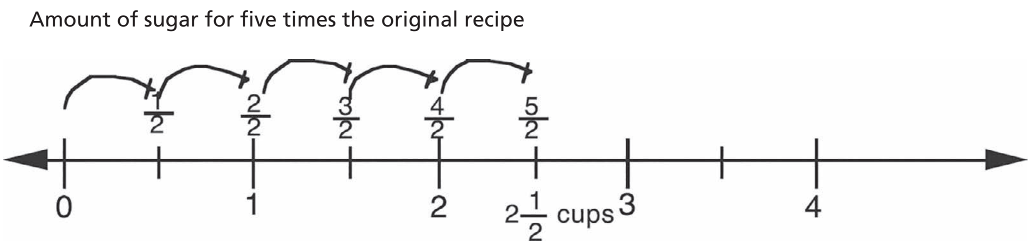
Figure 12.2 One-half of $\frac{3}{4}$ of a cup of flour is represented in area model B.



Model B also helps us see that multiplying by $\frac{1}{2}$ is the same as dividing by 2; that is, when we consider $\frac{1}{2}$ of $\frac{3}{4}$ of a cup of flour, we either divide $\frac{3}{4}$ of a cup of flour by 2 or multiply $\frac{1}{2}\times\frac{3}{4}$ of a cup of flour. Both operations result in the correct answer, $\frac{3}{8}$ of a cup of flour. In either situation, the answer, $\frac{3}{8}$, is less than $\frac{3}{4}$, the original amount of flour.

One must not overgeneralize that anytime a fraction is involved in multiplication the product will be less than the original number. For example, if one is quintupling a recipe and the original amount of sugar in the recipe is $\frac{1}{2}$ cup of sugar, the calculation would be $5\times\frac{1}{2}$ and the recipe would require more than $\frac{1}{2}$ cup of sugar (see Figure 12.3).

Figure 12.3 This number line shows how scaling up $\frac{1}{2}$ cup of sugar by a factor of 5 results in $2\frac{1}{2}$ cups of sugar. That is $5\times\frac{1}{2}=5\left(\frac{1}{2}\right)=\frac{5}{2}$.

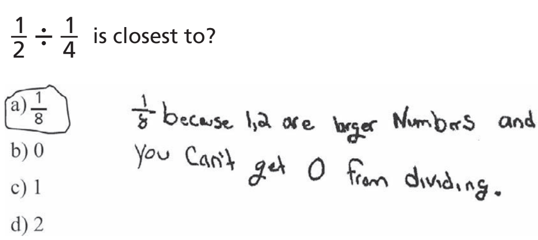


Notice that the answer, $2\frac{1}{2}$ cups of sugar, is greater than the original amount of sugar, $\frac{1}{2}$ cup. This is because multiplication by a number greater than 1 (in this case 5) results in a product that is greater than the other factor, even if that factor is a fraction. That is, the recipe was scaled up five times. Therefore, the amount of each ingredient, even if the amount is fractional, is five times greater than the original amount.

Impact of Division by a Fraction

As we described earlier in the chapter, students' experiences with the division of whole numbers sometimes leads them to believe that the operation of division always makes something smaller. Glen's work in Figure 12.4 is an example of a student who is bringing a whole number notion of the impact of division to dividing by a fraction.

Figure 12.4 Glen's response. Glen chose the non-zero number less than $\frac{1}{2}$ and $\frac{1}{4}$ using his whole number understanding of the impact of division on a quotient.



To investigate the impact of dividing by a fraction, we will use two problems. In each problem, we are dividing by a fraction between 0 and 1. Here is the first problem:

Carly has $\frac{1}{2}$ of a pound of jellybeans. She filled bags with $\frac{1}{4}$ of a pound of jellybeans. How many bags did Carly fill?

The answer can be found by making the calculation $\frac{1}{2}\div\frac{1}{4}$. In this interpretation of division, one is asking, "How many $\frac{1}{4}$ pounds of jellybeans are in $\frac{1}{2}$ a pound of jellybeans?" or "How many $\frac{1}{4}$s are in $\frac{1}{2}$?" Figure 12.5 illustrates that there are two $\frac{1}{4}$s in $\frac{1}{2}$, so $\frac{1}{2} \div \frac{1}{4} = 2$. Notice that the quotient, 2, is greater than the dividend, $\frac{1}{2}$. When a number is divided by a fraction less than 1, the quotient will be greater than the dividend.

Now let's consider the second problem: $\frac{1}{4} \div \frac{1}{2}$.

The number line in Figure 12.6 illustrates the impact of dividing a fraction by a fraction less than 1 when the divisor is greater than the dividend $\left(\frac{1}{4} \div \frac{1}{2}\right)$. Using the same interpretation of division that we used in the example in Figure 12.5, one can translate the expression $\frac{1}{4} \div \frac{1}{2}$ into "How much of $\frac{1}{2}$ is in $\frac{1}{4}$?"

Figure 12.5 Given the context in Carly's problem, the division expression $\frac{1}{2} \div \frac{1}{4}$ can be interpreted as "How many $\frac{1}{4}$s are in $\frac{1}{2}$?" There are two $\frac{1}{4}$s in $\frac{1}{2}$; therefore, Carly can fill two bags with jellybeans.

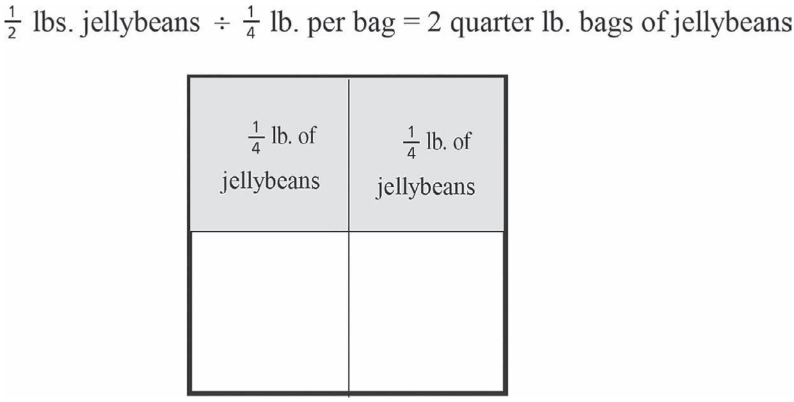
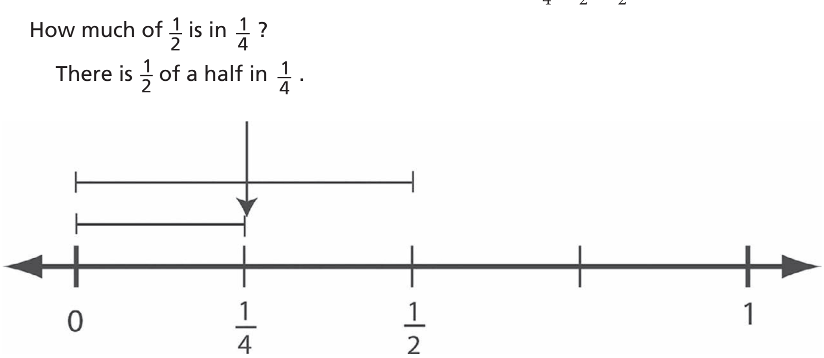


Figure 12.6 Number line illustrating the calculation $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$.



In this case, as in the previous example, the quotient, $\frac{1}{2}$, is greater than the dividend, $\frac{1}{4}$.

It is worth repeating that for many elementary school students, the idea that the quotient in a division problem can be greater than the dividend, or that multiplication can result in a smaller number, is counterintuitive. Students will only come to this understanding after many opportunities to visualize the impact of dividing and multiplying by a positive fraction less than 1. Solving contextual problems that involve making sense of the impact of the multiplication and division play important roles in the development of these ideas.

Justina's response in Figure 12.7 shows evidence of understanding the impact of multiplying by a positive fraction less than 1 on a product.

Figure 12.7 Justina's response. Justina's response shows evidence of understanding that multiplying a number less than 1, such as $\frac{4}{5}$, results in a smaller number.

The product of $3 \times \frac{4}{5}$ is?

a) greater than 3

b) less than 3

less than 3 because in order for it to be more than 3, it would have to be multiplied by something greater than 1 but it is only multiplied by 4/5.

Using Estimation to Strengthen Understanding of the Impact of Multiplication and Division

To continue to reinforce students' understanding of the impact of multiplication and division, students should be asked to make estimates of solutions before solving problems and to reflect on their final solutions through this estimation lens. Students should also be asked to solve both contextual and noncontextual problems that do not require exact answers but instead ask for estimates based on reasoning. Figures 12.8 and 12.9 provide examples of these types of problems.

Figure 12.8 Example of a contextual problem that requires an estimation, not an exact answer.

Liam is baking cookies. The recipe calls for $1\frac{2}{3}$ cups of flour. Liam wants to triple the recipe. He has only 4 cups of flour. Does he have enough flour?

Figure 12.9 This noncontextual problem requires an understanding of the impact of multiplication involving fractions without asking for an exact numerical answer to each of the problems.

Make each statement true using the greater than (>), less than (<), or equal to (=) sign. Support your answer.

A) $(476 \times \frac{3}{3}) \; \square \; (476 \times \frac{1}{3})$

(476×1) $\boxed{>}$ $476 \times \frac{1}{3}$ ← not one whole

is a whole.

B) $(476 \times \frac{5}{3}) \; \square \; (476 \times \frac{2}{3})$

$476 \times 1\frac{2}{3}$ $\boxed{>}$ $476 \times \frac{2}{3}$ ← less than 1

multiplying by more than 1

Anchoring Procedural Fluency in Understanding

As discussed in Chapter 11 and at the beginning of this chapter, students may struggle with the use and understanding of formal algorithms when their knowledge is dependent primarily on memory, rather than anchored with a deeper understanding of the foundational concepts (Kieren, as cited in Huinker, 2002). Some researchers even suggest that instruction focused on rules may present unintended consequences, because rule-based instruction does not encourage students to think about the meaning of the operation. Mastery in the use of operations learned through rule-based instruction is quickly lost (Aksu, 1997).

This research is particularly important considering the difficulties students have with multiplication and division of fractions. Operations with fractions, specifically division of fractions, are considered by some researchers to be the least understood topics in elementary school mathematics (Fendel, as cited in Tirosh, 2000).

Multiplication of Fractions: Developing Fluency Through Understanding

Students often learn to multiply fractions by simply multiplying the numerators and multiplying the denominators. Unfortunately, this type of instruction often leaves students with little conceptual understanding of this procedure. Fraction computation can be taught as a series of rules; however, this focus on rote learning can result in artificial feelings of accomplishment (Aksu, 1997).

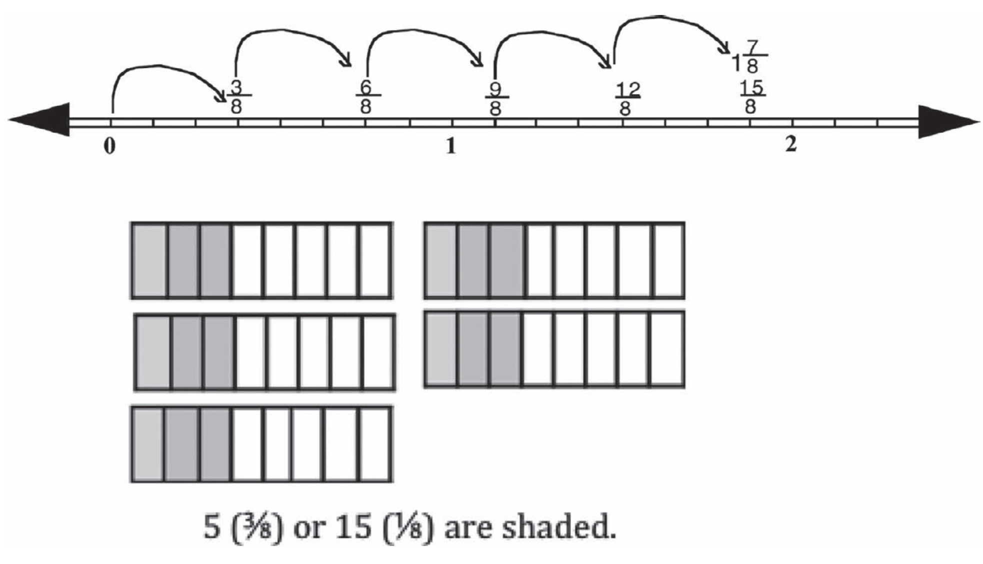
Multiplication of Whole Numbers by Fractions

By combining understanding of whole number multiplication and unit fraction understanding, students develop understanding and fluency when multiplying whole numbers by fractions. When learning multiplication with whole numbers, students represent the multiplication of 5 groups of 8 as $5 \times 8 = 8 + 8 + 8 + 8 + 8$. This understanding can be extended to fraction multiplication to see fractions as composed of unit fractions. That is, a fraction can be understood as the numerator multiplied by a unit fraction $\left(\text{e.g.,} \frac{5}{8} = 5\left(\frac{1}{8}\right) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$.

Review the problem and solution in Figure 12.10. This illustrates how the understanding of whole number multiplication and unit fractions can be combined. Also notice how the context helps bring meaning to the operation. That is, the candy bar problem involves scaling up the amount of candy bar by finding five times the amount that each student received. Given this context, it makes sense

Figure 12.10 The numerical solution, the number line, and the visual area models support an understanding of the relationship between repeated addition and multiplication.

If 5 students each get $\frac{3}{8}$ of a candy bar, how many candy bars do they have altogether?



to multiply the number of three-eighths directly, as shown in Figure 12.10 (5 students × $\frac{3}{8}$ of a candy bar for each student = total amount of candy bars).

The ultimate goal, however, is not for students to use repeated addition or to construct visual models to solve multiplication problems, but to use the understanding derived from the visual models and unit fraction understanding to understand that $n \times \frac{a}{b} = \frac{n \times a}{b}$, where n is any number and $b \neq 0$.

Study the student solutions in Figures 12.11 and 12.12. These solutions approached the multiplication of a whole number by a fraction in a different way than the solution in Figure 12.10. In these cases, the students used unit fraction understanding by determining how many students represented $\frac{1}{3}$ of the class and then multiplied that by 2 to determine the number of students in $\frac{2}{3}$ of the class.

Figure 12.11 Tamara's solution. Tamara partitioned a visual area model into thirds and then determined how many students represented $\frac{1}{3}$ of the class. She then determined the number of students who had brown eyes in $\frac{2}{3}$ of the class by multiplying 2 × 12.

There are thirty-six students in Ashley's class. Two-thirds of the students in Ashley's class have brown eyes. How many students have brown eyes in Ashley's class?

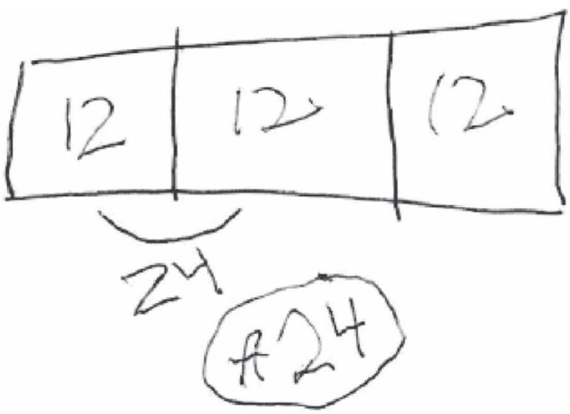
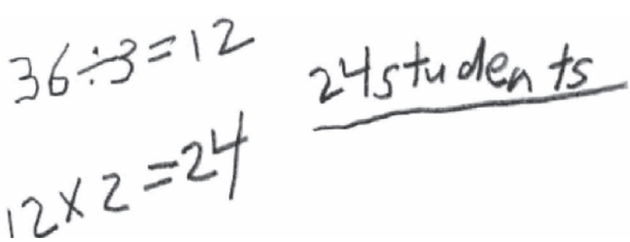


Figure 12.12 Zoe's response shows her understanding that finding $\frac{1}{3}$ of a quantity is the same as dividing the quantity by 3. She then used this understanding to determine the number of students in $\frac{2}{3}$ of the class.

There are thirty-six students in Ashley's class. Two-thirds of the students in Ashley's class have brown eyes. How many students have brown eyes in Ashley's class?



Also notice how the context of the problem brings meaning to the operation, but in a different way than the candy bar problem did in Figure 12.10. That is, in the classroom problem, we are finding a fractional part of a whole rather than scaling up. When determining $\frac{2}{3}$ of 36 students, it makes sense to find the number of students in $\frac{1}{3}$ of the class then multiply this number by 2 since the problem involves $\frac{2}{3}$ of the students ($\frac{2}{3} \times 36$ students [whole] = number of students with brown eyes in the class [part]).

It appears that Tamara applied similar reasoning using her visual model as Zoe did in her more efficient solution. One can imagine asking Tamara to explain how she determined the number of students in each third of the class and why she decided that 24 students have brown eyes. From that discussion, one could then ask Tamara to write equations or expressions to represent her thinking. With more opportunities to solve problems like these, Tamara has the potential to quickly transition from relying on visual models to using a more efficient strategy.

Students need to be flexible in their use of efficient strategies as they make sense of problem contexts drawing on their understanding of the problem and mathematics concepts underlying the problem. To support the development of flexibility, instruction should engage students in different types of multiplication problems with fractions (e.g., equal groups and measures, parts of wholes, scaling up, area, volume, data), allowing for students to use strategies that make sense for the problem situations and the quantities involved.

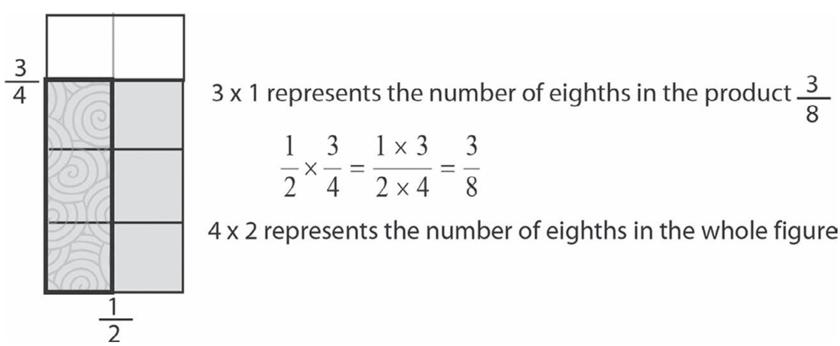
Multiplication of Fractions by Fractions

The common algorithm for multiplying fractions involves multiplying numerators of the factors and the denominators to obtain a product.

$$\frac{1}{2} \times \frac{3}{4} = \frac{1 \times 3}{2 \times 4} = \frac{3}{8}$$

The calculation $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$ is illustrated by the visual area model in Figure 12.13. Models such as these can be used to help students bring meaning to the procedure. There are two regions to consider in the representation in Figure 12.13: 1) the area that represents the whole, indicated by the large rectangle; and 2) the area that represents $\frac{1}{2}$ of $\frac{3}{4}$, represented by the swirled shading. The product of the denominators ($4 \times 2 = 8$) indicates the number of equal parts in the whole and the size of one of those equal parts, $\frac{1}{8}$. The product of the numerators ($1 \times 3 = 3$) indicates the number of $\frac{1}{8}$s in the product: $\frac{3}{8}$.

Figure 12.13 Area model showing that $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$.



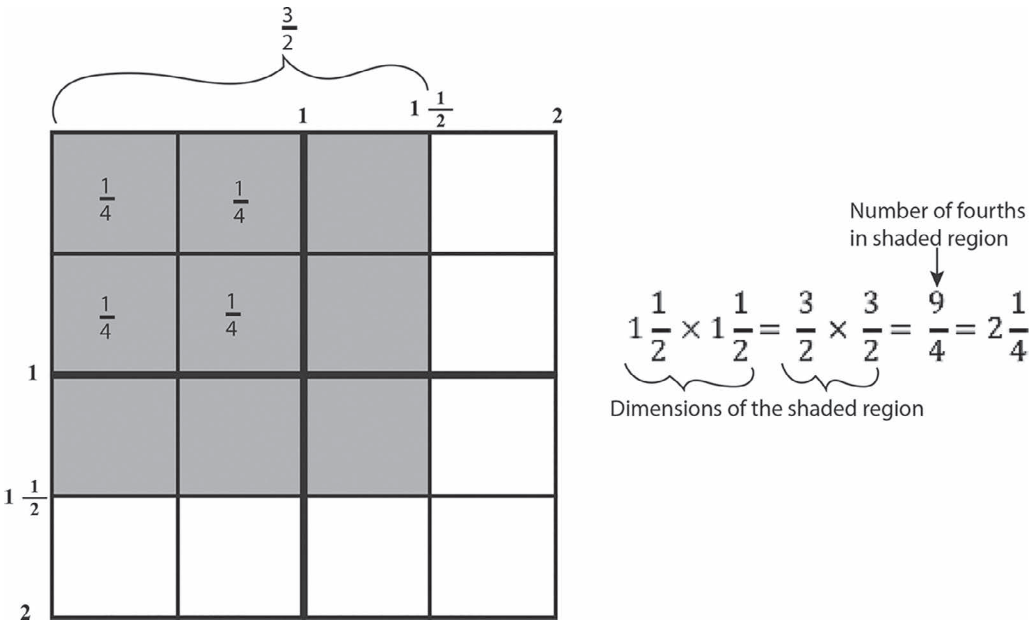
Understanding procedures for multiplying a mixed number by a mixed number are also important.

$$1\frac{1}{2} \times 1\frac{1}{2} = \frac{3}{2} \times \frac{3}{2} = \frac{9}{4} = 2\frac{1}{4}$$

A visual model like the one in Figure 12.14 can help bring meaning to the procedure by focusing on the area of a square that is $1\frac{1}{2}$ by $1\frac{1}{2}$. The dimensions $\left(1\frac{1}{2}\, or\, \frac{3}{2}\right)$ are along the sides of the visual model. One square unit (1 × 1) is partitioned into fourths. Next notice that the number of fourths in the $1\frac{1}{2}$ by $1\frac{1}{2}$ shaded region is $\frac{9}{4}$.

The diagram in Figure 12.14 also clearly illustrates the distributive property in the context of fractions. Using this property, one can see four partial products: $1\frac{1}{2}\times 1\frac{1}{2}=(1\times 1)+\left(1\times\frac{1}{2}\right)+\left(\frac{1}{2}\times 1\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)$. The sum of these four partial products is the area of the $1\frac{1}{2}$ by $1\frac{1}{2}$ square.

Figure 12.14 Visual model based on the area of a square $1\frac{1}{2}$ by $1\frac{1}{2}$ illustrates some important points. The whole (1 × 1 square) is partitioned into fourths. Therefore, the region representing the area equals $\frac{9}{4}$ or $2\frac{1}{4}$.



Division of Fractions

The two most widely used algorithms for division of fractions are the common denominator algorithm and the more traditional invert-and-multiply algorithm.

To understand how and why the common denominator algorithm works mathematically, we will look closely at the bike problem, the fraction problem, and the model in Figure 12.15.

Bike Problem

Chris rode her bike 10 miles. Kim rode her bike 2 miles. How many times more miles did Chris ride than Kim?

$$10 \text{ miles} \div 2 \text{ miles} = \frac{10 \cancel{\text{ miles}}}{2 \cancel{\text{ miles}}} = 5$$, or Chris rode 5 times as many miles as Kim.

The units in the bike problem are miles. By dividing miles by miles, we determined how many times more miles Chris rode than Kim. Notice that the units, miles, cancel out; the answer, 5, is *not* in miles, rather it is a scale factor representing the amount one would scale the number of miles Kim rode to equal the number of miles Chris rode.

Fraction Problem

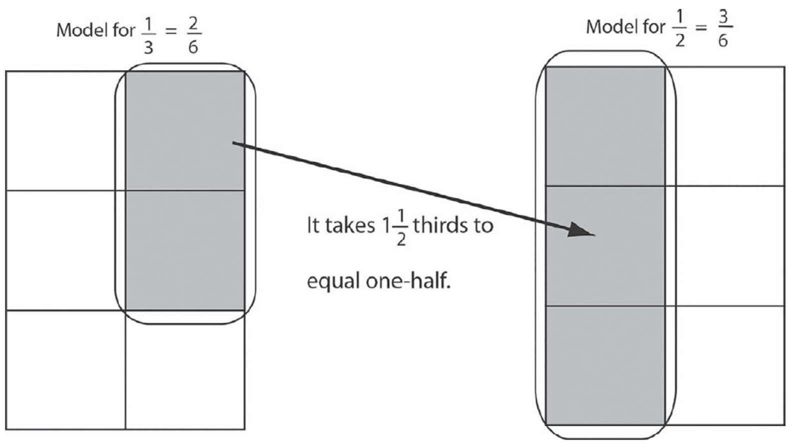
How many times greater is $\frac{1}{2}$ than $\frac{1}{3}$?

$\frac{1}{2} \div \frac{1}{3} = \frac{3}{6} \div \frac{2}{6} = \frac{3\ \cancel{\text{sixths}}}{2\ \cancel{\text{sixths}}} = 3 \div 2 = 1\frac{1}{2}$, or $\frac{1}{2}$ is $1\frac{1}{2}$ times greater than $\frac{1}{3}$.

The units in the fraction problem are sixths. Dividing sixths by sixths results in how many times greater $\frac{3}{6}$ is than $\frac{2}{6}$. Notice that the units cancel out; the answer is *not* in sixths. As with the Bike problem above, $1\frac{1}{2}$ is the amount one would scale $\frac{1}{3}$ for it to equal $\frac{1}{2}$.

The area model in Figure 12.15 helps to visualize the unit, sixths. Notice that each sixth represents the same area. The model also helps us understand that the answer will not be sixths, but in the number of times greater 3-sixths is than 2-sixths (3-sixths ÷ 2-sixths = $1\frac{1}{2}$). This means that $\frac{1}{2}$ is $1\frac{1}{2}$ times greater than $\frac{1}{3}$.

Figure 12.15 Illustrating the common denominator algorithm for the fraction problem $\frac{1}{2} \div \frac{1}{3}$.



Earlier in this chapter (see Figure 12.2), we provided a visual model that showed that multiplying a number by $\frac{1}{2}$ is the same as dividing the number by 2. This reciprocal relationship between $\frac{1}{2}$ and 2 can be used to understand the invert-and-multiply procedure for the division of fractions. Let us consider the following problem.

Toby hiked $\frac{3}{4}$ of a mile in $\frac{1}{3}$ of an hour.

Sam hiked $\frac{3}{4}$ of a mile in $\frac{2}{3}$ of an hour.

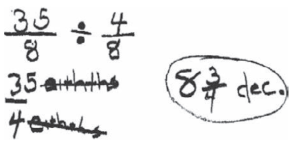
How many miles did each one hike in one hour?

The context of the problem and the fractions used are important elements that can help bring meaning to the invert-and-multiply procedure for dividing fractions that is based on the inverse relationship between multiplication and division.

Tania (Figure 12.16) used the common denominator approach to dividing $4\frac{3}{8}$ yards of wire by $\frac{1}{2}$. In this example, Tania explicitly treated eighths as the unit for the calculation and did not lose sight of the context of the problem that involved finding the number of decorations.

Figure 12.16 Tania's response. She calculated $35 \div 8$ to determine that Jim could make $8\frac{3}{4}$ decorations.

Jim is making decorations. He has $4\frac{3}{8}$ yards of wire. Each decoration needs $\frac{1}{2}$ of a yard of wire. How many decorations can Jim make?



Toby's Hike

Since we are finding Toby's unit rate (miles per hour), the problem can be conceptualized as $\frac{3}{4}$ (the distance) divided by $\frac{1}{3}$ (the time) or ($\frac{3}{4}$ miles $\div$ $\frac{1}{3}$ hour). However, the fractions used in the problem allow us to see that the distance Toby will hike in one hour (unit rate) will be three times the distance that he can hike in one-third of an hour since $3(\frac{1}{3}$ hours) = 1 hour. Therefore, the distance Toby will hike in one hour is $\frac{3}{4}$ mile $\times$ 3 = $2\frac{1}{4}$ miles.

$$\text{Toby: } \tfrac{3}{4} \text{ of a mile} \div \tfrac{1}{3} \text{ hour} = \tfrac{3}{4} \text{ mile} \times 3 = \tfrac{9}{4} = 2\tfrac{1}{4} \text{ miles in one hour}$$

The relationship between multiplication and division illustrated by this example is actually the invert-and-multiply algorithm for dividing fractions because $\frac{3}{1}$ is the reciprocal of $\frac{1}{3}$. The double number line in Figure 12.17 shows the relationships between distance hiked and hours and illustrates the multiplication of $\frac{3}{4}$ mile by 3.

Sam's Hike

Sam's hiking rate allows us to explore the invert-and-multiply algorithm for dividing by a non-unit fraction.

The double number line labeled Sam (bottom) and Toby (top) in Figure 12.17 shows that Sam takes twice the time to hike the same distance as Toby. That is, Sam hikes $\frac{3}{4}$ mile in $2\left(\frac{1}{3}\right)$ of an hour while Toby hikes $\frac{3}{4}$ mile in $\frac{1}{3}$ of an hour. Therefore, Sam hikes half the distance in the same amount of time because his hiking speed (rate) is half of Toby's hiking speed. If Toby hiked $2\frac{1}{4}$ miles in one hour, Sam must have hiked half that distance ($2\frac{1}{4} \div 2 = 1\frac{1}{8}$ miles).

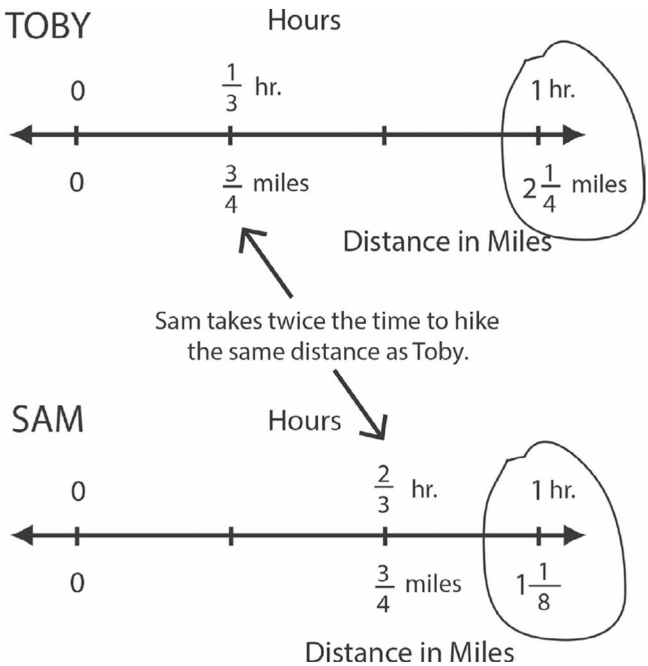
The following equation represents Sam's distance as half of Toby's distance in one hour of hiking. That is, $\frac{3}{4}$ miles $\times$ 3 is the distance Toby hikes in 1 hour. The division of that distance by 2 is the distance that Sam hikes in one hour, which is $\frac{1}{2}$ of $2\frac{1}{4}$ miles or $1\frac{1}{8}$ miles.

$$\tfrac{3}{4}\text{mile} \div \tfrac{2}{3}\text{hour} = \tfrac{3}{4} \times \tfrac{3}{2} = \left(\tfrac{3}{4} \times 3\right) \div 2 = 1\tfrac{1}{8}\text{miles in one hour}$$

The context, the fractions in the problem, and the representations shown in Figure 12.17 are examples of strategies that can help bring meaning to division and specifically to the invert-and-multiply algorithm for dividing fractions.

As we observed with the examples in Figures 12.2 and 12.17 and with using a context to bring meaning to a procedure, dividing by the fraction $\frac{a}{b}$ is the same as multiplying by its reciprocal, $\frac{b}{a}$. Prior to being introduced to the invert-and-multiply algorithm, students should be given plenty of opportunities to explore this relationship with a variety of problems in which they construct visual models and reason about division.

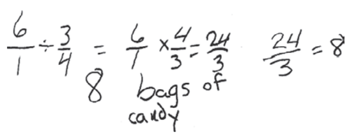
Figure 12.17 Sam takes twice the time ($2\left(\frac{1}{3}\right)$ hour) to hike the same distance as Toby. Therefore, Sam hikes half the distance ($1\frac{1}{8}$ mile) as Toby in 1 hour.



Selma used the invert-and-multiply procedure to solve the problem in Figure 12.18.

Figure 12.18 Selma's response. Selma multiplied 6 pounds of candy by the reciprocal of $\frac{3}{4}$.

Ashley bought 6 pounds of candy. She put the candy into bags that each hold $\frac{3}{4}$ of a pound of candy. How many bags of candy did Ashley fill?



Abby's and Troy's incorrect application of algorithms in Figures 12.19 and 12.20 may be examples of premature reliance on procedures without conceptual understanding.

Figure 12.19 Abby's response. Abby divided 32 by the numerator and by the denominator of the fraction given.

There are 32 students in the 6th grade class.

$\frac{5}{8}$ of the students are boys.

How many boys are in the sixth grade class?

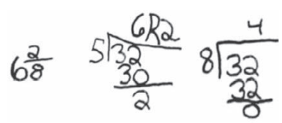
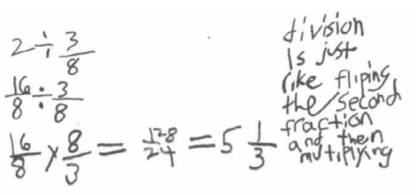


Figure 12.20 Troy's response. Troy calculated $2 \div \frac{3}{8}$ instead of $\frac{3}{8} \div 2$.

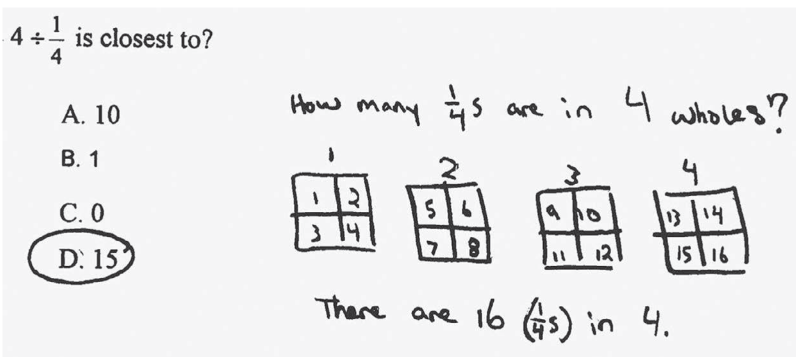
Two friends equally share $\frac{3}{8}$ of a ball of yarn.

How much of the yarn does each friend get?



Contrast Abby's and Troy's responses to Kelyn's solution in Figure 12.21. Kelyn's statement "How many $\frac{1}{4}$s are in four wholes?" and her visual model provide evidence that Kelyn understands the meaning of dividing four by one-fourth.

Figure 12.21 Kelyn's response. Kelyn interpreted the question as "How many $\frac{1}{4}$ s are in 4 wholes?"



Proficiency with multiplication and division of fractions requires both a conceptual understanding of the meaning of the operations and the ability to use efficient strategies flexibly. It appears that Kelyn is ready to move to a more efficient procedure because there is evidence that she can bring the meaning of division to the procedure.

Go-To See Chapter 2, Visual Models, and the discussion associated with Figure 2.26 for ways to help Kelyn and others build on their understanding represented in their visual models.

Examining Evidence in Student Solutions for Developing Understandings

Considering the vital role that visual models and unit fraction understanding play in learning fraction concepts such as equivalence and magnitude and addition and subtraction, it will come as no surprise that the same holds true for the importance of visual models as transitional strategies in the teaching and learning of multiplication and division of fractions.

Researchers indicate that teachers need knowledge of concrete models to help students' transition from multiplication by whole numbers to multiplication by fractions. Teachers must give closer consideration to division of fractions (Taber, cited in Yetkiner & Capraro, 2009).

Figures 12.22 and 12.23 provide examples of students effectively using visual models in their responses to fraction division problems. Although the visual models are different from one another, notice that each model accurately represents the context stated in the problem and leads to a correct response.

Figure 12.22 Corey used a number line partitioned into $24\left(\frac{1}{4}\text{s}\right)$ to determine that 8 bags can be filled.

Ashley bought 6 pounds of candy. She put the candy into bags that each hold $\frac{3}{4}$ of a pound of candy. How many bags of candy did Ashley fill?

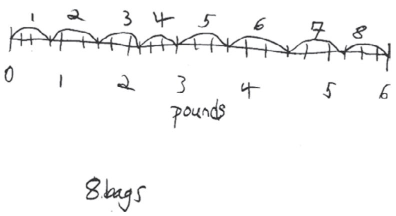
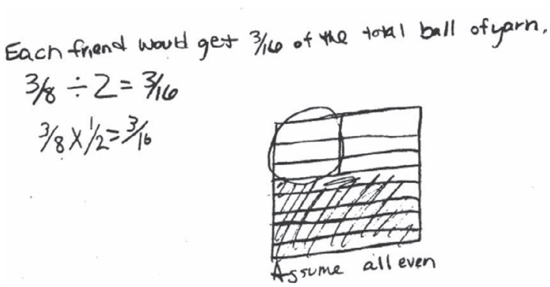


Figure 12.23 Jonathan's response. Jonathan partitioned $\frac{3}{8}$ of his model into 2 parts to represent $\frac{3}{8} \div 2$.

Two friends equally share $\frac{3}{8}$ of a ball of yarn.

How much of the ball of yarn did each friend get?



Mack (2001) suggests that one cannot assume that students' prior experiences with models have prepared them to conceptualize more complex mathematical ideas such as multiplication and division of fractions, and students may require guidance to reconsider these understandings.

Tracy's and Gail's responses in Figures 12.24 and 12.25 may exemplify this point. Each drew and interpreted a visual model in a way that may have been appropriate when considering other fraction concepts. However, Tracy and Gail did not use the models to solve the division problems given, as Kelyn did in Figure 12.21.

Figure 12.24 Tracy's response. Instead of determining how many $\frac{3}{4}$s are in 6 pounds, Tracy shaded $\frac{3}{4}$ of each bag.

Ashley bought 6 pounds of candy. She put the candy into bags that each hold $\frac{3}{4}$ of a pound of candy. How many bags of candy did Ashley fill?

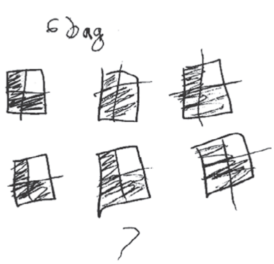
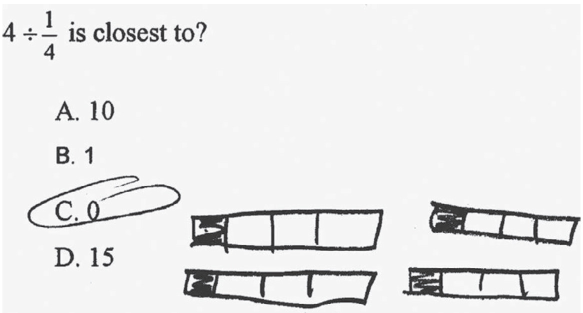


Figure 12.25 Gail's response. Gail shaded $\frac{1}{4}$ of four figures instead of finding how many $\frac{1}{4}$s are in 4.



Even though Tracy's and Gail's models do not fully represent the problems they were solving, they can be used as starting points for instruction. Both Tracy and Gail:

- correctly represented the number of wholes in their problems (although Tracy incorrectly labeled her wholes as bags not pounds)
- partitioned the wholes into fourths, which could have allowed them each to solve their problems
- shaded a fractional part of their wholes (e.g., Gail shaded $\frac{1}{4}$ of each of her wholes)

However, in both cases, the students did not interpret the problems as division (e.g., Tracy found $\frac{3}{4}$ of each whole by shading $\frac{3}{4}$ of each figure. She did not find the number of $\frac{3}{4}$s in 6). Clearly, instruction for Tracy and Gail should focus on the meaning of division and the ability to recognize division in a problem situation involving fractions. However, instruction should capitalize on their abilities to represent the whole with a visual model and to partition into fourths.

Researchers suggest that students should experience a variety of situations in which they need to recognize the appropriate fraction operation (Huinker, 2002). We have found that not only do some students struggle with recognizing when a context calls for a multiplicative solution, but students sometimes confuse the meaning and the procedures for multiplication and division of fractions (OGAP, 2005–2009). Figures 12.26 and 12.27 are examples of this confusion.

Figure 12.26 Alejandro's response. Alejandro used a visual model and described $\frac{1}{2} \times \frac{1}{4}$, not $\frac{1}{2} \div \frac{1}{4}$.

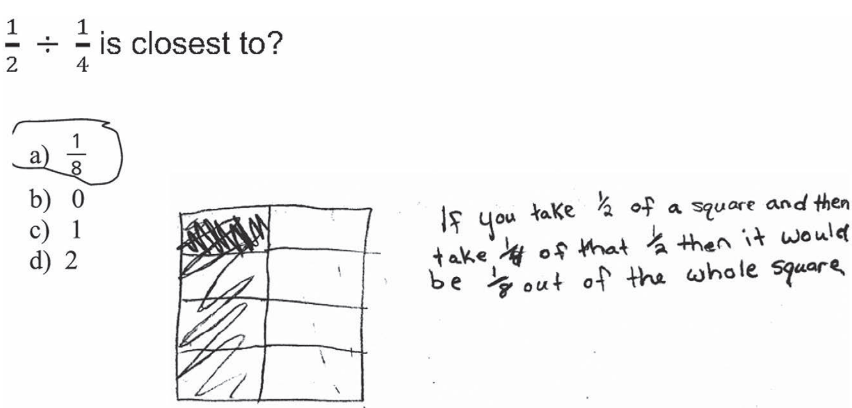
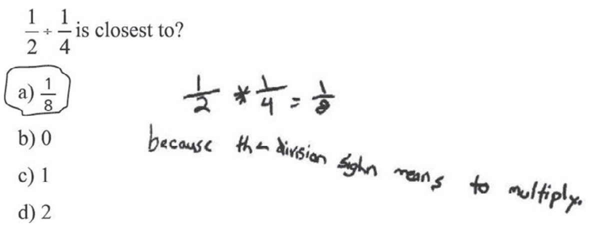


Figure 12.27 Claudia's response. Claudia believes that the division sign indicates multiplication.



Both pieces of evidence imply that Alejandro and Claudia may not be clear about the difference between multiplying and dividing by a fraction. One wonders if either student has a conceptual knowledge of the meaning of division as it relates to fractions.

The solutions in Figures 12.28 and 12.29 show evidence of using additive strategies in multiplication and division problems. Maya used repeated addition until she reached 6 pounds. Dom used repeated subtraction until he used up the 6 pounds of candy.

Figure 12.28 Maya's response. Maya used repeated addition to determine that Ashley could fill 8 bags.

Ashley bought 6 pounds of candy. She put the candy into bags that each hold $\frac{3}{4}$ of a pound of candy. How many bags of candy did Ashley fill?

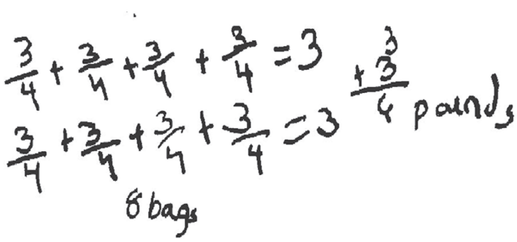
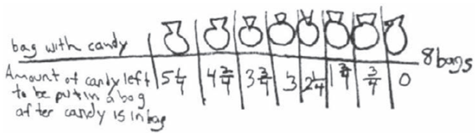


Figure 12.29 Dom's response. Dom used repeated subtraction to solve the problem.

Ashley bought 6 pounds of candy. She put the candy into bags that each hold $\frac{3}{4}$ of a pound of candy. How many bags of candy did Ashley fill?



While Maya's and Dom's solutions are correct, they are not efficient. That is, the use of repeated addition and repeated subtraction will limit students' abilities to solve problems as the numbers increase in magnitude or complexity. Imagine Maya and Dom using these strategies if the question included 120 pounds of candy instead of 6. Viewed in this light, Tania's (Figure 12.16) and Selma's (Figure 12.18) procedures are more efficient for solving fraction division problems. Students who have not developed multiplicative strategies to solve multiplication and division problems are at a significant disadvantage when faced with these more complex ideas and numbers.

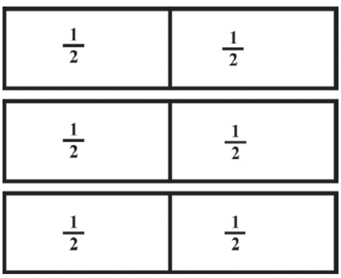
Partitive and Quotative Division

A discussion of division of fractions would be incomplete without an examination of partitive and quotative division. Partitive and quotative division are recognized as two different conceptual models for division (Graeber & Tanenhaus, as cited in Oksuz & Middleton, 2007).

Interpretation of a Fraction as Division (Partitive Division)

The interpretation of a fraction as division of the numerator by the denominator builds on an understanding of division as sharing and is referred to as *partitive division*. For example, six people sharing three candy bars equally can be represented as the fraction $\frac{3}{6} = 3 \div 6 = \frac{1}{2}$ of a candy bar for each person. In Figure 12.30, the three candy bars are divided into six equal parts. Each person gets $\frac{1}{2}$ of a candy bar.

Figure 12.30 Six people sharing three candy bars equally. Each person gets $\frac{1}{2}$ of a candy bar.



Partitive division can also be represented by a fraction when the numerator is greater than the denominator. For example, six people sharing 20 pounds of coffee equally can be represented as the fraction $\frac{20}{6} = 20 \div 6 = 3\frac{1}{3}$ pounds of coffee for each person.

Notice in both these cases, the number of groups is known (e.g., six people) and the total amount to be shared is known (e.g., three candy bars; 20 pounds of coffee). However, the amount in each group is not known (the amount of a candy bar or the amount of coffee each person will receive). Partitive problem situations involve this relationship and can be represented by the following equation:

Total ÷ number of groups = number in each group

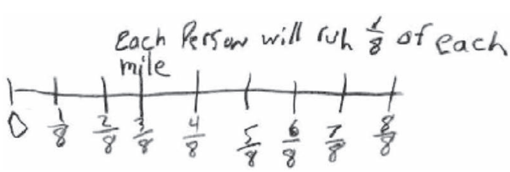
Study the student solutions in Figures 12.31 through 12.33 to the following partitive problem:

A relay race is 7 miles long. There are eight people on the relay race team and each person will run an equal distance. What fraction of the race will each person run?

Notice that the solutions in Figures 12.31 and 12.32 are built on unit fraction understanding. Kaitlyn's solution shows evidence of understanding that each person will run $\frac{1}{8}$ of each mile but does not extend that understanding to 7 miles.

Figure 12.31 Kaitlyn's response. Kaitlyn used a number line to determine the fraction of each mile each person will run.

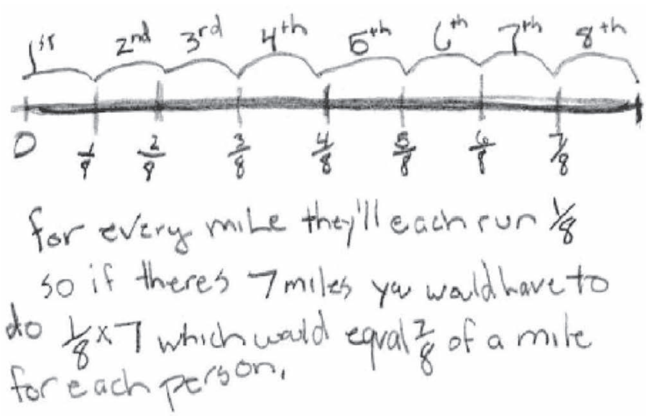
A relay race is 7 miles long. There are eight people on the relay race team and each person will run an equal distance. What fraction of the race will each person run?



Tyler's response in Figure 12.32 shows evidence of an effective use of a number line to determine the fraction of a mile each person will run.

Figure 12.32 Tyler's response. Tyler used his understanding of division, and that division can be represented as a fraction.

A relay race is 7 miles long. There are eight people on the relay race team and each person will run an equal distance. What fraction of the race will each person run?

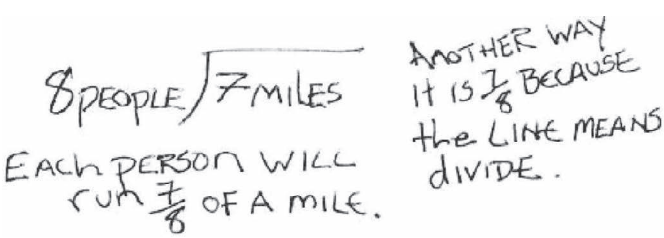


Unlike Kaitlyn, Jayden extended this understanding to 7 miles (7 miles $\times \frac{1}{8}$ mile for each mile $= 7\left(\frac{1}{8}\right) = \frac{7}{8}$). His solution shows evidence of understanding the relationship between unit fractions and division: "for every mile they'll each run $\frac{1}{8}$. So, if with 7 miles you multiply $\frac{1}{8}$ by 7, which would equal $\frac{7}{8}$ of a mile for each person."

Tyler's solution in Figure 12.33 shows evidence of understanding that division can be represented as fractions.

Figure 12.33 Jayden's response. Tyler used a number line to determine the fraction of a mile that each person will run and extended it to 7 miles.

A relay race is 7 miles long. There are eight people on the relay race team and each person will run an equal distance. What fraction of the race will each person run?



Partitive division can present unique challenges for students. In partitive division problems, students should consider two questions: 1) How much is one share? and 2) What part of the unit is that share? (Lamon, 1999).

In Figure 12.34, Cameron finds both the fraction of a pizza (one share) and the fraction of the pizzas (the unit).

Figure 12.34 Cameron's response. Cameron determined that each person would get $\frac{7}{10}$ of a pizza and $\frac{1}{5}$ of the pizzas.

Five friends equally share $3\frac{1}{2}$ pizzas.

a) What fraction of a pizza does each friend get?

$3\frac{1}{2} = \frac{7}{2}$ or $\frac{35}{10}$ $\frac{1}{5}$ of $\frac{35}{10} = \frac{7}{10}$ So each person would get $\frac{7}{10}$ of a pizza.

b) What fraction of all the pizzas does each friend get?

5 people so each person gets 1/5 of the pizzas.

In Figure 12.35, Brody does not correctly identify the size of one share and the fraction of all the pizzas that each friend receives.

Figure 12.35 Brody's response. Brody concludes that each person would receive $\frac{1}{5}$ of a pizza and $\frac{1}{5}$ of the pizzas.

Five friends equally share $3\frac{1}{2}$ pizzas.

a) What fraction of a pizza does each friend get?

each get $\frac{1}{5}$

b) What fraction of all the pizzas does each friend get?

They get $\frac{1}{5}$ of all the pizzas because if $3\frac{1}{2}$ is the whole then they each get $\frac{1}{5}$ of the whole

Brody's model shows $\frac{1}{5}$ of one pizza but is incomplete when considering the $3\frac{1}{2}$ pizzas. It is possible that he is wrestling with the conceptualization of a single pizza versus $3\frac{1}{2}$ pizzas.

Quotative Division

In quotative division, the total and the group size are known, but the number of groups is unknown. The question asked in this situation is: How many groups are there in all? All quotative problem situations involve this relationship. Quotative problems can be represented by the following equation:

Total ÷ number in each group = number of groups

An example of quotative division is shown in Figure 12.36.

Figure 12.36 Example of quotative division.

There are 6 yards of cloth.

Each pattern needs $\frac{2}{3}$ of a yard of cloth.

How many patterns can you make?

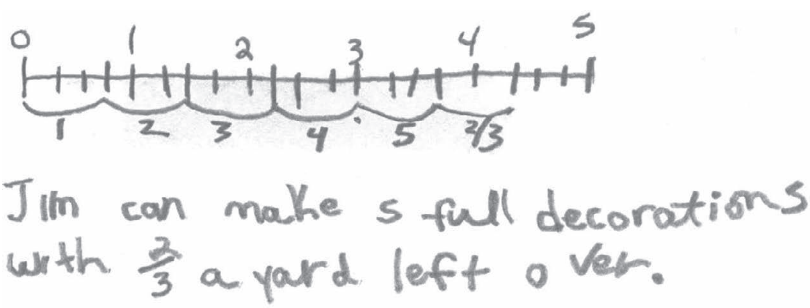
Quotative division can also pose challenges for students. More specifically, students sometimes have a difficult time identifying the unit in quotative division problems (Lamon, 1999). This difficulty can make it tricky for students to interpret a remainder.

In Figure 12.37, Cheney correctly used a number line to determine the number of full decorations. However, the evidence in Cheney's response indicates confusion with the remainder. The $\frac{2}{3}$ left is not $\frac{2}{3}$ of a yard, but $\frac{2}{3}$ of a decoration.

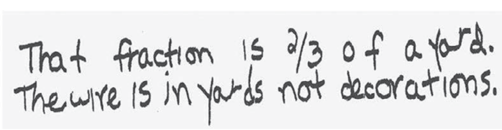
Figure 12.37 Cheney's response. Cheney used a number line to determine the number of full decorations but misinterpreted the remainder.

Jim is making decorations. He has $4\frac{1}{4}$ yards of wire. Each decoration needs $\frac{3}{4}$ of a yard of wire.

a) How many full decorations can Jim make?



b) Is the fraction left over a fraction of a decoration or a fraction of a yard of wire?



In contrast, Abigail's solution in Figure 12.38 suggests an understanding that $\frac{2}{3}$ refers to the fraction of a decoration that is left, and that there is $\frac{1}{2}$ of a yard of wire remaining.

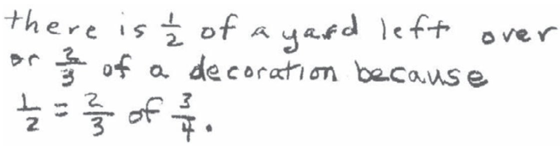
In the decoration problem, students had to make meaning of a remainder in terms of the problem context. Several studies from Silver and colleagues have shown that "students' failure to solve division problems with remainders can be attributed, at least in part, to their failure to relate the computational results to the situation in the problem." (Silver et al., 1993, p. 118)

Figure 12.38 Abigail's response. Abigail uses a model to conclude that five full decorations can be made with $\frac{2}{3}$ of a decoration and $\frac{1}{2}$ of a yard of wire left over.

Jim is making decorations. He has $4\frac{1}{4}$ yards of wire. Each decoration needs $\frac{3}{4}$ of a yard of wire.

a) How many full decorations can Jim make?

b) Is the fraction left over a fraction of a decoration or a fraction of a yard of wire?



From an instructional perspective it is important that the types of division problems that students solve vary (Van de Walle, 2004). Students should encounter partitive problems, quotative problems, problems in which the remainders are fractions, problems in which remainders are whole numbers, problems with no remainders, and so on. Mixing problems in this way will help to ensure that students do not overgeneralize one way to think about division or one way to interpret remainders.

OGAP Fraction Progression and Multiplication and Division

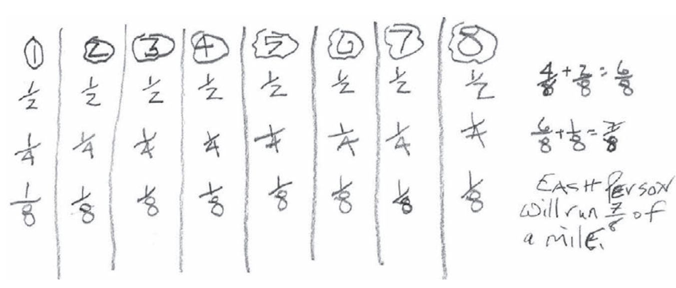
At the *Fractional Strategies* level, students' solutions show evidence of understanding the impact of multiplication and division as exemplified in Justina's response in Figure 12.7. Solutions also show evidence of using efficient strategies or reasoning when solving multiplication and division problems, as exemplified in the solutions in Figures 12.16, 12.18, and 12.33.

Transitional Strategies involve the effective use of visual models to make sense of the operation (Figures 12.21 and 12.22). At this level there can be evidence of inefficient procedures, as exemplified in Maya's solution in Figure 12.39.

The evidence at the *Early Fractional Strategy* level includes partial solutions to problems (Figure 12.31) or visual models with errors that interfere with arriving at a correct solution.

Figure 12.39 Maya's solution. While the solution is correct ("each person will run $\frac{7}{8}$ of a mile"), the strategy of "dealing out" and building up is inefficient.

A relay race is 7 miles long. There are 8 people on the relay race team and each person will run an equal distance. What fraction of the race will each person run?



Case Study—Pre-assess and Incorporate Findings into Subsequent Lessons

Before beginning a unit on multiplication of fractions, Mr. Bart, a fifth grade teacher, wanted to gather some information about his students' ideas related to multiplication of a whole number by a fraction. He decided to administer the formative assessment question shown in Figure 12.40, and use the student evidence from this item to inform his unit planning. Study the question in Figure 12.40. Why do you think Mr. Bart decided to administer this question before his students had any formal instruction on the multiplication of fractions?

Figure 12.40 Formative assessment question

Josh spends $\frac{3}{8}$ of his salary on rent each month. His monthly salary is $2,400.00. How much does Josh spend each month?

You probably noticed a few things about this question. First, it involves multiplication of a whole number by a fraction. Second, the whole number is evenly divisible by 8 which makes the problem accessible to more students. Lastly, the problem can be solved in several ways including the use of unit fraction reasoning or visual models. Mr. Bart also choose this problem because it involved a situation that provides students a helpful context in which to situate their solutions.

After completing the *OGAP Sort* Mr. Bart recorded the evidence in the *OGAP Evidence Collection Sheet* in Figure 12.41. What do you notice about the results?

Mr. Bart analyzed the student solutions. He noticed that a few students effectively used a visual model to solve the problem as exemplified in Harper's solution in Figure 12.42. Some students used the incorrect operation as exemplified in Maria's solution in Figure 12.43. Most of the students only found $\frac{1}{8}$ of the salary as shown in Elijah's solution in Figure 12.44. Study each of these solutions. Given this evidence, identify several potential first instructional steps.

Figure 12.41 *OGAP Evidence Collection Sheet.*

Items #	Content	Fractional	Transitional	Early Fractional Strategy	Non-fractional Reasoning
1	Josh's salary – fraction × whole number		HARPER Amelia MASon Michael Effectively used a visual model	Amy ★ Evelyn★ Elijah Ava ★ Charlotte Oliver★ Sophie ★ Isabelle★ Lucas Liam Ben Emma ★ William★ Mia★ ★used visual Model Only found 1/8 of the salary	Albert MARIA Thomas Incorrect operation

General Errors	Rule based, not understanding ______ Size of whole error ______ Partitioning error ______ Misinterprets model ______ Calculation error ______ Property relationship error ______	Inappropriate whole number reasoning ______ Unit missing or incorrect ______ Remainder error ______ Concept error All student solutions in Early fractional Incorrect operation Albert, MARIA, Thomas Equation error ______

Figure 12.42 Harper finds $\frac{1}{8}$ of \$2,400 and then uses an area model to determine $\frac{3}{8}$ of \$2,400 is \$900.

Josh spends $\frac{3}{8}$ of his salary on rent each month. He has a monthly salary of \$2,400. How much does Josh spend on rent each month?

Figure 12.43 The evidence indicates that Maria interpreted $\frac{3}{8}$ incorrectly as 2.66 and then divided \$2,400 by 2.66 with errors.

Josh spends $\frac{3}{8}$ of his salary on rent each month. He has a monthly salary of \$2,400. How much does Josh spend on rent each month?

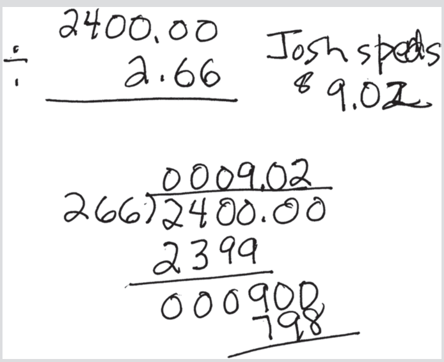
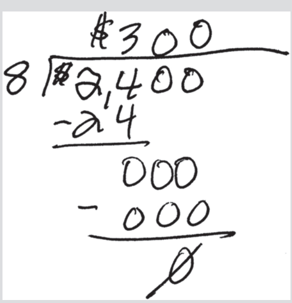


Figure 12.44 Elijah's found $\frac{1}{8}$ of \$2,400 not $\frac{3}{8}$ of \$2,400.

Josh spends $\frac{3}{8}$ of his salary on rent each month. He has a monthly salary of \$2,400. How much does Josh spend on rent each month?



After studying the evidence collection sheet Mr. Bart considered the three OGAP planning questions that follow:

1. *What are developing understandings that can be built upon?* All but a few students found $\frac{1}{8}$ of $2,400 and many used visual models. Some students effectively used visual models to find $\frac{3}{8}$ of $2,400.
2. *What issues her concern were in the student work?* Most students only found $\frac{1}{8}$ of $2,400 and a few students used the incorrect operation.
3. *What are potential next instructional steps for the whole class, small groups, or individual students?* Incorporate these findings into the next couple of lessons by building on unit fraction understanding and visual models and using contexts to bring meaning to solutions.

Mr. Bart began the next class by bringing in a pan of brownies cut into twelve equal pieces. He asked the class to consider then discuss the number of brownies in $\frac{1}{3}, \frac{1}{6}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ of the pan of brownies.

After this activity, Mr. Bart introduced the four problems shown in Figure 12.45 to the class and gave them time to work on them with a partner or small group. He reminded students that they needed to justify their answers with equations, visual models, and or words. What do you notice about the sequence of problems?

Figure 12.45 Four problems Mr. Bart introduced.

1. There are 32 students in Mark's class. One-fourth of the students in Mark's class have blue eyes. How many students have blue eyes in Mark's class?
2. There are 30 students in Anna's class. Two-thirds of the students in Anna's class have blue eyes. How many students have blue eyes in Ashley's class?
3. Max has $84.00. He decides to open a saving account and put $\frac{3}{4}$ of his money into it? What fraction of the $84.00 will he keep for spending? How much money will he put in a savings bank?
4. Three-hundred sixty people attended an outdoor concert. Five-ninths of the people sat on blankets. How many people sat on blankets?

Mr. Bart then pulled the class together for a discussion, displaying carefully chosen student solutions for each question and asking students to make sense of those solutions to see different ways each problem could be solved. The main goal of this discussion was to engage his students in thinking about how the different solutions are mathematically connected.

Mr. Bart purposefully sequenced the opening activity and problems to engage all students regardless of the level of their response on the pre-assessment question. The opening activity provided a concrete model on which to reason about their answers. All the problems he chose for this activity involved multiplying a whole number by a fraction. The first question utilizes a unit fraction while questions 2, 3, and 4 use non-unit fractions.

At the end of this lesson Mr. Bart had his students solve the exit question shown in Figure 12.46. Why do you think Mr. Bart choose this problem?

Figure 12.46 Mr. Bart's exit question

Three-fifths of the students in Danville walk to school. There are 350 students in Danville.

Three-fourths of the students in Westport walk to school. There are 320 students in Westport.

In which school do the most students walk to school? Show your work.

You probably noticed that the fraction of students who walk to school in Danville is less than the fractional amount who walk to school in Westmore. You probably also noticed that the number of students in Danville is less than the number of students in Westport. Mr. Bart was interested to see how his students apply their unit fraction reasoning to each situation without initially being influenced by the values in the problem. In this way he was looking to see if they could apply their understanding of multiplication of a fraction by a whole number to a new and slightly more challenging situation.

This case study illustrates how formative assessment questions can be used as a pre-assessment, before beginning a topic, and the evidence used to inform the planning of subsequent lessons. Ongoing use of exit questions that ask students to apply their understanding to new contexts and situations is another important feature of the OGAP formative assessment system.

Chapter Summary

This chapter focused on research related to multiplying and dividing fractions. In particular, we examined:

- the need to build procedural fluency through conceptual understanding, not through instruction focused solely on rote application of algorithms
- the importance of contextual problems in building understanding of multiplication and division of fractions
- the use of visual models and partitioning to help build multiplication and division concepts
- the difficulties students encounter as they contemplate the impact of multiplication and division involving fractions on the magnitude of a product or quotient
- the need for students to interact with a variety of situations and contexts that include both partitive and quotative division requiring different interpretations of remainders.

Looking Back

1. Mr. Way gave his class a pre-assessment prior to the upcoming unit on multiplication and division of fractions. He is concerned about Claudia's response to the division problem in Figure 12.27. Help Mr. Way by answering the following questions.
   a. What are some possible explanations for Claudia's apparent belief that "the division sign means to multiply"?
   b. What are some questions, lessons, or activities that Mr. Way could use to help Claudia develop an understanding of the similarities and differences between multiplication of fractions and division of fractions?

2. The strategy shown in Figure 12.22 is representative of Corey's solutions for fraction division problems. Corey's teacher, Mrs. Rousseau, would like Corey to use her understanding

of models to develop a more efficient approach for division of fraction problems. Examine Corey's response in Figure 12.22 and answer the questions that follow.

a. Based on the evidence, what concepts related to division of fractions does Corey appear to understand?
b. How could Mrs. Rousseau use the developing understandings you identified in question and her facility with models to help Corey develop an efficient approach for solving division of fractions problems?

3. Despite the fact that Ms. Altrui's class can use models effectively to solve equivalence, magnitude, addition, and subtraction problems, the group is struggling with using models to solve more complex multiplication and division of fraction problems. Help Ms. Altrui by studying Tracy's solution in Figure 12.24 and answering the questions that follow.
   a. What context does Tracy's model appear to represent?
   b. How could Tracy's model be modified or reinterpreted to answer the question posed in the problem?
   c. Identify questions, activities, or lessons that Ms. Altrui could use to help her class extend their models of equivalence, magnitude, and addition and subtraction problems to include effective models for multiplication and division.

4. Although Alejandro can solve both multiplication and division of fraction problems, he tends to confuse the two operations. He often misinterprets problems requiring a division strategy and solves them using multiplication. Figure 12.26 is an example of his confusion.
   a. How could Alejandro's model be altered or reinterpreted to answer the question $\frac{1}{2} \div \frac{1}{4} = ?$
   b. How might you help Alejandro to conceptualize the similarities and differences between division by a fraction and multiplication by a fraction?

5. One of Selma's typical responses to division of fraction problems is shown in Figure 12.18. Mr. Latham, Selma's teacher, wants to be sure that Selma possesses the needed conceptual understanding of division of fractions to go along with her algorithmic knowledge.
   a. Based on the evidence in her response, what does Selma appear to understand about division of fractions?
   b. What questions might Mr. Latham ask Selma to help him to determine her conceptual understanding of division of fractions?

6. Mr. Alberti is preparing for an upcoming lesson on partitive division. As part of the lesson, he plans to use the following problem. Mr. Alberti is contemplating a model that clearly shows the answers to both parts of the question.

   Five friends equally share $3\frac{1}{2}$ pizzas.

   a. What fraction of a pizza does each friend get?
   b. What fraction of all the pizzas does each friend get?
   c. Draw a model clearly showing that each friend gets $\frac{7}{12}$ of a pizza and $\frac{1}{5}$ of the pizzas.
   d. Explain how you might connect the model you drew for part (a) to the mathematical calculations $3\frac{1}{2} \div 5$ and $1 \div 5$.

7. Cheney's solution to a quotative division problem is shown in Figure 12.37.
   a. What do the numbers on the top of Cheney's number line represent?

b. What do the numbers on the bottom of Cheney's number line represent?
c. What instructional strategies might you use to help Cheney understand that the fraction $\frac{2}{3}$ represents $\frac{2}{3}$ of a decoration, not $\frac{2}{3}$ of a yard?
d. How might you show how this problem results in both $\frac{1}{2}$ a yard and $\frac{2}{3}$ of a decoration left over?

Instructional Link: Your Turn

Use the questions in Table 12.1 to help you think about how your instruction and mathematics programs provide students the opportunity to develop understandings of fraction multiplication and division.

Table 12.1 Strategies to Support Development of Concepts Related to Multiplication and Division of Fractions

Do you (or does your program):	*Yes/No*
1. Provide opportunities for students to develop conceptual understanding of multiplication and division of fractions before introducing formal algorithms?	
2. Provide opportunities for students to create and interact with visual models to help them transition from multiplication and division by whole numbers to multiplication and division by fractions?	
3. Build on students' prior experiences with models?	
4. Provide opportunities for students to consider why both multiplication and division can "make smaller" and "make larger"?	
5. Provide a variety of situations in which students are asked to recognize the appropriate fraction operation?	
6. Provide students with contexts to reason multiplicatively without relying on additive strategies?	
7. Provide opportunities at the appropriate time for students to develop computational procedures that are efficient, accurate, and result in correct answers?	
8. Provide opportunities for students to translate mathematical ideas between real-world situations, manipulatives, pictures, spoken symbols, and written symbols?	
9. Provide opportunities for students to solve fraction problems involving both partitive and quotative division?	

What gaps in your instruction or mathematics program did you identify?
How might you address these gaps?

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